ON THE ZEROS OF A CONFORMAL VECTOR FIELD

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1. Introduction

In [1] S. Kobayashi showed that the connected components of the set of zeros of a Killing vector field on a Riemannian manifold \((M^n, g)\) are totally geodesic submanifolds of \((M^n, g)\) of even codimension including the case of isolated singular points. The purpose of this short note is to give a simple proof of the corresponding result for conformal vector fields on compact Riemannian manifolds. In particular we prove the following

**Theorem.** Let \((M^n, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\). Let \(F\) be the set of zeros of a conformal vector field \(\xi\) and let \(F = \bigcup V_i\) where the \(V_i\)'s are the connected components of \(F\). Then each \(V_i\) is either an umbilical submanifold of \((M^n, g)\) of even codimension including the case of isolated singular points or an isolated singular point of a conformal non-Killing vector field on a Euclidean sphere.

The idea of our proof is to reduce the problem to Kobayashi's case by a simple application of a theorem of M. Obata characterizing a sphere to conformality. In Section 2 we discuss Obata's result and then prove our theorem in Section 3.

2. Preliminaries

A Riemannian metric \(\bar{g}\) is said to be conformal to \(g\) if there exists a smooth function \(\rho\) on \(M^n\) such that \(\bar{g} = e^{2\rho}g\). Let \(f: M^n \to M^n\) be a diffeomorphism of \(M^n\) onto itself; we say \(f\) is a conformal diffeomorphism if \(f^*g\) is conformal to \(g\).

Let \(C(M^n, g)\) denote the Lie group of all conformal diffeomorphisms of \((M^n, g)\) and \(C_0(M^n, g)\) the connected component of the identity. A

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subgroup $G$ of $C(M^n, g)$ is said to be essential if it does not become a group of isometries under any conformal change of metric, and a conformal vector field is said to be essential if its one-parameter group is essential. In [2] and [3] M. Obata obtained the following results.

**THEOREM** (Obata [2]). Let $(M^n, g)$ be a compact Riemannian manifold of dimension $n > 2$. Then $C_0(M^n, g)$ is essential if and only if $(M^n, g)$ is conformally diffeomorphic to a Euclidean sphere.

**THEOREM** (Obata [3]). Let $\xi$ be an essential conformal vector field on a Euclidean sphere. Then $\xi$ has either exactly one or exactly two singular points.

3. **Proof of the Theorem**

The proof of the theorem is, for $n > 2$, by cases using Obata’s result. If $C_0(M^n, g)$ is inessential, then there exists a conformal change of metric, say $\bar{g} = e^{2\xi}g$, such that $C_0(M^n, g)$ is a group of isometries with respect to $\bar{g}$. Thus given a conformal vector field $\xi$ on $(M^n, g)$, $\xi$ is Killing on $(M^n, \bar{g})$ and hence by Kobayashi’s Theorem each $V_t$ is a totally geodesic submanifold of $(M^n, \bar{g})$ of even codimension. Thus it remains only to show that $V_t$ is umbilical in $(M^n, g)$. To this end let $\bar{V}$ and $\bar{F}$ be the Riemannian connexions of $g$ and $\bar{g}$ respectively and let $P = \text{grad } \rho$. Then

\[ \bar{V}_XY = F_XY + (X\rho)Y + (Y\rho)X - g(X, Y)P . \]  

(3.1)

Now let $\iota$ denote the imbedding of $V_t$ in $M^n$. Considering $V_t$ as a submanifold of $(M^n, g)$ with $g'$ and $F'$ denoting the induced Riemannian metric and connexion, choose a local orthonormal frame $\eta_1, \ldots, \eta_k$ of normal vector fields on $V_t$, $k = \text{codim } V_t$, and let $h^*$ denote the corresponding second fundamental forms. Then the Gauss equation is

\[ \bar{V}_{\iota_*X}\iota_*Y = \iota_*F'_X Y + h^*(X, Y)\eta_* \]  

(3.2)

summed on $\alpha$. Considering $V_t$ as a submanifold of $(M^n, \bar{g})$ with $\bar{F}'$ denoting the induced Riemannian connexion, the Gauss equation is

\[ \bar{V}_{\iota_*X}\iota_*Y = \iota_*\bar{F}'_X Y . \]  

(3.3)

Thus using (3.1), (3.2) and (3.3) we have

\[ \iota_*F'_X Y + h^*(X, Y)\eta_* = \iota_*\bar{F}'_X Y - (\iota_*X\rho)\iota_*Y - (\iota_*Y\rho)\iota_*X + g(\iota_*X, \iota_*Y)P . \]
Now taking the $g$ inner product with $\eta_\beta$ we have

$$h^\beta(X, Y) = (\eta_\beta P)g'(X, Y)$$

and hence that $V_\xi$ is umbilical in $(M^n, g)$.

If on the other hand $C_\xi(M^n, g)$ is essential then $(M^n, g)$ is conformally diffeomorphic to a Euclidean sphere, but a conformal vector field remains conformal under a conformal change of metric. Thus if a conformal vector field $\xi$ is essential, its zeros are isolated. If $\xi$ is inessential then again there exists a conformal change of metric with respect to which $\xi$ becomes a Killing vector field.

Finally if $n = 2$, there exist local isothermal parameters with respect to which $(M^2, g)$ becomes a Hermitian (complex) manifold. If now $\xi$ is a conformal vector field on $(M^2, g)$, the conformal transformations of its oneparameter group are given by analytic functions. Thus by the identity theorem for analytic functions, their fixed points are isolated and hence the zeros of $\xi$ are isolated.

References


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