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GEOMETRIC APPLICATIONS OF CRITICAL POINT THEORY TO SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACE

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Section 0—Introduction.

In a recent paper, [6], Nomizu and Rodriguez found a geometric characterization of umbilical submanifolds $M^n \subset \boldsymbol{R}^{n+p}$ in terms of the critical point behavior of a certain class of functions L_p, $p \in \boldsymbol{R}^{n+p}$, on M^n. In that case, if $p \in \boldsymbol{R}^{n+p}$, $x \in M^n$, then $L_p(x) = (d(x,p))^2$, where d is the Euclidean distance function.

The result of Nomizu and Rodriguez can be expressed as follows. Let M^n $(n \geqslant 2)$ be a connected, complete Riemannian manifold isometrically immersed in $\boldsymbol{R}^{n+p}$. Suppose there exists a dense subset D on $\boldsymbol{R}^{n+p}$ such that every function of the form L_p, $p \in D$, has index 0 or n at any of its non-degenerate critical points. Then M^n is an umbilical submanifold, that is M^n is embedded in $\boldsymbol{R}^{n+p}$ as a Euclidean subspace, $\boldsymbol{R}^n$, or a Euclidean n-sphere, S^n.

Since the set of all points $p \in \boldsymbol{R}^{n+p}$ such that L_p is a Morse function is a dense subset of $\boldsymbol{R}^{n+p}$, the above theorem could also have been stated in terms of Morse functions of the form L_p.

In this paper, we prove results analogous to those of Nomizu and Rodriguez for submanifolds of complex projective space, $P^m(\boldsymbol{C})$, endowed with the standard Fubini-Study metric.

Let M^n be a complex n-dimensional submanifold of $P^{n+p}(\boldsymbol{C})$. For $p \in P^{n+p}(\boldsymbol{C})$, $x \in M^n$, the function $L_p(x)$ which we define is essentially the distance in $P^{n+p}(\boldsymbol{C})$ from p to x. In section 2, we define the concept of a focal point of (M^n, x). We then prove an Index Theorem for L_p which states that the index of L_p at a non-degenerate critical point x is equal to the number of focal points of (M^n, x) on the geodesic in $P^{n+p}(\boldsymbol{C})$ from x to p.

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In the process, we find that if $L_p(x) = \pi/2$, then L_p has a degenerate critical point at x. Because of this, it is impossible to state the following result in terms of Morse functions of the form L_p.

Our main result is the following. Let M^n $(n \geqslant 2)$ be a connected, complete, complex n-dimensional Kählerian manifold which is holomorphically and isometrically immersed in $P^{n+p}(\boldsymbol{C})$. Assume there exists a dense subset D of $P^{n+p}(\boldsymbol{C})$ such that every function of the form L_p, $p \in D$, has index 0 or n at any of its non-degenerate critical points. Then M^n is $P^n(\boldsymbol{C})$ or $Q^n(\boldsymbol{C})$. Here $P^n(\boldsymbol{C})$ denotes a totally geodesic submanifold of $P^{n+p}(\boldsymbol{C})$, and $Q^n(\boldsymbol{C})$ is the standard complex quadric hypersurface of a totally geodesic $P^{n+1}(\boldsymbol{C}) \subset P^{n+p}(\boldsymbol{C})$.

In section 3, we prove the above result for co-dimension $p = 1$; and in section 4, we extend the result to arbitrary co-dimensions. Section 5 is devoted to a detailed study of the interesting special case $Q^n(\boldsymbol{C}) \subset P^{n+1}(\boldsymbol{C})$. We find, among other things, that the set of focal points is $P^{n+1}(\boldsymbol{R})$, a real $(n+1)$-dimensional projective space naturally embedded in $P^{n+1}(\boldsymbol{C})$.

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Section 1—Preliminaries.

We first recall the construction of the Fubini-Study metric on $P^m(\boldsymbol{C})$ (see [4], vol. II, p. 273–78 and [7], p. 514–515, for more detail). We consider $P^m(\boldsymbol{C})$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4 (we choose 4 instead of 1 for the curvature to make calculations easier).

Consider $\boldsymbol{C}^{m+1}$ with natural basis $e_0, \cdots, e_m$. The natural Hermitian inner product on $\boldsymbol{C}^{m+1}$ is defined by

$$(z, w) = \sum_{k=0}^{m} z^k \overline{w}^k$$

where

$$z = \sum_{k=0}^{m} z^k e_k \quad \text{and} \quad w = \sum_{k=0}^{m} w^k e_k \, .$$

The Euclidean metric g on $\boldsymbol{C}^{m+1}$ is given by

$$g(z, w) = \operatorname{Re}(z, w) \qquad \text{for } z, w \in \boldsymbol{C}^{m+1} \, .$$

The unit sphere

$$S^{2m+1} = \{z \in C^{m+1} \mid (z, z) = 1\}$$

is a principal fibre bundle over $P^m(C)$ with structure group S^1 and projection π. With the natural identification between vectors tangent to S^{2m+1} and vectors in C^{m+1}, one can show that for $z \in S^{2m+1}$, the tangent space to S^{2m+1} at z, which we denote as $T_z(S^{2m+1})$, is given by

$$T_z(S^{2m+1}) = \{w \in C^{m+1} \mid g(z, w) = 0\} .$$

If we define T'_z by

$$T'_z = \{w \in C^{m+1} \mid g(z, w) = g(iz, w) = 0\} ,$$

then T'_z is a subspace of $T_z(S^{2m+1})$ whose orthogonal complement is $\{iz\}$, the 1-dimensional subspace spanned by the vector iz. The distribution T' defines a connection in the principal fibre bundle $S^{2m+1}(P^m(C), S^1)$, in that T'_z is complementary to the subspace $\{iz\}$ tangent to the fibre through z, and T' is invariant by the action of S^1. Thus the projection π induces a linear isomorphism π_* of T'_z onto $T_{\pi(z)}(P^m(C))$, and π_* maps $\{iz\}$ into 0 for each $z \in S^{2m+1}$.

We define the Fubini-Study metric, $\tilde{g}$, of constant holomorphic sectional curvature 4 by the equation

$$\tilde{g}(X, Y) = g(X', Y')$$

where $X, Y \in T_p(P^m(C))$ and X', Y' are their respective horizontal lifts at z where $\pi(z) = p$. Since g is invariant by the action of S^1, the definition is independent of the choice of z. The complex structure on T'_z defined by multiplication by i induces the canonical complex structure, J, on $P^m(C)$ by means of the isomorphism π_*. Finally, π_* induces the Kählerian connection, $\tilde{\nabla}$, on $P^m(C)$ in the following way. Let X, Y be vector fields on $P^m(C)$, and let X', Y' be their respective horizontal lifts. Then for ∇' the covariant derivative on S^{2m+1}, the equation

$$\tilde{\nabla}_X Y = \pi_*(\nabla'_{X'} Y')$$

defines the Kählerian connection on $P^m(C)$.

Section 2—Focal points, the functions L_p, and the Index Theorem.

Let M^n be a connected, complex n-dimensional Kählerian manifold, and let f be a holomorphic and isometric immersion of M^n into $P^{n+p}(C)$.

Let $N(M^n)$ denote the normal bundle of M^n. Any point of $N(M^n)$ can be represented by a pair $(x, r\xi)$ where $x \in M^n$, $r \in \boldsymbol{R}$, and ξ is a unit-length vector in $T_x^{\perp}(M^n)$, the normal space to M^n at $f(x)$.

We define $\gamma(x, \xi, r)$, $-\infty < r < \infty$, to be the geodesic in $P^{n+p}(\boldsymbol{C})$ parametrized by arc-length parameter r such that

$$\gamma(x, \xi, 0) = f(x) \quad \text{and} \quad \vec{\gamma}(x, \xi, 0) = \xi \ .$$

In terms of the vector representation of $P^{n+p}(\boldsymbol{C})$, $\gamma(x, \xi, r)$ can be described as follows. Let $w \in S^{2(n+p)+1}$ such that $\pi(w) = f(x)$, and let $\xi' \in T'_w$ such that $\pi_*(\xi') = \xi$. Then

$$\gamma(x, \xi, r) = \pi(\cos r\, w + \sin r\, \xi') \ .$$

Of course, $\gamma(x, \xi, r)$ does not depend on the choice of w.

We define a map $F\colon N(M^n) \to P^{n+p}(\boldsymbol{C})$ by

$$F(x, r\xi) = \gamma(x, \xi, r) \ .$$

We note that for any values of x, ξ and r the following holds,

$$F(x, (r + \pi)\xi) = F(x, r\xi) \ .$$

Thus we may restrict the range of values of r to $-\pi/2 \le r \le \pi/2$.

For $\xi \in T_x^{\perp}(M^n)$, let A_ξ denote the symmetric endomorphism of $T_x(M^n)$ corresponding to the second fundamental form of M^n at x in the direction of ξ. We first prove the following proposition.

PROPOSITION 1. *Let $(x, r\xi) \in N(M^n)$. Then F_*, the Jacobian of F, is degenerate at $(x, r\xi)$ in precisely the following cases:*

(i) *If $r = \pm\pi/2$, then F_* is degenerate.*

(ii) *For $-\pi/2 \le r \le \pi/2$, there is a contribution of $\nu > 0$ to the nullity of F_* at $(x, r\xi)$ if*

$$\cot r = k$$

where k is an eigen-value of multiplicity ν of A_ξ.

Proof. Fix the point $(x, r\xi) \in N(M^n)$; we want to examine the nullity of F_* at $(x, r\xi)$. We assume for the moment that $r \neq 0$, and by replacing ξ by $-\xi$ if necessary, we may assume $r > 0$.

Let U be a local co-ordinate neighborhood of x in M^n with local co-ordinates $u^1, u^2, \cdots, u^{2n}$. Choose orthonormal normal vector fields $\xi_1, \cdots,$

$\xi_p, J\xi_1, \cdots, J\xi_p$ on U such that $\xi_1(x) = \xi$. For ease in notation, we let $\xi_{p+j} = J\xi_j$ for $1 \leq j \leq p$. For $u \in U$, $\eta \in T_u^\perp(M^n)$, we can write

$$\eta = \mu\left(\left(1 - \sum_{j=2}^{2p} (t^j)^2\right)^{1/2} \xi_1 + t^2\xi_2 + \cdots + t^{2p}\xi_{2p}\right)$$

where $0 \leq \mu < \infty$, $0 \leq |t^j| \leq 1$ for all j, and $\sum_{j=2}^{2p} (t^j)^2 \leq 1$. The t^j are the direction cosines of η and $\mu = \|\eta\|$. The co-ordinates $u^1, \cdots, u^{2n}, \mu, t^2, \cdots, t^{2p}$ are local co-ordinates for $N(U)$.

Let $w \in S^{2(n+p)+1}$. To avoid confusion, we will denote the map $\pi_*: T'_w \to T_{\pi(w)}(P^{n+p}(C))$ by $(\pi_*)_w$ when such precision is required.

Now let $w \in S^{2(n+p)+1}$ such that $\pi(w) = f(x)$. We define $z \in S^{2(n+p)+1}$ by the vector equation

$$z = \cos r\, w + \sin r\, \xi'$$

where $(\pi_*)_w(\xi') = \xi$. Then $F(x, r\xi) = \pi(z)$. For any j, $2 \leq j \leq 2p$, the definition of F implies that

$$F_*\left(\frac{\partial}{\partial t^j}\right)\bigg|_{(x,r\xi)} = (\pi_*)_z(\vec{\eta}(t^j))\bigg|_{t^j=0}$$

where $\eta(t^j)$ is a curve on $S^{2(n+p)+1}$ defined by

$$\eta(t^j) = \cos r\, w + \sin r((1 - (t^j)^2)^{1/2}\xi'_1 + t^j\xi'_j) ,$$

where ξ'_1, ξ'_j are the horizontal lifts of ξ_1, ξ_j respectively to T'_w. We see that $\eta(0) = z$ for any j.

If $r = \pm\pi/2$, we will show $F_*(\partial/\partial t^{p+1})|_{(x,r\xi)} = 0$. In that case, $\xi_{p+1} = J\xi_1$ and for $r = \pi/2$

$$\vec{\eta}(t^{p+1})|_{t^{p+1}=0} = i\eta(0) = iz ,$$

and

$$F_*\left(\frac{\partial}{\partial t^{p+1}}\right)\bigg|_{(x,r\xi)} = (\pi_*)_z(iz) = 0 .$$

The case $r = -\pi/2$ is handled similarly. This proves (i).

For $|r| < \pi/2$, a straight-forward calculation which we omit shows,

$$F_*\left(\frac{\partial}{\partial t^j}\right)\bigg|_{(x,r\xi)} = \sin r\,(\pi_*)_z(\xi'_j) \neq 0 , \qquad \text{for } 2 \leq j \leq 2p ,$$

and

$$F_*\left(\frac{\partial}{\partial\mu}\right)\Big|_{(x,r\xi)} = (\pi_*)_z(\sin r\, w + \cos r\, \xi_1') \neq 0 .$$

In fact, these computations show that if

$$V = a_1\left(\frac{\partial}{\partial\mu}\right) + \sum_{j=2}^{2p} a_j\left(\frac{\partial}{\partial t^j}\right) \in T_{(x,r\xi)}(N(U)) ,$$

then $F_*(V) = 0$ only if $a_j = 0$ for all j. If we let

$$X = \sum_{j=1}^{2n} b_j \left(\frac{\partial}{\partial u^j}\right) \in T_{(x,r\xi)}(N(U)) ,$$

we shall next compute $F_*(X)$. That computation and the above will show that

$$F_*(X + V) = 0 \qquad \text{only if } V = 0 .$$

(We remark that if $r = 0$, we must choose a slightly different co-ordinate system to obtain the same result.)

Consider a vector $X = \sum_{j=1}^{2n} b_j(\partial/\partial u^j) \in T_{(x,r\xi)}(N(U))$. If $r = 0$, one easily shows $F_*(X) = X$ and so F_* is non-degenerate at $(x, 0)$. Assume again, then, that $r > 0$. Considering $T_{(x,r\xi)}(N(U))$ as $T_x(U) \oplus \boldsymbol{R}^{2p}$, we can write $X = (Y, 0)$ where $Y \in T_x(U)$. To facilitate the computation of $F_*(X)$, we assume that the vector field ξ_1 defined above has been chosen so that

$$\nabla_Y^\perp \xi_1 = 0$$

where $\nabla^\perp$ is the connection in the normal bundle.

Locally, i.e. for some $\varepsilon > 0$, there is a curve $\beta(t)$, $-\varepsilon < t < \varepsilon$, in M^n such that $\beta(0) = x$ and $\vec{\beta}(0) = Y$. Let $\alpha(t)$ be the lift of $\beta(t)$ to $S^{2(n+p)+1}$ so that $\alpha(0) = w$, and $\pi(\alpha(t)) = f(\beta(t))$ for $-\varepsilon < t < \varepsilon$.

If we define the curve $\eta(t)$ in $S^{2(n+p)+1}$ by

$$\eta(t) = \cos r\, \alpha(t) + \sin r\, \xi_1'(\alpha(t)) ,$$

then $\eta(0) = z$, and

(1) $$F_*(X) = (\pi_*)_z(\vec{\eta}(0)) .$$

We need to find the component of $\vec{\eta}(0)$ in T_z'. Considering $\eta(t)$ as a curve in $\boldsymbol{C}^{n+p+1}$, we find

(2) $$\vec{\eta}(t) = \cos r\, \vec{\alpha}(t) + \sin r\, D_{\vec{\alpha}(t)}\xi_1'$$

where D is the Euclidean covariant derivative in C^{n+p+1}. Since $g(\vec{\alpha}(t), \xi_1'(\alpha(t))) = 0$ for $-\varepsilon < t < \varepsilon$, we have $D_{\vec{\alpha}(t)}\xi_1' = \nabla'_{\vec{\alpha}(t)}\xi_1'$. Thus we have by evaluating (2) at $t = 0$,

$$\vec{\eta}(0) = \cos r\,\vec{\alpha}(0) + \sin r\,\nabla'_{\vec{\alpha}(0)}\xi_1' \,. \tag{3}$$

One can show by a straight-forward calculation that

$$g(\vec{\eta}(0), z) = 0 = g(\vec{\eta}(0), iz) \,,$$

and hence $\vec{\eta}(0) \in T_z'$. Since (π_*) is an isomorphism on T_z', we have shown

$$(\pi_*)_z\vec{\eta}(0) = 0 \qquad \text{if and only if } \vec{\eta}(0) = 0 \,. \tag{4}$$

To find when $\vec{\eta}(0) = 0$, we proceed as follows. We displace the vector $\vec{\eta}(0) \in T_z'$ by Euclidean parallelism and consider $\vec{\eta}(0) \in T_w(S^{2(n+p)+1})$. Equation (3) shows that, in fact, $\vec{\eta}(0) \in T_w'$ since $\vec{\alpha}(t)$ and $\xi_1'(\alpha(t)) \in T'_{\alpha(t)}$ for all t. Now, applying the isomorphism $(\pi_*)_w$ we have

$$(\pi_*)_w(\vec{\alpha}(0)) = \vec{\beta}(0) = Y \tag{5}$$

and

$$(\pi_*)_w(\nabla'_{\vec{\alpha}(0)}\xi_1') = \tilde{\nabla}_Y\xi_1 \,. \tag{6}$$

But $\tilde{\nabla}_Y\xi_1 = -A_{\xi_1}Y + \nabla_Y^{\perp}\xi_1$, and since $\xi_1(x) = \xi$ and $\nabla_Y^{\perp}\xi_1 = 0$, we have

$$\tilde{\nabla}_Y\xi_1 = -A_\xi Y \,. \tag{7}$$

Thus, using (5), (6), (7) and applying $(\pi_*)_w$ to (3) we have

$$(\pi_*)_w\vec{\eta}(0) = \cos r\, Y - \sin r\, A_\xi Y \,. \tag{8}$$

Since $\vec{\eta}(0) \in T_w'$, we know $(\pi_*)_w\vec{\eta}(0) = 0$ if and only if $\vec{\eta}(0) = 0$. From (8) we see that $\vec{\eta}(0) = 0$ if and only if $k = \cot r$ is an eigen-value of A_ξ and Y is an eigen-vector of k. From (1) and (4) we see that this also gives necessary and sufficient conditions under which $F_*(X) = 0$. If $\cot r$ is an eigen-value of multiplicity ν, then it is clear that F_* vanishes on a ν-dimensional subspace of $T_{(x,r\xi)}N(M^n)$, i.e. F_* has nullity ν.

Q.E.D.

Since the degeneracies of F_* of type (i) in Proposition 1 depend only on $r = \pm\pi/2$ and not on M^n or the point $x \in M^n$, they provide no information about M^n itself. Thus such degeneracies will not be included in the following definition of a focal point of (M^n, x). In the definition it is understood, as above, that ξ is a unit vector in $T_x^{\perp}(M^n)$ and $-\pi/2 \leq r \leq \pi/2$.

DEFINITION. A point $p \in P^{n+p}(C)$ is called a focal point of (M^n, x) of multiplicity ν if $p = F(x, r\xi)$ and $\cot r$ is an eigen-value of multiplicity $\nu > 0$ of A_ξ. (We say p is a focal point of M^n if p is a focal point of (M^n, x) for some $x \in M^n$.)

We now proceed to define the functions L_p. For $p, q \in P^{n+p}(C)$, and $z, w \in S^{2(n+p)+1}$ such that $\pi(z) = p$, $\pi(w) = q$, we define

$$L_p(q) = \cos^{-1}(|(z, w)|^2) ,$$

where $0 \leq \cos^{-1}(\ \ ) \leq \pi/2$. One easily checks that the definition of $L_p(q)$ is independent of the choice of z, w.

We remark that $L_p(q)$ is essentially $d(p, q)$ the distance in $P^{n+p}(C)$ from p to q which is given by

$$d(p, q) = \cos^{-1}(|(z, w)|) .$$

We use $L_p(q)$ rather than $d(p, q)$ to gain differentiability at points q such that $L_p(q) = \pi/2$. i.e. $(z, w) = 0$.

For $p \in P^{n+p}(C)$, $x \in M^n$, we define $L_p(x) = L_p(f(x))$. If $p \notin f(M^n)$, then the restriction of L_p to M^n is a differentiable function on M^n. From this point on, we will only consider L_p such that $p \notin f(M^n)$. For such a point p, the following proposition describes the critical points of the function L_p on M^n.

PROPOSITION 2. *Let $p \in P^{n+p}(C)$, and $x_0 \in M^n$ such that $f(x_0) \neq p$. Then x_0 may be a critical point of L_p in precisely the following 2 ways.*

(i) *If $L_p(x_0) = \pi/2$, then L_p has a degenerate maximum at x_0.*

(ii) *If $L_p(x_0) < \pi/2$, L_p has a critical point at x_0 if and only if p can be expressed as $F(x_0, r\xi)$ where ξ is a unit vector in $T_{x_0}^{\perp}(M^n)$ and $0 < r < \pi/2$. In this case,*

(a) *x_0 is a degenerate critical point if and only if* $\cot r$ *is an eigen-value of A_ξ.*

(b) *The index of L_p at a non-degenerate critical point x_0 equals the number of eigen-values, k_i, of A_ξ such that $k_i >$* $\cot r$. *Each k_i is counted with its multiplicity.*

Proof. Fix $x_0 \in M^n$, and let $p \in P^{n+p}(C)$. Fix $z_0 \in S^{2(n+p)+1}$ such that

$\pi(z_0) = p$. Let X be a vector field on M^n, and let X' be the horizontal lift of X. For $x \in M^n$ and $w \in S^{2(n+p)+1}$ such that $\pi(w) = x$, we have

$$\begin{aligned} XL_p(x) &= (\pi_* X')L_p(x) = X'(L_p \circ \pi)(w) \\ &= X'(\cos^{-1}(|(z_0, w)|^2)) = X'(\cos^{-1}(g(z_0, w)^2 + g(z_0, iw)^2)) \\ &= \frac{-[2g(z_0, w)X'(g(z_0, w)) + 2g(z_0, iw)X'(g(z_0, iw))]}{(1 - [g(z_0, w)^2 + g(z_0, iw)^2]^2)^{1/2}} . \end{aligned}$$

But $X'(g(z_0, w)) = g(z_0, X'_w)$, and we obtain

$$XL_p(x) = \frac{-2[g(z_0, w)g(z_0, X'_w) + g(z_0, iw)g(z_0, iX'_w)]}{(1 - [g(z_0, w)^2 + g(z_0, iw)^2]^2)^{1/2}} . \tag{9}$$

In particular, to find $XL_p(x_0)$, we can choose $w_0 \in S^{2(n+p)+1}$ such that $\pi(w_0) = x_0$, and such that $g(z_0, iw_0) = 0$ and $0 \leq g(z_0, w_0) < 1$. We know $g(z_0, w_0) < 1$ since $p \neq f(x_0)$. From (9) we then obtain,

$$XL_p(x_0) = \frac{-2[g(z_0, w_0)g(z_0, X'_{w_0})]}{(1 - g(z_0, w_0)^4)^{1/2}} . \tag{10}$$

From (10) we see that to have $XL_p(x_0) = 0$, we must have either,

(i) $g(z_0, w_0) = 0$ or

(ii) $g(z_0, X'_{w_0}) = 0$.

In case (i) x_0 is obviously a maximum of L_p since $L_p(x_0) = \pi/2$ which is the maximum value L_p attains on $P^{n+p}(C)$. A direct calculation of the Hessian of L_p at x_0 would show that the Hessian is degenerate, and hence x_0 is a degenerate maximum of L_p. We omit that argument here and appeal instead to the following geometric argument. The set of points

$$P^{n+p-1}(C) = \{q \in P^{n+p}(C) \,|\, L_p(q) = \pi/2\}$$

is a totally geodesic hypersurface of $P^{n+p}(C)$ given by the image under the projection π of $S^{2(n+p)-1}$ where

$$S^{2(n+p)-1} = S^{2(n+p)+1} \cap \{w \in C^{n+p+1} \,|\, (z_0, w) = 0\} .$$

This $P^{n+p-1}(C)$ is the set of zeroes of an analytic function on $P^{n+p}(C)$. If $f(x_0) \in f(M^n) \cap P^{n+p-1}$, then in a neighborhood U of x_0 in M^n, the set $f(U) \cap P^{n+p-1}$ is the set of zeroes of an analytic function on U. It follows essentially from the Weierstrass Preparation Theorem (see [1], p. 37–43) that $f(U) \cap P^{n+p-1}(C)$ is a sub-variety of U of dimension j, where $j \geq n-1$. For $n \geq 2$, this illustrates that x_0 is not an isolated maximum of L_p on M^n; clearly then, x_0 is a degenerate maximum. This proves (i).

Now we assume $g(z_0, w_0) > 0$, i.e. $L_p(x_0) < \pi/2$. Since $L_p(x_0) \neq 0$, we know $g(z_0, w_0) < 1$; and so there exists r, $0 < r < \pi/2$, so that $\cos r = g(z_0, w_0)$. Then it is easy to show,

$$z_0 = \cos r\, w_0 + \sin r\, \xi' \tag{11}$$

where $\xi' \in T'_{w_0}$ and $\|\xi'\| = 1$. Then,

$$g(z_0, X'_{w_0}) = \sin r\, g(\xi', X'_{w_0})$$

for X'_{w_0} the horizontal lift of $X \in T_{x_0}(M^n)$. This and (10) imply that if $L_p(x_0) < \pi/2$, then x_0 is a critical point of L_p if and only if $\pi_*(\xi') = \xi \in T^\perp_{x_0}(M^n)$; in that case, $p = F(x_0, r\xi)$ and we have proven (ii).

Now for $p = F(x_0, r\xi)$, $0 < r < \pi/2$, we wish to prove (a) and (b). We first compute the Hessian of L_p at x_0. Let X, Y be vector fields on M^n and X', Y' their respective horizontal lifts. We have shown

$$XL_p(x) = \frac{-2[g(z_0, w)g(z_0, X'_w) + g(z_0, iw)g(z_0, iX'_w)]}{(1 - [g(z_0, w)^2 + g(z_0, iw)^2]^2)^{1/2}} \tag{9}$$

where $\pi(w) = x$.

We now find $YXL_p(x_0)$. For w_0 as chosen above,

$$g(z_0, X'_{w_0}) = g(z_0, iX'_{w_0}) = 0 \quad \text{and} \quad g(z_0, iw_0) = 0\ .$$

We also know that

$$Y'(g(z_0, X'_w)) = g(z_0, D_{Y'}X')$$

where D is the Euclidean covariant derivative in C^{n+p+1}. Using these facts we differentiate (9) to find $YXL_p(x)$ and then evaluate at x_0 obtaining

$$YXL_p(x_0) = \frac{-2g(z_0, w_0)g(z_0, D_{Y'}X')\,|_{w_0}}{(1 - g(z_0, w_0)^4)^{1/2}}\ . \tag{12}$$

But we know $g(z_0, w_0) = \cos r$ so

$$1 - g(z_0, w_0)^4 = 1 - \cos^4 r = \sin^2 r(1 + \cos^2 r)\ ,$$

and we re-write (12) as

$$(13) \qquad YXL_p(x_0) = \frac{-2\cos r\, g(z_0, D_{Y'}X')|_{w_0}}{(1+\cos^2 r)^{1/2}\sin r} .$$

From well-known properties of the embedding of $S^{2(n+p)+1}$ in C^{n+p+1}, we know that for any $w \in S^{2(n+p)+1}$,

$$(14) \qquad D_{Y'}X'|_w = \nabla'_{Y'}X'|_w - g(X', Y')w .$$

We can also write

$$(15) \qquad \nabla'_{Y'}X' = W + \alpha'(X', Y')$$

where $\pi_*(W) = \nabla_Y X$, where ∇ is the covariant derivative on M^n, and

$$\pi_*(\alpha'(X', Y')) = \alpha(X, Y) ,$$

where $\alpha(X, Y)$ is the second fundamental form of the immersion f. Now since $\pi_*(\xi') \in T^\perp_{x_0}(M^n)$, we have $g(\xi', W) = 0$. Since $\xi', W \in T'_{w_0}$, we know

$$g(w_0, \xi') = 0 = g(w_0, W) .$$

Thus (11), (14), and (15) yield,

$$(16) \qquad g(z_0, D_{Y'}X')|_{w_0} = \sin r\, g(\xi', \alpha'(X', Y'))|_{w_0} - \cos r\, g(X', Y')|_{w_0} .$$

But

$$\begin{aligned} g(\xi', \alpha'(X', Y'))|_{w_0} &= \tilde{g}(\xi, \alpha(X, Y))|_{x_0} \\ &= \tilde{g}(A_\xi X, Y)|_{x_0} . \end{aligned}$$

Thus (16) becomes

$$g(z_0, D_{Y'}X')|_{w_0} = \sin r\, \tilde{g}(A_\xi X, Y) - \cos r\, \tilde{g}(X, Y)|_{x_0}$$

and (13) becomes

$$(17) \qquad YXL_p(x_0) = \frac{2\cos r}{(1+\cos^2 r)^{1/2}} \tilde{g}((-A_\xi + \cot r\, I)X, Y)|_{x_0}$$

where I is the identity endomorphism on $T_{x_0}(M^n)$.

From this expression for the terms of the Hessian of L_p at x_0, we conclude that x_0 is a degenerate critical point of L_p, if and only if $\cot r = k$ for k an eigen-value of A_ξ. This proves (a).

The index of L_p at a non-degenerate critical point x_0 is defined to be the number of negative eigen-values of the Hessian of L_p at x_0. For $\cot r \neq k_i$ for any eigen-value k_i of A_ξ, we see from (17) that the index

of L_p at x_0 is the number of k_i such that $k_i > \cot r$. This proves (b).
Q.E.D.

Propositions (1) and (2) yield immediately the following theorem:

THEOREM 1 (Index Theorem for L_p). *Let $p = F(x, r\xi)$ for $0 < r < \pi/2$. Suppose L_p has a non-degenerate critical point at x. Then the index of L_p at x equals the number of focal points of (M^n, x) which lie on the geodesic in $P^{n+p}(C)$ from $f(x)$ to p. Each focal point is counted with its multiplicity.*

Section 3—A Characterization of $P^n(C)$ and $Q^n(C)$.

We now proceed to the main result of this article which we state here.

THEOREM 2. *Let M^n $(n \geq 2)$ be a connected, complete, complex n-dimensional Kählerian manifold which is holomorphically and isometrically immersed in $P^{n+p}(C)$. If there exists a dense subset D of $P^{n+p}(C)$ such that every function of the form L_p, $p \in D$, has index 0 or n at any of its non-degenerate critical points, then M^n is embedded in $P^{n+p}(C)$ as $P^n(C)$ or $Q^n(C)$.*

In the above statement, $P^n(C)$ stands for a totally geodesic submanifold of $P^{n+p}(C)$, and $Q^n(C)$ is the standard complex quadric hypersurface of some totally geodesic $P^{n+1}(C)$. In $P^{n+1}(C)$ has homogeneous co-ordinates $(z_0, \cdots, z_{n+1})$, then $Q^n(C)$ is defined by the equation

$$z_0^2 + \cdots + z_{n+1}^2 = 0 .$$

In the remainder of this section we assume that M^n satisfies the hypotheses of Theorem 2. To begin the proof of Theorem 2, we state the following proposition. Its proof, which we omit here, depends on Propositions 1 and 2. With minor changes, the proof is identical to the corresponding proposition for submanifolds of $\boldsymbol{R}^m$ proven by Nomizu and Rodriguez ([6], p. 199).

PROPOSITION 3. *Let D be a dense subset of $P^{n+p}(C)$. Assume that for $p \in P^{n+p}(C)$, L_p has a non-degenerate critical point of index j at $x \in M^n$. Then there exists $q \in D$, $y \in M^n$ such that L_q has a non-degenerate critical point of index j at y (q and y may be chosen as close to p and x, respectively, as desired).*

Using Proposition 3 and the Index Theorem, we now prove the following proposition which is sufficient to complete the proof of Theorem 2 for the case of co-dimension $p = 1$.

PROPOSITION 4. *Let $x \in M^n$ and ξ be a unit-length vector in $T_x^\perp(M^n)$. Then there exists $\lambda \geq 0$ such that $A_\xi^2 = \lambda^2 I$ on $T_x(M^n)$.*

Proof. Fix $x \in M^n$ and ξ a unit-length vector in $T_x^\perp(M^n)$. If A_ξ has no non-zero eigen-values, then $A_\xi = 0$ and the proof is complete.

Suppose A_ξ has at least one non-zero eigen-value. It is known that A_ξ must have the form

$$A_\xi = \begin{bmatrix} k_1 & & & & & & & 0 \\ & k_2 & & & & & & \\ & & \ddots & & & & & \\ & & & k_n & & & & \\ & & & & -k_1 & & & \\ & & & & & -k_2 & & \\ & & & & & & \ddots & \\ 0 & & & & & & & -k_n \end{bmatrix}$$

when diagonalized for $k_i \geq 0$, $1 \leq i \leq n$. Let λ be the largest of the eigen-values. If $k_i = \lambda$ for $1 \leq i \leq n$, then $A_\xi^2 = \lambda^2 I$ and the proof is finished. If $k_i \neq \lambda$ for some i, let $\beta \geq 0$ be the second largest of the non-negative eigen-values. Choose $r, 0 < r < \pi/2$, such that $\beta < \cot r < \lambda$. For $p = F(x, r\xi)$, Proposition 2 implies that L_p has a non-degenerate critical point of index j at x where $0 < j \leq 2n$. Since $\lambda > \cot r > k_i$, for any $k_i \neq \lambda$, Proposition 2 also implies that j equals the multiplicity of λ.

For D as in Theorem 2, Proposition 3 implies that there exists $q \in D$ and $y \in M^n$ such that L_q has a non-degenerate critical point of index j at y. Since $j > 0$, the hypothesis on the index of $L_q, q \in D$, at a non-degenerate critical point implies that $j = n$. Thus λ has multiplicity equal to n, and again we conclude $A_\xi^2 = \lambda^2 I$. Q.E.D.

Remark 1. For the case when M^n is a hypersurface of $P^{n+1}(C)$, Proposition 4 yields the proof of Theorem 2 in the following way.

The condition that $A_\xi^2 = \lambda^2 I$ for any $\xi \in T_x^\perp(M^n)$ and any $x \in M^n$ implies that M^n is an Einstein manifold. This is clear from the following equation (see [8], p. 253). For $S(X, Y)$, the Ricci tensor of M^n, it is true that

$$\begin{aligned} S(X, Y) &= -2\tilde{g}(A_\xi^2 X, Y) + 2(n+1)\tilde{g}(X, Y) \\ &= 2(n+1-\lambda^2)\tilde{g}(X, Y) . \end{aligned}$$

Since the real dimension of M^n exceeds 2, a classical theorem (see [4], Vol. I, p. 292) implies that $2(n+1-\lambda^2)$ is indeed constant on M^n. Thus M^n is an Einstein manifold. Theorem 2 then follows from the following result of Brian Smyth ([8], p. 265).

THEOREM (Smyth). *For $n \geq 2$, $P^n(C)$ and $Q^n(C)$ are the only complex hypersurfaces of $P^{n+1}(C)$ which are complete and Einstein.*

(end of Remark 1).

Section 4—Reducing the co-dimension.

To complete the proof of Theorem 2 for arbitrary co-dimensions, we will show that under the hypotheses of Theorem 2, M^n is actually a hypersurface of a totally geodesic $P^{n+1}(C) \subset P^{n+p}(C)$.

We first must introduce the concept of the first normal space of M^n at $x \in M^n$.

DEFINITION. For $x \in M^n$, *the first normal space*, $N_1(x)$, is the orthogonal complement in $T_x^\perp(M^n)$ of the set

$$N_0(x) = \{\xi \in T_x^\perp(M^n) \mid A_\xi = 0\} .$$

We define a new inner product, $\langle\ ,\ \rangle$, on $N_1(x)$ by

$$\langle \xi, \eta \rangle = \operatorname{trace} A_\xi A_\eta \qquad \text{for } \xi, \eta \in N_1(x) .$$

One easily checks that $\langle\ ,\ \rangle$ is a positive definite inner product on $N_1(x)$, and that for $\xi, \eta \in N_1(x)$,

$$\langle J\xi, J\eta \rangle = \langle \xi, \eta \rangle \tag{18}$$

and

$$\langle \xi, J\xi \rangle = 0 . \tag{19}$$

For $\xi \in N_1(x)$, Proposition 4 implies $A_\xi^2 = \lambda^2 I$ for $\lambda > 0$. Then it is easy to see that $T_x(M^n)$ can be decomposed as

$$T_x(M^n) = T_\xi^+ \oplus T_\xi^-$$

where

$$T_\xi^+ = \{X \in T_x(M^n) \,|\, A_\xi X = \lambda X\}$$

and

$$T_\xi^- = \{X \in T_x(M^n) \,|\, A_\xi X = -\lambda X\} .$$

It is a simple matter to show that if $X \in T_\xi^+$, then $JX \in T_\xi^-$; and if $X \in T_\xi^-$, then $JX \in T_\xi^+$. We employ the inner product $\langle\ ,\ \rangle$ in the following proposition to prove that $N_1(x)$ has complex dimension no larger than 1 for all $x \in M^n$.

PROPOSITION 5. *Let $x \in M^n$ and let k be the complex dimension of $N_1(x)$. Then $k \leq 1$.*

Proof. Assume $k > 1$. Choose $\xi_1, \cdots, \xi_k$ so that *with respect to the inner product* $\langle\ ,\ \rangle$, the vectors $\xi_1, \cdots, \xi_k$, $J\xi_1, \cdots, J\xi_k$ from an orthonormal basis for $N_1(x)$.

We know there is a positive function λ on $N_1(x)$ such that $A_\xi^2 = \lambda^2(\xi)I$ for any $\xi \in N_1(x)$. If $e_1, \cdots, e_n$ are an orthonormal basis for $T^+ = T_{\xi_1}^+$, then $Je_1, \cdots, Je_n$ are an orthonormal basis for $T^- = T_{\xi_1}^-$. With respect to the basis Ω for $T_x(M^n)$,

$$\Omega = \{e_1, \cdots, e_n, Je_1, \cdots, Je_n\} ,$$

the endomorphism A_{ξ_1} is represented by the matrix

$$A_{\xi_1} = \begin{bmatrix} \lambda(\xi_1)I_n & 0 \\ 0 & -\lambda(\xi_1)I_n \end{bmatrix} \tag{20}$$

where I_n is an $n \times n$ identity matrix.

Fix $j, 2 \leq j \leq k$. Consider $X \in T^+$, and suppose $A_{\xi_j}X = Y + Z$ where $Y \in T^+$, $Z \in T^-$. First of all, we have

$$A_{\xi_1+\xi_j}^2 X = \lambda^2(\xi_1 + \xi_j)X . \tag{21}$$

But also we find,

$$\begin{aligned} A_{\xi_1+\xi_j}^2 X &= A_{\xi_1+\xi_j}A_{\xi_1+\xi_j}X = A_{\xi_1}^2 X + (A_{\xi_1}A_{\xi_j} + A_{\xi_j}A_{\xi_1})X + A_{\xi_j}^2 X \\ &= \lambda^2(\xi_1)X + \lambda^2(\xi_j)X + \lambda(\xi_1)(Y - Z) + \lambda(\xi_1)(Y + Z) \\ &= (\lambda^2(\xi_1) + \lambda^2(\xi_j))X + 2\lambda(\xi_1)Y . \end{aligned} \tag{22}$$

Then (21) and (22) yield

$$Y = \mu X \,, \qquad \text{where } \mu = [\lambda^2(\xi_1 + \xi_j) - \lambda^2(\xi_1) - \lambda^2(\xi_j)]/2\lambda(\xi_1) \,. \tag{23}$$

Since we see that μ does not depend on the choice of X, we have shown that for any $X \in T^+$,

$$A_{\xi_j}X = \mu X + Z \qquad \text{where } Z \in T^- \,. \tag{24}$$

From (24) we can also compute for $X \in T^+$,

$$A_{\xi_j}JX = -JA_{\xi_j}X = -J(\mu X + Z) = -\mu JX - JZ \,. \tag{25}$$

Equations (24) and (25) and the fact that A_{ξ_j} is symmetric imply that with respect to the basis Ω, A_{ξ_j} has the form

$$A_{\xi_j} = \begin{bmatrix} \mu I_n & {}^tB \\ B & -\mu I_n \end{bmatrix} \tag{26}$$

where B is an $n \times n$ matrix.

Since ξ_1 and ξ_j are orthogonal with respect to $\langle\ ,\ \rangle$, we know

$$\operatorname{trace} A_{\xi_1}A_{\xi_j} = 0 \,. \tag{27}$$

However, equations (20) and (26) imply that with respect to the basis Ω,

$$A_{\xi_1}A_{\xi_j} = \begin{bmatrix} \lambda(\xi_1)\mu I_n & {}^tB \\ B & \lambda(\xi_1)\mu I_n \end{bmatrix} \,. \tag{28}$$

From (28) we compute $\operatorname{trace} A_{\xi_1}A_{\xi_j} = 2n\lambda(\xi_1)\mu$. Comparing this with (27), we conclude $\mu = 0$, since $\lambda(\xi_1) > 0$. Hence (26) becomes

$$A_{\xi_j} = \begin{bmatrix} 0 & {}^tB \\ B & 0 \end{bmatrix} \,. \tag{29}$$

From the fact that $\tilde{\nabla}$ is a Kählerian connection, one easily shows that $A_{J\xi_j} = JA_{\xi_j}$. From (29), we see that as a matrix,

$$A_{J\xi_j} = JA_{\xi_j} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} 0 & {}^tB \\ B & 0 \end{bmatrix} = \begin{bmatrix} -B & 0 \\ 0 & {}^tB \end{bmatrix} \,.$$

This shows that $A_{J\xi_j}$ maps T^+ into T^+ and T^- into T^-. This fact and computations similar to those leading to (23) show that for $X \in T^+$,

$$A_{J\xi_j}X = \nu X \,,$$

where

$$\nu = [\lambda^2(\xi_1 + J\xi_j) - \lambda^2(\xi_1) - \lambda^2(J\xi_j)]/2\lambda(\xi_1) \,.$$

Thus we can represent $A_{J\xi_j}$ as,

$$A_{J\xi_j} = \begin{bmatrix} \nu I_n & 0 \\ 0 & -\nu I_n \end{bmatrix} . \tag{30}$$

Now equations (20) and (30) imply that $A_{\xi_1} A_{J\xi_j} = \lambda(\xi_1)\nu I$ on $T_x(M^n)$, and

$$\operatorname{trace} A_{\xi_1} A_{J\xi_j} = 2n\lambda(\xi_1)\nu . \tag{31}$$

But $\langle \xi_1, J\xi_j \rangle = 0$, and so $\operatorname{trace} A_{\xi_1} A_{J\xi_j} = 0$. Comparing this with (31), we conclude $\nu = 0$. Then (30) implies $A_{J\xi_j} = 0$ which implies $A_{\xi_j} = 0$, and $\xi_j \notin N_1(x)$. This is true for $2 \leq j \leq k$, and we have obtained a contradiction if we assume $k > 1$. Thus, $k \leq 1$. Q.E.D.

We first want to make it clear that we have no further use for the inner product $\langle\ ,\ \rangle$. Any subsequent references to metric properties such as orthogonality are made with respect to the metrics g or $\tilde{g}$.

We now begin to reduce the co-dimension. The argument is similar to that used by Cartan to show that an umbilical submanifold of $\boldsymbol{R}^m$ which is not totally geodesic must be a Euclidean sphere embedded in $\boldsymbol{R}^m$ (see [2], p. 231).

Proposition 5 enables us to define a function λ on M^n in the following way. Let $\alpha(X, Y)$ be the second fundamental form of M^n in $P^{n+p}(\boldsymbol{C})$. If $\alpha(X, Y) = 0$ at $x \in M^n$, we set $\lambda(x) = 0$. If $\alpha(X, Y) \neq 0$ at $x \in M^n$, then by Proposition 5, $N_1(x)$ has complex dimension 1. We define $\lambda(x)$ to be the well-defined positive number such that $A_\xi^2 = \lambda^2(x) I$ for any unit vector ξ in $N_1(x)$. It is easy to show from the obvious dependence of λ on $\alpha(X, Y)$ that λ is continuous on M^n. We omit that proof here, however, and next prove the following.

PROPOSITION 6. *Let $x \in M^n$ and suppose the second fundamental form $\alpha(X, Y) \neq 0$ at x. Then there is a neighborhood U of x in M^n on which the function λ is constant.*

Proof. Let U be a neighborhood of x on which $\alpha(X, Y) \neq 0$. Then by Proposition 5, $N_1(u)$ has constant dimension 1 on U. It is easy to show, then, that there exists a unit-length vector field ξ_1, on U such that

$$N_1(u) = \operatorname{span}\{\xi_1, J\xi_1\} \qquad \text{for every } u \in U .$$

Let $\xi_2, \cdots, \xi_p$ be unit-length normal vector fields on U such that $\xi_1, \xi_2, \cdots, \xi_p, J\xi_1, \cdots, J\xi_p$ are an orthonormal basis for $T_u^\perp(M^n)$ for any $u \in U$.

Fix an arbitrary point $u \in U$. The following equation defines the the tensors s_{kj} and t_{kj} on $T_u(M^n)$,

$$\nabla^{\perp}_X \xi_j = \sum_{k=1}^{p} s_{kj}(X)\xi_k + \sum_{k=1}^{p} t_{kj}(X)J\xi_k \qquad \text{for } X \in T_u(M^n) . \tag{32}$$

The fact that $\nabla^{\perp}$ is a Kählerian connection readily implies

$$s_{kj}(X) = -s_{jk}(X) \tag{33}$$

and

$$t_{kj}(X) = t_{jk}(X) . \tag{34}$$

Now we know $A_{\xi_j} = A_{J\xi_j} = 0$ for $2 \leq j \leq p$. This fact and (33) imply that Codazzi's equation for A_{ξ_1} reduces to

$$(\nabla_X A_{\xi_1})(Y) - t_{11}(X)JA_{\xi_1}(Y) = (\nabla_Y A_{\xi_1})(X) - t_{11}(Y)JA_{\xi_1}(X) . \tag{35}$$

Let $X, Y \in T^+ = T^+_{\xi_1}(u)$ such that X, Y are linearly independent, and suppose

$$\begin{aligned} \nabla_X Y &= X_1 + X_2 \qquad \text{for } X_1 \in T^+, X_2 \in T^- , \\ \nabla_Y X &= Y_1 + Y_2 \qquad \text{for } Y_1 \in T^+, Y_2 \in T^- . \end{aligned}$$

Using the above equations and recalling the following equations,

$$\begin{aligned} A_{\xi_1} Z &= \lambda Z \qquad \text{for } Z \in T^+ , \\ A_{\xi_1} Z &= -\lambda Z \qquad \text{for } Z \in T^- , \end{aligned}$$

we find after some calculation that (35) becomes

$$(X\lambda)Y + 2\lambda X_2 + t_{11}(X)\lambda JY = (Y\lambda)X + 2\lambda Y_2 + t_{11}(Y)\lambda JX . \tag{36}$$

But X_2, Y_2, JX, JY are in T^-, and the component of (36) in T^+ is,

$$(X\lambda)Y = (Y\lambda)X . \tag{37}$$

The linear independence of X and Y implies that $X\lambda = 0$. This is true for any $X \in T^+$. A similar calculation shows $X\lambda = 0$ for any $X \in T^-$. So we have $X\lambda = 0$ for any $X \in T_u(M^n)$ for any $u \in U$. This is implies λ is constant on U. Q.E.D.

Proposition 6 enables us to prove that $N_1(x)$ has constant dimension on M^n as follows.

PROPOSITION 7. *$N_1(x)$ has constant dimension on M^n.*

Proof. If the second fundamental form $\alpha(X, Y) = 0$ for all $x \in M^n$, then $N_1(x)$ has constant dimension 0; and the proof is complete.

Suppose $\alpha(X, Y) \neq 0$ at $x_0 \in M^n$. Consider the set S defined by

$$S = \{x \in M^n \mid \lambda(x) = \lambda(x_0)\} .$$

Since λ is continuous on M^n, we know S is closed. However Proposition 6 implies S is open. Since $x_0 \in S$, we know $S \neq \phi$; so the connectedness of M^n implies $S = M^n$. Hence $\lambda = \lambda(x_0)$ on M^n, and $N_1(x)$ has constant dimension 1 on M^n. Q.E.D.

In the case where $N_1(x)$ has constant dimension 0, M^n is totally geodesic, and hence $M^n = P^n(C)$. To complete the proof of Theorem 2, we must show that when $N_1(x)$ has constant dimension 1, we can reduce the co-dimension to 1.

Let U be any co-ordinate neighborhood of M^n. As before we choose orthonormal vector fields $\xi_1, \cdots, \xi_p$ so that $\xi_1, \cdots, \xi_p, J\xi_1, \cdots, J\xi_p$ span $T_u^\perp(M^n)$ for any $u \in U$, and such that $\xi_1, J\xi_1$ span $N_1(u)$ for any $u \in U$. We then prove, .

PROPOSITION 8. *For any $x \in U$ and $X \in T_x(M^n)$ the following equations are true:*

(i) $\nabla_X^\perp \xi_1 = t_{11}(X) J\xi_1$

(ii) *For $j \geq 2$, $\nabla_X^\perp \xi_j$ and $\nabla_X^\perp J\xi_j \in \operatorname{span}\{\xi_k, J\xi_k \mid 2 \leq k \leq p\}$, i.e. $N_1(x)$ and $N_0(x)$ are invariant with respect to $\nabla^\perp$.*

Proof. For ease of notation, let $A_j = A_{\xi_j}$, $1 \leq j \leq p$. For any fixed $j, 2 \leq j \leq p$, Codazzi's equation says the following,

$$(\nabla_X A_j)(Y) - \sum_{k=1}^{p} s_{kj}(X) A_k(Y) - \sum_{k=1}^{p} t_{kj}(X) J A_k(Y)$$

is symmetric in X and Y.

Since $A_j = 0$, then $(\nabla_X A_j) = 0$ and Codazzi's equation can be written as:

$$s_{1j}(X) A_1(Y) + t_{1j}(X) J A_1(Y) = s_{1j}(Y) A_1(X) + t_{1j}(Y) J A_1(X) . \tag{38}$$

Choose X, Y linearly independent vectors in $T_{\xi_1}^+(x)$; then since $A_1(X) = \lambda X$ and $A_1(Y) = \lambda Y$, (38) becomes

$$s_{1j}(X) \lambda Y + t_{1j}(X) \lambda J Y = s_{1j}(Y) \lambda X + t_{1j}(Y) \lambda J X . \tag{39}$$

But X, Y, JX, JY are linearly independent, so (39) implies

$$s_{1j}(X) = t_{1j}(X) = 0\ , \qquad 2 \leq j \leq p\ . \tag{40}$$

A similar calculation shows that (40) holds for $X \in T^{-}_{\xi_1}(x)$, and hence (40) holds for all $X \in T_x(M^n)$. We recall that for $1 \leq j \leq p$,

$$\nabla^{\perp}_X \xi_j = \sum_{k=1}^{p} s_{kj}(X)\xi_j + \sum_{k=1}^{p} t_{kj}(X)J\xi_k\ . \tag{32}$$

Then $s_{kj} = -s_{jk}$ and $t_{kj} = t_{jk}$ and (40) imply that for $j = 1$, (32) becomes

$$\nabla^{\perp}_X \xi_1 = t_{11}(X)J\xi_1 \tag{41}$$

proving (i). For the same reasons, for $j > 1$, (32) becomes

$$\nabla^{\perp}_X \xi_j = \sum_{k=2}^{p} s_{kj}(X)\xi_k + \sum_{k=2}^{p} t_{kj}(X)J\xi_k\ . \tag{42}$$

Then $\nabla^{\perp}_X J\xi_j = J(\nabla^{\perp}_X \xi_j)$ and (42) prove (ii). Q.E.D.

Finally Proposition 8 and the fact that $N_1(x)$ has constant complex dimension 1 will imply that $f(M^n) \subset P^{n+1}(C)$ after we prove the following proposition. We note that J. Erbacher, [3], has proven a corresponding result for real submanifolds of real space forms. With minor changes, the following proposition can be proven for submanifolds of C^{n+p} and the complex hyperbolic space form, $H^{n+p}(C)$.

PROPOSITION 9. *Let $f: M^n \to P^{n+p}(C)$ be a holomorphic and isometric immersion of a connected, complete, complex n-dimensional Kählerian manifold M^n into $P^{n+p}(C)$. Suppose the first normal space $N_1(x)$ has constant dimension k, and is parallel with respect to the normal connection. Then there is a totally geodesic $(n+k)$-dimensional submanifold, $P^{n+k}(C)$, such that $f(M^n) \subset P^{n+k}(C)$.*

Proof. We first remark that since $N_1(x)$ is parallel with respect to $\nabla^{\perp}$, so is its complement $N_0(x)$. Let U be a co-ordinate neighborhood of M^n and fix $x_0 \in U$.

Choose $\xi_1, \cdots, \xi_p \in T^{\perp}_{x_0}(M^n)$ so that the following equations hold for $x = x_0$,

$$N_1(x) = \text{span}\,\{\xi_j, J\xi_j \mid 1 \leq j \leq k\} \tag{43}$$

and

(44) $$N_0(x) = \operatorname{span}\{\xi_j, J\xi_j \mid k+1 \le j \le p\}\ .$$

Extend $\xi_1, \cdots, \xi_p$ to vector fields on U by parallel translation with respect to $\nabla^\perp$ along geodesics of M^n. Then (43) and (44) hold for any $x \in U$.

Let ξ'_j denote the horizontal lift to T'_w of $\xi_j(\pi(w))$ where $\pi(w) \in U$. Fix $w_0 \in S^{2(n+p)+1}$ so that $\pi(w_0) = x_0$. Let V_{w_0} be the real affine subspace of C^{n+p+1} through w_0 given by

$$V_{w_0} = \operatorname{span}\{\xi'_j(w_0), i\xi'_j(w_0) \mid k+1 \le j \le p\}\ .$$

Let W_{w_0} be the real affine space through w_0 of real dimension $2(n+k+1)$ which is orthogonal to V_{w_0}. Since the vector $-w_0 \in W_{w_0}$, we know that the affine space W_{w_0} passes through the origin in C^{n+p+1}. Hence the set

$$S^{2(n+k)+1} \equiv W_{w_0} \cap S^{2(n+p)+1}$$

is a great $(2(n+k)+1)$-dimensional sphere in $S^{2(n+p)+1}$. The set $P^{n+k}(C) = \pi(S^{2(n+k)+1})$ is an $(n+k)$-dimensional totally geodesic submanifold of $P^{n+p}(C)$. We will show that $f(M^n) \subset P^{n+k}(C)$.

We first prove $f(U) \subset P^{n+k}(C)$. Fix $u \in U$, and let $x(t)$, $0 \le t \le t_0$, be a curve in $f(U)$ from $f(x_0)$ to $f(u)$. Let $w(t)$ be the lift of $x(t)$ to $S^{2(n+p)+1}$ so that $w(0) = w_0$ and $\pi(w(t)) = x(t)$, $0 \le t \le t_0$.

We know that for $0 \le t \le t_0$ we have

$$\tilde{\nabla}_{\vec{x}(t)}\xi_j = \pi_*(\nabla'_{\vec{w}(t)}\xi'_j) \qquad \text{for } 1 \le j \le p\ .$$

We also know

$$\tilde{\nabla}_{\vec{x}(t)}\xi_j = -A_{\xi_j}(\vec{x}(t)) + \nabla^\perp_{\vec{x}(t)}\xi_j\ .$$

For $j > k$, however, $A_{\xi_j} = 0$ and

$$\nabla^\perp_{\vec{x}(t)}\xi_j \in \operatorname{span}\{\xi_m, J\xi_m \mid k+1 \le m \le p\}\ ,$$

and thus

$$\tilde{\nabla}_{\vec{x}(t)}\xi_j \in \operatorname{span}\{\xi_m, J\xi_m \mid k+1 \le m \le p\}\ .$$

A similar result holds for $\tilde{\nabla}_{\vec{x}(t)}J\xi_j$. If we let

$$V_t = \operatorname{span}\{\xi'_m(w(t)), i\xi'_m(w(t)) \mid k+1 \le m \le p\}\ ,$$

then by the isomorphism π_*, we have for each t,

(45) $$\nabla'_{\vec{w}(t)}\xi'_j \quad \text{and} \quad \nabla'_{\vec{w}(t)}i\xi'_j \in V_t\ .$$

Since $g(w(t), \xi'_j) = 0$, for $0 \leq t \leq t_0$, we have $D_{\vec{w}(t)}\xi'_j = \nabla'_{\vec{w}(t)}\xi'_j$, where D is the Euclidean covariant derivative in C^{n+p+1}.

This fact and (45) imply that for all t, and for $k+1 \leq j \leq p$,

$$D_{\vec{w}(t)}\xi'_j \quad \text{and} \quad D_{\vec{w}(t)}i\xi'_j \in V_t \,.$$

Thus V_t is a parallel Euclidean subspace along $w(t)$, i.e. for each t, V_t is parallel to V_{w_0} in the sense of Euclidean parallelism.

For each t, let W_t be the $2(n+k+1)$-dimensional real affine space through $w(t)$ which is orthogonal to V_t. Since V_t is parallel to V_{w_0} for each t, W_t is parallel to W_{w_0} for each t, in the Euclidean sense of parallelism. However, for each t, $-w(t) \in W_t$, and thus W_t passes through the origin for each t. Hence we conclude $W_t = W_{w_0}$ for $0 \leq t \leq t_0$.

Since $\vec{w}(t)$ is orthogonal to V_t for all t, we have $\vec{w}(t) \in W_t = W_{w_0}$. Since $w(0) \in W_{w_0}$, this shows that $w(t) \in W_{w_0}$ for all t; and so $w(t) \in W_{w_0} \cap S^{2(n+p)+1} = S^{2(n+k)+1}$ for $0 \leq t \leq t_0$. Applying π, we get $x(t) \in P^{(n+k)}(C)$ for all t. In particular, $f(u) = x(t_0) \in P^{n+k}(C)$. Since $u \in U$ was arbitrary, we have shown $f(U) \subset P^{n+k}(C)$.

To prove the global result we use the connectedness of M^n. Let U_1, U_2 be co-ordinate neighborhoods of M^n such that $U_1 \cap U_2 \neq \phi$. We have shown above that there exist 2 totally geodesic $(n+k)$-dimensional submanifolds of $P^{n+p}(C)$, call them P_1^{n+k} and P_2^{n+k}, such that $f(U_1) \subset P_1^{n+k}$ and $f(U_2) \subset P_2^{n+k}$.

Suppose $P_1^{n+k} \neq P_2^{n+k}$. Then, $P_1^{n+k} \cap P_2^{n+k} = P^{n+k-1}$, a totally geodesic $(n+k-1)$-dimensional submanifold of $P^{n+p}(C)$, and $f(U_1 \cap U_2) \subset P^{n+k-1}$. This implies that for $z \in U_1 \cap U_2$, the first normal space $N_1(z)$ has dimension $k-1$. This contradicts the assumption that $N_1(x)$ has constant dimension k on M^n. Thus we conclude $P_1^{n+k} = P_2^{n+k} = P^{n+k}(C)$. Using this, one easily proves from the connectedness of M^n that $f(M^n) \subset P^{n+k}(C)$.

Q.E.D.

Now Propositions 7, 8, and 9 combine to imply that under the hypotheses of Theorem 2, $f(M^n) \subset P^{n+1}(C)$, a totally geodesic $(n+1)$-dimensional submanifold of $P^{n+p}(C)$. The proof of Theorem 2 then follows from Remark 1.

Section 5—The Special Case $Q^n \subset P^{n+1}(C)$.

In this section we make a detailed study of the case $Q^n \subset P^{n+1}(C)$. The main results are contained in Theorem 3. We first discuss some necessary preliminaries.

Consider $\boldsymbol{C}^{n+2}$ with natural basis $e_0, \cdots, e_{n+1}$. We denote by $H(z, w)$ the complex bi-linear form defined by

$$H(z, w) = \sum_{k=0}^{n+1} z^k w^k , \qquad \text{where } z = \sum_{k=0}^{n+1} z^k e_k \text{ and } w = \sum_{k=0}^{n+1} w^k e_k .$$

Then Q^n is defined as

$$Q^n = \{\pi(z) \mid z \in S^{2(n+1)+1} \text{ and } H(z, z) = 0\} ,$$

where π is the projection from $S^{2(n+1)+1}$ to $P^{n+1}(\boldsymbol{C})$. We continue to assume that $P^{n+1}(\boldsymbol{C})$ has constant holomorphic sectional curvature 4.

Let $q \in Q^n$ and ξ be a unit-length vector in $T_q^{\perp}(Q^n)$. Then Smyth ([8], p. 263–265) shows that A_ξ has the following form when diagonalized,

$$A_\xi = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} ,$$

where again I_n is an $n \times n$ identity matrix.

With these remarks aside, we first prove the following elementary proposition.

PROPOSITION 10. *Let* $z = \sum_{k=0}^{n+1} z^k e_k \in S^{2(n+1)+1}$. *Then* $H(z, z) = 1$ *if and only if* z^k *is real for* $0 \leq k \leq n + 1$.

Proof. $H(z, z) = \sum_{k=0}^{n+1} (z^k)^2$; and if each z^k is real, then $H(z, z) = \|z\|^2 = 1$. Conversely, suppose $H(z, z) = 1$. Then letting $\bar{z} = \sum_{k=0}^{n+1} \bar{z}^k e_k$, we have

$$|(z, \bar{z})|^2 = \left|\sum_{k=0}^{n+1} (z^k)^2\right| = 1 = \|z\|^2 \cdot \|\bar{z}\|^2 . \tag{46}$$

The Schwarz inequality for the inner product (,) implies that (46) can be true only if $\bar{z} = cz$ for some $c \in \boldsymbol{C}$.

But then since $(z, z) = 1$,

$$1 = \sum_{k=0}^{n+1} z^k \bar{z}^k = \sum_{k=0}^{n+1} z^k c z^k = c \sum_{k=0}^{n+1} (z^k)^2 = c .$$

Hence $c = 1$ and so $\bar{z} = z$ and z is real, i.e. z^k is real for $0 \leq k \leq n + 1$.

Q.E.D.

Let $\boldsymbol{R}^{n+2}$ denote the real vector space spanned by $e_0, \cdots, e_{n+1}$. Then S^{n+1}, defined by $S^{n+1} = \boldsymbol{R}^{n+2} \cap S^{2(n+1)+1}$, is an $(n + 1)$-dimensional Euclidean sphere. The projection π takes the antipodal points z and $-z \in S^{n+1}$ onto the same point $p = \pi(z) \in P^{n+1}(\boldsymbol{C})$. This is the only identification on S^{n+1} induced by π, and we see that $\pi(S^{n+1}) = P^{n+1}(\boldsymbol{R})$, a real $(n+1)$-dimensional projective space naturally embedded in $P^{n+1}(\boldsymbol{C})$. Let $p \in P^{n+1}(\boldsymbol{R})$, and let $z \in S^{n+1}$ such that $\pi(z) = p$. We define a set S_p^n by

$$S_p^n = \left\{\pi\left(\frac{x + iz}{\sqrt{2}}\right) \middle| x \in S^{n+1}, g(x, z) = 0\right\}$$

One easily shows that S_p^n is independent of the choice of z.

PROPOSITION 11. *Let $p \in P^{n+1}(\boldsymbol{R})$, then S_p^n is the image of a Euclidean n-sphere of radius $1/\sqrt{2}$ isometrically embedded in $P^{n+p}(\boldsymbol{C})$.*

Proof. Let $z \in S^{n+1}$ such that $\pi(z) = p$. We define $\boldsymbol{R}^{n+1}$ by

$$\boldsymbol{R}^{n+1} = \{w \in \boldsymbol{R}^{n+2} \mid g(z, w) = 0\} .$$

Let $\bar{\boldsymbol{R}}^{n+2} \equiv \boldsymbol{R}^{n+1} \times \{iz\}$ where $\{iz\}$ is the 1-dimensional real subspace spanned by the vector iz. Then

$$\bar{S}^{n+1} \equiv \bar{\boldsymbol{R}}^{n+2} \cap S^{2(n+1)+1}$$

is a Euclidean $(n+1)$-sphere of radius 1. Then

$$S \equiv \left\{\frac{x + iz}{\sqrt{2}} \middle| x \in S^{n+1}, g(x, z) = 0\right\} \subset \bar{S}^{n+1} .$$

In fact, it is easy to see that S is a small-sphere of dimension n with center $iz/\sqrt{2}$ and radius $1/\sqrt{2}$ contained in $\bar{S}^{n+1}$. One checks that no two points of S are identified under the projection π. Thus π is a one-to-one isometry on S, and $\pi(S) = S_p^n$ is the image of a Euclidean n-sphere of radius $1/\sqrt{2}$ isometrically embedded in $P^{n+p}(\boldsymbol{C})$. Q.E.D.

The following theorem describes the focal point behavior for $Q^n \subset P^{n+1}(\boldsymbol{C})$.

THEOREM 3. (i) *The set of focal points of $Q^n \subset P^{n+1}(\boldsymbol{C})$ is $P^{n+1}(\boldsymbol{R})$.*
(ii) *Let $p \in P^{n+1}(\boldsymbol{R})$; then*

$$\{q \in Q^n \mid p \textit{ is a focal point of } (Q^n, q)\} = S_p^n .$$

Proof. To prove (i), we first show that the set of focal points of Q^n is contained in $P^{n+1}(\boldsymbol{R})$.

Let $p \in P^{n+1}(\boldsymbol{C})$ be a focal point of (Q^n, q) for some $q \in Q^n$. By

Proposition 1, $p = F(q, r\xi)$ where ξ is a unit-length vector in $T_q^{\perp}(Q^n)$ and $\cot r = \lambda$ for some eigen-value λ of A_ξ. As we remarked at the beginning of this section, $\lambda = \pm 1$ for any such q and ξ. Choosing the sign of ξ properly we may assume $\cot r = 1$, and then

$$F\left(q, \frac{\pi}{4}\xi\right) = \pi\left(\frac{w}{\sqrt{2}} + \frac{\xi'}{\sqrt{2}}\right) \qquad \text{where } \pi(w) = q \text{ and } \pi_*(\xi') = \xi .$$

It is known (see [4], Vol. II, p. 279) that there exist unique real vectors x, y of length $1/\sqrt{2}$, with $g(x, y) = 0$, such that $w = x + iy$. Then $T_q^{\perp}(Q^n)$ is spanned by $\pi_*(ix + y)$ and $\pi_*(-x + iy)$. Thus we can express ξ' as

$$\xi' = \cos\phi(ix + y) + \sin\phi(-x + iy) \qquad \text{for some } \phi, 0 \leq \phi \leq 2\pi .$$

Thus $p = \pi(z)$ where

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}(w + \cos\phi(ix + y) + \sin\phi(-x + iy)) \\ &= \frac{x}{\sqrt{2}}[(1 - \sin\phi) + i\cos\phi] + \frac{y}{\sqrt{2}}[\cos\phi + (1 + \sin\phi)i] . \end{aligned}$$

Using the defining properties of x and y, we compute

$$H(z, z) = -\sin\phi + i\cos\phi = e^{i(\phi + \pi/2)} .$$

Let $z' = e^{-i(\phi+\pi/2)/2}z$; then $\pi(z') = p$, but

$$H(z', z') = e^{-i(\phi+\pi/2)}H(z, z) = 1 .$$

Thus by Proposition 10, z' is real, and so $p \in P^{n+1}(\boldsymbol{R})$.

Conversely, suppose $p = \pi(z)$ where $z \in S^{n+1}$. Let $x \in S^{n+1}$ such that $g(x, z) = 0$. Let $w = (x + iz)/\sqrt{2}$. Then,

$$H(w, w) = 0 , \quad \text{and} \quad q = \pi(w) \in Q^n .$$

One easily shows that $\xi' = (-x + iz)/\sqrt{2} \in T'_w$ and $\pi_*(\xi') \in T_q^{\perp}(Q^n)$. If we let

$$z' = \frac{1}{\sqrt{2}}\left(\frac{x + iz}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{-x + iz}{\sqrt{2}}\right) = iz ,$$

then by Proposition 1, $\pi(z')$ is a focal point of (Q^n, q). But $\pi(z') = \pi(iz) = p$, and so the proof of (i) is complete.

To prove (ii) we let $p = \pi(z)$ for $z \in S^{n+1}$. Let

$$S = \{(x + iz)/\sqrt{2} \mid x \in S^{n+1}, g(x, z) = 0\}$$

and

$$T = \{q \in Q^n \mid p \text{ is a focal point of } (Q^n, q)\} .$$

By definition $S_p^n = \pi(S)$, and in the above proof of (i) we showed that $S_p^n \subset T$. To complete the proof of (ii), we show $T \subset S_p^n$.

Suppose $q \in T$. Let $w \in S^{2(n+1)+1}$ such that $\pi(w) = q$. Then $w = (x + iy)/\sqrt{2}$ for a unique choice of $x, y \in S^{n+1}$ such that $g(x, y) = 0$. By (i) we know $p \in P^{n+1}(\boldsymbol{R})$, so there is $z \in S^{n+1}$ such that $\pi(z) = p$. We first show

$$z = \cos\alpha\, x + \sin\alpha\, y \qquad \text{for some } \alpha, 0 \leq \alpha \leq 2\pi .$$

We know that $T_q^{\perp}(Q^n)$ is spanned by

$$\pi_*\Big(\frac{-x + iy}{\sqrt{2}}\Big) \quad \text{and} \quad \pi_*\Big(\frac{ix + y}{\sqrt{2}}\Big) .$$

By Proposition 1, any focal point of (Q^n, q) can be expressed as $\pi(u)$ where

$$u = \frac{1}{\sqrt{2}}\Big(\frac{x + iy}{\sqrt{2}}\Big) + \frac{1}{\sqrt{2}}\Big(\cos\phi\Big(\frac{-x + iy}{\sqrt{2}}\Big) + \sin\phi\Big(\frac{ix + y}{\sqrt{2}}\Big)\Big) \tag{47}$$

for some $\phi, 0 \leq \phi \leq 2\pi$.

Since $\pi(z) = p$ is a focal point of (Q^n, q), we must have $z = e^{i\beta}u$ for some u as in (47), and for some β, $0 \leq \beta \leq 2\pi$. This implies that z is a real linear combination of x, y, ix and iy. Since x, y and z are all real, we must have

$$z = \cos\alpha\, x + \sin\alpha\, y \qquad \text{for some } \alpha, 0 \leq \alpha \leq 2\pi . \tag{48}$$

Consider $w' = (\sin\alpha + i\cos\alpha)[(x + iy)/\sqrt{2}]$. Then $\pi(w') = \pi(w) = q$. But from (48) we see

$$\begin{aligned} w' &= \frac{1}{\sqrt{2}}[(\sin\alpha\, x - \cos\alpha\, y) + i(\cos\alpha\, x + \sin\alpha\, y)] \\ &= \frac{1}{\sqrt{2}}[(\sin\alpha\, x - \cos\alpha\, y) + iz] . \end{aligned}$$

Thus $w' \in S$, and $q \in \pi(S) = S_p^n$. This is true for any $q \in T$, and we have $T \subset S_p^n$. Q.E.D.

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