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Nagoya Math. J.
Vol. 54 (1974), 165–178

AN APPLICATION OF THE MORSE THEORY TO FOLIATED MANIFOLDS

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In [5], R. Thom has started the study of the foliated structures by using the Morse theory. Recently K. Yamato [7] has studied the topological properties of leaves of a codimension one foliated manifold by investigating the "critical points" of variation equation of the given one-form.

In this note, using their methods we shall show that a codimension k foliation on a closed manifold is a "bundle foliation" under certain conditions (Theorem I), and give some topological properties of those leaves (Theorem II, III). By using Theorem I, we shall show the Stability Theorem of Reeb [3]. Furthermore, we shall show that bundle foliations satisfying some conditions, are stable under a small perturbation (Theorem IV). All manifolds, foliations and mappings considered here, are smooth (i.e., differentiable of class C^∞).

The author is grateful to Professors N. Shimada, M. Adachi, H. Imanishi and Mr. T. Matumoto for their kind advices.

§ 1. Definitions and statement of the results

Let V^n be a closed n-manifold, $\mathscr{F}^k$ $(0 < k < n)$ a codimension k foliation on V^n and $\{U_i, i = 1, 2, \cdots, \ell_0\}$ a distinguished neighborhood covering of V^n. The local coordinate of a distinguished neighborhood is $(u_1, \cdots, u_k, x_1, \cdots, x_{n-k})$ such that each plate is defined by $u_i =$ constant for $1 \leqq i \leqq k$. At first we take a smooth function f on V^n, and for each distinguished neighborhood U_i, we define a mapping F_i of U_i into $\boldsymbol{R}^{k+1}$ by

$$F_i(u_1, \cdots, u_k, x_1, \cdots, x_{n-k}) = (u_1, \cdots, u_k, f(u_1, \cdots, u_k, x_1, \cdots, x_{n-k})) .$$

Then, we define a subset $\Gamma(f)$ of V^n as follows: $\Gamma(f) \cap U_i = \{(u_1, \cdots, u_k,$

Received June 4, 1973.

$x_1, \cdots, x_{n-k}) \in U_i$; corank $(F_i) = 1$ at $(u_1, \cdots, u_k, x_1, \cdots, x_{n-k})\}$. This definition is well-defined.

PROPOSITION A (Thom [5]). *Let $f: V^n \to \boldsymbol{R}$ be a smooth function. Then there exists a smooth function g approximating f such that $\Gamma(g)$ is a closed k-manifold of V^n.*

Proof. We have only to approximate f in each distinguished neighborhood U_i in order that $\Gamma(f) \cap U_i$ is a k-manifold. Then using Thom's notation, we have $\Gamma(f) \cap U_i = S_1(F_i)$. Hence the proposition follows from Thom [6].

For $p \in \Gamma(f) \cap U_i$, let $(u_1, \cdots, u_k, x_1, \cdots, x_{n-k})$ be a local coordinate around p such that $u_i(p) = 0, x_j(p) = 0$. We can assume that f is described as follows:

$$f - f(p) = u_1 + \sum_{i,j} a_{i,j} x_i x_j + \sum_{s,t} c_s^t u_s x_t \cdots ,$$

where $(a_{i,j})$ is a symmetric $(n-k)$-matrix. Then the point p is said to be of type λ $(0 \leqq \lambda \leqq n-k)$ if the matrix $(a_{i,j})$ is non-singular and its signature is λ (i.e., the number of negative eigenvalues of $(a_{i,j})$ is equal to λ). Let $\Gamma_\lambda(f)$ denote the set of points of type λ. Next for each U_i, consider $F_i | \Gamma(f) \cap U_i : \Gamma(f) \cap U_i \to \boldsymbol{R}^{k+1}$. Then we define $S_1(\Gamma(f)) \cap U_i$ to be $S_1(F_i | \Gamma(f) \cap U_i)$. $S_1(\Gamma(f)) \cap U_i$ is the set of points where the above matrix $(a_{i,j})$ is singular. At $p \in S_1(\Gamma(f)) \cap U_i$, we can describe f "generically" as follows:

$$f - f(p) = u_1 - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n-k-1} x_j^2 + x_{n-k}^3 + u_1 x_{n-k} + \sum_{s,t} c_s^t u_s x_t + \cdots$$

(cf. [5]). Note that $S_1(\Gamma(f))$ is a $(k-1)$-dimensional submanifold of $\Gamma(f)$ and $\Gamma(f) = \Gamma_1(f) \cup \Gamma_2(f) \cup \cdots \cup \Gamma_{n-k}(f) \cup S_1(\Gamma(f))$.

THEOREM I. *Let a foliated manifold $(V^n, \mathscr{F}^k)$ $(n-k \geqq 2)$ be given. Suppose that there exists a smooth function $f: V^n \to \boldsymbol{R}$ such that $\Gamma_1(f) = \phi$. Then V^n is the total space of a fiber bundle over $\Gamma_0(f)$ such that* 1) *the fiber is a connected, simply-connected, closed $(n-k)$-manifold, and* 2) *$\mathscr{F}^k$ is a foliation such that each leaf is a fiber.*

For $k = 0$, this is only a simple example of the ordinary Morse theory. The proof of this theorem for $k \neq 0$ will be given in § 3.

THEOREM II. *Let a foliated manifold* $(V^n, \mathscr{F}^k)$ $(n-k \geqq 2)$ *be given. Suppose that there exists a smooth function* $f: V^n \to \boldsymbol{R}$ *such that* $\Gamma_i(f) = \phi$ *for* $1 \leqq i \leqq k$. *Then* $w^{n-k}(L) \neq 0$ *for any leaf* L *if and only if* $[\Gamma(f)]_2 \neq 0$ *in* $H_k(V^n: Z_2)$, *where* $w^{n-k}(L)$ *is the* $(n-k)$*-th Stiefel-Whitney class of* L *and* $[\Gamma(f)]_2$ *is a* mod 2 *homology class of* $\Gamma(f)$.

In § 4, we shall give the proof of Theorem II, and furthermore obtain the orientable case of Theorem II (Theorem III).

Let $\mathrm{FOL}^k(V^n)$ denote the space of codimension k foliations on V^n with C^∞-topology as usual. By definition $\mathscr{F}^k$ is said to be stable under a small perturbation if there exists a small neighborhood $N(\mathscr{F}^k)$ of $\mathscr{F}^k$ such that for any $'\mathscr{F}^k \in N(\mathscr{F}^k), \mathscr{F}^k$ is integrably homotopic to $'\mathscr{F}^k$. Remark that $\Gamma(f)$ is an invariant set modulo isotopy under a small perturbation of $\mathscr{F}^k$. Then we have the following corollary.

COROLLARY 1. *Under the assumption of Theorem* I, $\mathscr{F}^k$ *is stable under a small perturbation.*

Furthermore, $\mathscr{F}^k$ is said to be stable along a compact leaf L if there exist an open neighborhood U of L and a small neighborhood $N(\mathscr{F})$ of $\mathscr{F}$ such that for $\mathscr{F}' \in N(\mathscr{F})$, there are a neighborhood W of L, in U, and a homeomorphism $h: W \to U$ satisfying $h(\mathscr{F} | W) = \mathscr{F}' | h(W)$.

COROLLARY 2. *Let* $(V^n, \mathscr{F}^k)$ $(n-k \geqq 6)$ *be a (compact or non-compact) codimension* k *foliated manifold without boundary, and* L *a compact, simply-connected leaf. Then* $\mathscr{F}$ *is stable along* L.

Proof. By Reeb's argument [3], pp. 130–131, there exists an open neighborhood U of L which is diffeomorphic to the product $L \times \operatorname{int} D^k$, where D^k is a k-disk. We define a smooth function $f: U \equiv L \times \operatorname{int} D^k \to \boldsymbol{R}$ by $f(y, z) = g(y)$, where $g: L \to \boldsymbol{R}$ is a nice function without singular points of index 1. Let p_0 (resp. $p_{n-k}) \in L$ be a singular point of index 0 (resp. $n-k$). Thus we have $\Gamma_0(f, \mathscr{F}) \equiv p_0 \times \operatorname{int} D^k$, $\Gamma_1(f, \mathscr{F}) = \phi$ and $\Gamma_{n-k}(f, \mathscr{F}) = p_{n-k} \times \operatorname{int} D^k$ in U. Next, we choose a sufficiently small neighborhood $N(\mathscr{F})$ of $\mathscr{F}$. Since U is open, $\mathscr{F} | U$ is a codimension k foliation on U for any $\mathscr{F}$. Since $\Gamma(f)$ is an invariant set modulo isotopy under a small perturbation, we may see that $\Gamma_0(f, \mathscr{F}') \equiv \Gamma_0(f, \mathscr{F})$, $\Gamma_1(f, \mathscr{F}') = \phi$ and $\Gamma_{n-k}(f, \mathscr{F}') \equiv \Gamma_{n-k}(f, \mathscr{F})$. We discuss about $f | L'_{p_0 \times 0}; L'_{p_0 \times 0} \to \boldsymbol{R}$, where $L'_{p_0 \times 0}$ is a leaf of $\mathscr{F}'$ which contains $p_0 \times 0$. Since $\Gamma_1(f, \mathscr{F}') = \Gamma_{n-k-1}(f, \mathscr{F}') = \phi$, $f | L'_{p_0 \times 0}$ has not singular points of index 1 and also

$n-k-1$. Furthermore $L'_{p_0\times 0} \cap \Gamma_{n-k}(f, \mathscr{F}') \neq \phi$ because of compactness of L. So $f \mid L'_{p_0\times 0}$ has exactly one singular point of index 0 and also $n-k$. This function satisfies the "completeness" condition of section 2. Hence by Proposition B (see section 2) we see that $L'_{p_0\times 0}$ is compact and simply-connected. A natural projection $L \times \operatorname{int} D^k \to L$ induces a diffeomorphism $\varphi: L'_{p_0\times 0} \to L$. The rest of proof is easy.

COROLLARY 3 (The Stability Theorem of Reeb [3: BII, 21]). *Let $(V^n, \mathscr{F}^k)$ $(n-k \geqq 6)$ be a codimension k foliated manifold and L^{n-k} a compact leaf with a finite fundamental group. Given an open neighborhood U of L, there exist neighborhoods W of L, and $N(\mathscr{F})$ of $\mathscr{F}$ such that if $\mathscr{F}' \in N(\mathscr{F})$, then every leaf of $\mathscr{F}'$ meeting W is compact and has a finite fundamental group and is contained in U.*

Proof. Take a neighborhood W_1 of L which is contractible to L. Let $p: \tilde{W}_1 \to W_1$ be a universal covering map and $\tilde{\mathscr{F}} = p^{-1}(\mathscr{F})$ is a codimension k foliation on $\tilde{W}_1$. For each leaf L' of $\mathscr{F}$ in $W_1, p^{-1}(L')$ is a union of leaves of $\tilde{\mathscr{F}}$ in $\tilde{W}_1$. In particular, $p^{-1}(L) = \tilde{L}$ is a compact, simply-connected leaf of $\tilde{\mathscr{F}}$. Thus there exists an open neighborhood $\tilde{U}_1\,(\equiv \tilde{L} \times \operatorname{int} D^k)$ of $\tilde{L}$, in $\tilde{W}_1$, satisfying $p(\tilde{U}_1) \subset U$. Then we may prove this corollary by the same discussions used for the corollary 2.

THEOREM IV. *Let V^n be a total space of L^{n-k}-bundle over a closed k-manifold M^k such that L^{n-k} is a closed, connected, simply-connected $(n-k)$-manifold $(n-k \geqq 6)$, and the universal covering space of M^k is contractible. Then the foliation $\mathscr{F}^k$ such that each leaf is a fiber, is stable under a small perturbation.*

Remark. When L^{n-k} is a closed, connected $(n-k)$-manifold $(n-k \geqq 6)$ with a finite fundamental group, a foliation $'\mathscr{F}^k$, near to $\mathscr{F}^k$, is a foliation induced from a certain fiber bundle. I don't know whether $\mathscr{F}^k$ is integrably homotopic to $'\mathscr{F}^k$, but it seems true.

The proof of this theorem will be given in § 5. Finally, we may give some examples by using Corollary 1, 2 and Theorem IV.

EXAMPLES. 1) $V^n = L^{n-k} \times M^k$, where L is a closed, connected $(n-k)$-manifold $(n-k \geqq 6)$ with a finite fundamental group and M^k is a closed k-manifold. Then the foliation $\mathscr{F}^k$ such that each leaf is $L \times \{m\}$, $m \in M$, is stable under a small perturbation.

2) Let V^n be a total space of S^{n-k}-bundle over a closed manifold

M^k with $\mathrm{SO}(n-k)$ as structural group $(n-k \geqq 2)$. Then the foliation such that each leaf is a fiber, is stable under a small perturbation.

3) Let $(V^n, \mathscr{F}^1)$ $(n \geqq 7)$ be a closed, transversally orientable codimension one foliated manifold. Suppose that there exists a compact leaf with a finite fundamental group. Then $\mathscr{F}^1$ is stable under a small perturbation.

4) Let V^n be a total space of L^{n-k}-bundle over $(S^1)^k$ or $(S^1)^{k-2\ell} \times (M^2)^\ell$, where L^{n-k} is a closed, connected, simply-connected $(n-k)$-manifold $(n-k \geqq 6)$ and M^2 is a closed 2-manifold which is not diffeomorphic to a two dimensional sphere and a two dimensional projective space. Then the foliation such that each leaf is a fiber, is stable under a small perturbation.

Proof. 2) is obtained by finding a function on V satisfying the assumption of Corollary 1, and 4) is an immediate consequence of Theorem IV. For 1), consider $\tilde{L} \times M$, where $\tilde{L}$ is a universal covering space of L. By using the same way as in the proof of Corollary 2, the foliation $\tilde{\mathscr{F}}$ such that each leaf is $\tilde{L} \times \{m\}$, $m \in M$, is stable under a small perturbation. Hence $\mathscr{F}^k$ is so. 3) is a special case of the above Remark. The proof will be given in § 5.

§ 2. Morse theory on a non-compact manifold

The purpose of this section is to show the following Proposition B which plays an essential role in the proof of Theorem I. Our definition and argument in this section are based on Yamato [7]. Let M^m be a connected, paracompact, complete, Riemannian m-manifold without boundary, and $f: M^m \to \boldsymbol{R}$ a bounded smooth function such that all of its singular points are of Morse type. Denote by $\| \quad \|$ the norm of tangent vectors or covectors of M. f is said to be "complete" if the gradient vector field of f (denoted by grad f) is complete and if there exist two families $\{E_i\}$, $i \in I$, $\{\tilde{E}_i\}$, $i \in I$ of open sets of M satisfying the following conditions: 1) for each singular point p of f, there is $i \in I$ such that $p \in E_i$, 2) $E_i \subset \tilde{E}_i$ for each i, and $\tilde{E}_i \cap \tilde{E}_j = \phi$ if $i \neq j$, 3) there exist three positive constants a_0, b_0, c_0 such that (a) $\|(\operatorname{grad} f)_x\| > a_0$, for $x \in M - \bigcup_{i \in I} E_i$, (b) $\operatorname{dis}(E_i, M - \tilde{E}_i) > b_0$, for each i, and (c) $\operatorname{diam}(\tilde{E}_i) < c_0$, for each i.

PROPOSITION B. *Let $f: M^m \to \boldsymbol{R}$ be a bounded, "complete", smooth* *function such that all of its singular points are of Morse type. If f has no singular point of index* 1, M^m *is compact and simply connected. In particular, f has exactly one singular point of index* 0.

The proof of Proposition B will be preceded by some lemmas. Let $\{\psi_t; t \in \boldsymbol{R}\}$ be the one-parameter group of transformations generated by $\operatorname{grad} f$.

LEMMA 1. *There exist positive constants d, h such that for $x \in M$ and $\tau > 0$, if* $\operatorname{dis}(x, \psi_\tau(x)) > d$, *then*

$$f(\psi_\tau(x)) - f(x) > h .$$

Proof. It follows by putting $d = \max(b_0, c_0)$, $h = a_0 b_0$ and noting

$$f(\psi_\tau(x)) - f(x) = \int_0^\tau \|(\operatorname{grad} f)_{\psi_t(x)}\|^2 dt$$

and

$$\int_0^\tau \|(\operatorname{grad} f)_{\psi_t(x)}\| dt \geqq \operatorname{dis}(x, \psi_\tau(x)) .$$

LEMMA 2. *Let p be a non-singular point of f. Then* $\lim_{t\to\infty} \psi_t(p)$ *exists and is a singular point of f.*

Proof. By the boundedness of f and Lemma 1, we can easily see that $\bigcup_{t \geqq 0} \psi_t(p)$ is bounded in M. Since M is complete, there is an infinite sequence $t_1 < t_2 < \cdots < t_n < \cdots \to \infty$ such that $\lim_{i\to\infty} \psi_{t_i}(p)$ exists in M. The rest of proof is easy.

LEMMA 3. *f has at least one singular point of index* 0 *and also m.*

Proof. Suppose that f has no singular point of index m. Let p_1 be a singular point of index $\lambda_1 (\neq m)$, and E_{i_1} an open set containing p_1 in the above definition. Then there exists a point $x_1 \in E_{i_1}$ such that $\psi_t(x_1)$ $(-\infty < t < \infty)$ is a trajectory issued from p_1. By Lemma 1, 2, $\lim_{t\to\infty} \psi_t(x_1) = p_2$ is a singular point of index $\lambda_2 (\neq m)$ and the inequality $f(\lim_{t\to\infty} \psi_t(x_1)) - f(p) > h$ holds. Similarly, there exists a point $x_2 \in E_{i_2}$ such that $\psi_t(x_2)$ $(-\infty < t < \infty)$ is a trajectory issued from p_2, and $\lim_{t\to\infty} \psi_t(x_2)$ exists and the inequality $f(\lim_{t\to\infty} \psi_t(x_2)) - f(p_2) > h$ holds. After iterating this process q-times, we get the inequality $f(\lim_{t\to\infty} \psi_t(x_q)) - f(p_q) > h$ and hence the inequality $f(p_{q+1}) - f(p_1) > q \cdot h$. This process can be continued infinitely because of the absence of a singular point of index m. This contradicts the boundedness of f. Hence f has at least one singular point of index m. It is similar for the case of index 0.

LEMMA 4. *Suppose that there is a non-singular, compact, connected submanifold J of $f^{-1}(r)$ for some r. Let δ_0 be a positive number satisfying the following condition: for any $\delta \in (0, \delta_0)$ and any $x \in J$, there exists a positive number $\tau(x, \delta)$ such that $f(\psi_{\tau(x,\delta)}(x)) - f(x) = \delta$. Let S be the subset of J consisting of those points x such that for any τ, the inequality $f(\psi_\tau(x)) - f(x) < \delta_0$ holds. If $S \neq \phi$, then $S_0(J) = \{\lim_{t\to\infty} \psi_t(x) ; x \in S\}$ is a finite set.*

Proof. By the boundedness of f and Lemma 1, we can see that the set $S_0(J)$ is bounded in M. Since every element of $S_0(J)$ is a singular point of Morse type by Lemma 2, $S_0(J)$ must be a finite set.

LEMMA 5. *Under the same assumption as in Lemma 4, if $S \neq \phi$, and if $S_0(J)$ contains no singular point of index 1, then the set $\tilde{J} = S_0(J) \cup (\bigcup \{\psi_{\tau(x)} ; x \in J - S\})$ is a compact, connected singular submanifold of $f^{-1}(r + \delta_0)$, where $\tau(x)$ is a positive function satisfying the equality $f(\psi_{\tau(x)}(x)) - f(x) = \delta_0$.*

Proof. Using the trajectories issued from J, it is easily verified that $\tilde{J}$ is a compact, connected, singular manifold. Since $\tilde{J}$ has no singular point of index 1 by the assumption, $\tilde{J}$ is a connected component of $f^{-1}(r + \delta_0)$.

Remark. If W^m is a compact, connected m-dimensional submanifold of M^m with J as the boundary, then there exists a compact, connected m-dimensional submanifold $\tilde{W}^m$ of M such that $\mathrm{Int}(\tilde{W}^m) \supset W^m$ and $\partial\tilde{W}^m$ is a connected component of $f^{-1}(r + \delta_0 + \eta)$ for some $\eta > 0$. In particular we may suppose $\eta > h$, where h is a positive constant in Lemma 1.

Proof of Proposition B. By Lemma 3, f has at least one singular point of index 0. Let p be such a point and suppose $f(p) = 0$. For a sufficiently small $\varepsilon > 0$, a connected component of $f^{-1}([0, \varepsilon])$ which contains p, is a m-disk D^m. Since the boundary of D^m is an $(m - 1)$-sphere, there exists a compact, connected submanifold $W_1^m (\supset D^m)$ of M^m such that $f(\partial W_1^m) > h$, by Lemma 4, 5 and Remark. The boundary of W_1^m is a compact, connected, non-singular level submanifold of M. Again by the use of Lemma 4, 5 and Remark, there exists a compact, connected submanifold $W_2^m (\supset W_1^m)$ of M^m such that $f(\partial W_2^m) > 2h$. After iterating this process q-times, we obtain a compact, connected submanifold $W_q^m (\supset W_{q-1}^m)$ of M^m such that $f(\partial W_q^m) > q \cdot h$. But by the boundedness of f, this process must finish at finite steps, i.e., there is a positive integer q_0 such that $M - \operatorname{Int} D^m = W_{q_0}^m$. Hence M^m is compact. Since M has no 1-handle, it is simply connected. In particular, the number of the singular points of index 0 is equal to one.

§ 3. Proof of Theorem 1

In order to prove Theorem I, we prepare the following lemma.

LEMMA 6 (Under the same assumption as in Theorem I). *Let N be a sufficiently small neighborhood of $S_1(\Gamma(f))$ in $\Gamma(f)$. Then for any leaf L such that $N \cap L \neq \phi$, there exist a neighborhood A of $N \cap L$ in L and a modified function $g: L \to \boldsymbol{R}$ such that* 1) $g|L - A = (f|L)|L - A$, *where $f|L$ is the restricted function to L of f,* 2) *g has no singular point in A, and* 3) $\|\operatorname{grad} g\| > a_0$ *on A, where a_0 is a positive constant.*

Proof. Since $S_1(\Gamma(f))$ is compact, there is a finite number of distinguished neighborhoods U_i, $i = 1, \cdots, i_0$, such that $\bigcup_{i=1}^{i_0} U_i \supset S_1(\Gamma(f))$ and the norm of u-components in its local coordinate is small. In U_i which contains $p \in S_1(\Gamma(f))$, we may assume that f is described as follows:

$$f - f(p) = u_1 - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n-k-1} x_j^2 + x_{n-k}^3 + u_1 x_{n-k} + \sum_{s,t} c_s^t u_s x_t + \cdots .$$

Then for any leaf L such that $L \cap U_i \neq \phi, f|$(component of $L \cap U_i$) is the restricted function to the plate, $u_i =$ constant for $1 \leqq i \leqq k$, of the above function. Therefore, by Milnor's method [1] pp. 48–66, $f|L$ is modified in U_i to g' such that 1) $(f|L)|L - U_i = g'|L - U_i$, 2) g' has no singular point in $L \cap U_i$, and 3) $\|\operatorname{grad} g'\| > a_0$ on $L \cap U_i$. Performing the same process in each small distinguished neighborhood U_i, we produce a modified function g satisfying the required properties. Hence we have only to put $N = (\bigcup_{i=1}^{i_0} U_i) \cap \Gamma(f)$ and $A = (\bigcup_{i=1}^{i_0} U_i) \cap L$.

Now we are in a position to prove Theorem I. By Lemma 6, for any leaf L of $\mathscr{F}^k$, all singular points of $f|L$ or the modified function g are of Morse type. Let $T(\Gamma(f)) \subset \tilde{T}(\Gamma(f))$ be tubular neighborhoods of $\Gamma(f)$ in V such that $\operatorname{dis}(T(\Gamma(f)), V - \tilde{T}(\Gamma(f))) > b_0$, and $\operatorname{diam}(D_x) < c_0$ for each $x \in \Gamma(f)$, where D_x is a fiber of $T(\Gamma(f))$ over x. We can regard as $D_x \subset L_x$ for $x \in \Gamma(f) - N$, where L_x is a leaf which contains x. Putting $\bigcup_{i \in I} E_i = L \cap (T(\Gamma(f)) | \Gamma(f) - N)$, $\bigcup_{i \in I} \tilde{E}_i = L \cap (\tilde{T}(\Gamma(f)) | \Gamma(f) - N)$, we see that $f|L$ or g is "complete". Since $f|L$ or g is bounded, and has no singular point of index 1, $f|L$ or g satisfies the assumption of Proposition B. Therefore, each leaf of $\mathscr{F}^k$ has exactly one singular point of index 0. Hence every leaf of $\mathscr{F}^k$ intersects $\Gamma_0(f)$ with exactly one point. We define a mapping $\pi: V \to \Gamma_0(f)$ by

$$\pi(x) = \Gamma_0(f) \cap L_x \qquad \text{for any } x \in V .$$

This implies that $\Gamma_0(f)$ can be identified with the leaf space $V/\mathscr{F}^k$. It is clear that π is a submersion. Since V is compact, π is a fiber mapping.

§ 4. Some topological properties of leaves in Theorem I

Let $(V^n, \mathscr{F}^k)$ be a codimension k foliated manifold as in § 1, and f be a smooth mapping of V^n into $\boldsymbol{R}^p$. As in § 1, we can define a subset $S_r(f, p; \mathscr{F}^k)$ of V^n, but in this section, we give the definition in another way. Let $\iota: V^n \to \boldsymbol{R}^m$ be an imbedding and $g: V^n \to \boldsymbol{R}^{m+p} = \boldsymbol{R}^m \times \boldsymbol{R}^p$ be an imbedding defined by $g(v) = (\iota(v), f(v))$ for $v \in V^n$. The Grassmann manifold of all $(n-k)$-dimensional vector subspaces of the space $\boldsymbol{R}^{m+p}$ we denote by $G_{m+p-n+k, n-k}$. Then we define a mapping $\bar{g}$ of V^n into $G_{m+p-n+k, n-k}$ by $\bar{g}(v) = T_v$, where T_v is the element of $G_{m+p-n+k, n-k}$ parallel to the tangent vector space of a leaf L_v at $g(v)$. Let F_r denote the set $\{X \in G_{m+p-n+k, n-k}; \dim (X \cap R^m \times 0) = n-k-p+r\}$. This F_r is a set of generic points of Schubert variety of type

$$(\underbrace{m+p-n+k-r, \cdots, m+p-n+k-r}_{n-k-p+r}, \underbrace{m+p-n+k, \cdots, m+p-n+k}_{p-r})$$

(cf. [2]). Then we define a subset $S_r(f, p; \mathscr{F}^k)$ of V to be $\bar{g}^{-1}(F_r)$. The following proposition is easily obtained by t-regularity theorem.

PROPOSITION C. *Let $f: V^n \to \boldsymbol{R}^p$ be a smooth mapping. Then there exists a smooth mapping g approximating f such that $S_r(g, p; \mathscr{F}^k)$ is a regular submanifold of V^n.*

Remark. $S_1(f, 1; \mathscr{F}^k)$ is equal to $\Gamma(f)$ defined in § 1. We can easily see that $S_1(f, p; \mathscr{F}^k)$ is a closed submanifold of V^n if $n > 2p + 2k - 2$ (cf. [6]).

Let $\xi: \boldsymbol{R}^{n-k} \to E(\xi) \to V^n$ be a completely integrable $(n-k)$-plane bundle which defines $\mathscr{F}^k$. Then by the usual argument, we have the following proposition.

PROPOSITION D. *The cohomology class dual to the* mod 2 *homology class of* $S_1(f, p; \mathscr{F}^k)$ $(1 \leqq p \leqq n-k)$ *is the* $(n-k-p+1)$*-th Stiefel-Whitney class* $W^{n-k-p+1}(\xi)$ *of* ξ.

Proof of Theorem II. By Theorem I, V^n is the total space of a fiber bundle over $\Gamma_0(f)$. Furthermore from the assumption, the fiber is a k-connected, closed $(n-k)$-manifold. Therefore by Proposition D, we have only to show the following lemma.

LEMMA 7. *Let* $L^{n-k} \xrightarrow{\iota} E^n \xrightarrow{\pi} M^k$ *be a smooth fiber bundle and suppose that* L *is* k*-connected. Then* $\iota^*: H^{n-k}(E; Z_2) \to H^{n-k}(L; Z_2)$ *is isomorphic.*

Proof. At first, note that $H^{n-k}(L; Z_2) = Z_2$, $H^{n-k}(E; Z_2) \cong H_k(E; Z_2)$. Since L is k-connected, $\pi_*: \pi_i(E) \to \pi_i(M)$ is isomorphic for $i \leqq k$. Therefore, $\pi_*: H_i(E; Z_2) \to H_i(M; Z_2)$ is so for $i \leqq k$. In particular $H_k(E; Z_2) \cong H_k(M; Z_2) = Z_2$. Next we shall use the cohomology spectral sequence for the fiber bundle. Since L is k-connected, we can easily see that $\iota^*: H^{n-k}(E; Z_2) \to H^{n-k}(L; Z_2)$ is epimorphic.

Finally we consider the orientable case. Let V^n be an oriented, closed n-manifold and $\mathscr{F}^k$ a transversally orientable codimension k foliation on V^n. At first we orient $\Gamma(f)$ as follows. Let $p \in \Gamma(f)$ and U be a distinguished neighborhood at p, whose local coordinate is $(u_1, \cdots, u_k, x_1, \cdots, x_{n-k})$. The orientation of V^n at p is given by

$$\left\{\left(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_k}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-k}}\right)_p\right\},$$

where $\{\ \}$ is an equivalence class of the basis $(\ )$. For $p \in \Gamma_\lambda(f)$, we can suppose that $T(\Gamma_\lambda(f))_p$ is spanned by $(\partial/\partial u_1, \cdots, \partial/\partial u_k)_p$, where $T(\Gamma_\lambda(f))_p$ is the tangent vector space of $\Gamma_\lambda(f)$ at p. Then we define the orientation $\eta(p)$ of $T(\Gamma(f))_p$ at $p \in \Gamma(f) - S_1(\Gamma(f))$ such that $\eta(p) = (-1)^\lambda\{(\partial/\partial u_1, \cdots, \partial/\partial u_k)_p\}$ for $p \in \Gamma_\lambda(f)$. The orientation $\eta(p)$ of $T(\Gamma(f))_p$ at $p \in S_1(\Gamma(f))$ is naturally determined. This definition is well-defined. In the following Theorem, we shall suppose that $\Gamma(f)$ is oriented by η.

THEOREM III. *Let* $(V^n, \mathscr{F}^k)$ $(n-k \geqq 2)$ *be a transversally orientable* *codimension k foliated manifold. Suppose that there exists a smooth function $f: V^n \to \boldsymbol{R}$ such that $\Gamma_i(f) = \phi$ for $1 \leqq i \leqq k$. Then $[\Gamma(f)] = \chi(L)[\Gamma_0(f)]$ in $H_k(V; Z)$ for any leaf L, where $\chi(L)$ is the Euler characteristic of L and $[\Gamma(f)]$ is the integral homology class of $\Gamma(f)$ with the above orientation.*

Proof. Let $\pi: V^n \to \Gamma_0(f)$ be a fiber map. $\Gamma_0(f)$ is already oriented as above. As in the proof of Theorem II, we see that $H_k(V; Z) = Z$ and $[\Gamma_0(f)]$ is its generator. Denote by π_1 the restriction of π to $\Gamma(f)$. Let $b \in \Gamma_0(f)$. We may assume that the map $\pi_1: \Gamma(f) \to \Gamma_0(f)$ is transversal to b. Let m_+ (or m_-) be the number of point p in $\pi_1^{-1}(b)$ such that $\pi_{1*}(\eta(p))$ is equal to $\eta(b)$ (or $\pi_{1*}(\eta(p))$ is not equal to $\eta(b)$). We define $\deg \pi_1$ to be m if $\pi_{1*}([\Gamma(f)]) = m \cdot [\Gamma_0(f)]$, where $\pi_{1*}: H_k(\Gamma(f); Z) \to H_k(\Gamma_0(f); Z)$. Note that $\deg \pi_1 = m_+ - m_-$.

On the other hand, the Euler characteristic of leaf $\pi^{-1}(b)$ is equal to $\sum_{i=1}^{n-k} (-1)^i c_i$, where c_i is the number of singular points of index i of $f|\pi^{-1}(b)$. It is easily checked that $m_+ = \sum_{i:\text{ even}} c_i$, $m_- = \sum_{i:\text{ odd}} c_i$. Therefore, we obtain $[\Gamma(f)] = (\deg \pi_1)[\Gamma_0(f)] = (m_+ - m_-)[\Gamma_0(f)] = (\sum_{i=0}^{n-k} (-)^i c_i)$ $[\Gamma_0(f)] = \chi(L) \cdot [\Gamma_0(f)]$.

§ 5. Proof of Theorem IV

Let $\tilde{M}$ be a universal covering space of M and q its projection. $\tilde{V}$ is a total space of a fiber bundle induced by q from the fiber bundle $\pi: V \to M$. Thus we have a commutative diagram:

$$\begin{array}{ccccc} L & \xrightarrow[\tilde{\iota}]{} & \tilde{V} & \xrightarrow[\tilde{\pi}]{} & \tilde{M} \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ L & \xrightarrow[\iota]{} & V & \xrightarrow[\pi]{} & M \end{array} .$$

Since $\tilde{M}$ is contractible, there exists a diffeomorphism d of $L \times \tilde{M}$ onto $\tilde{V}$ such that a following diagram commutes:

$$\begin{array}{ccc} L \times \tilde{M} & \xrightarrow[\mathrm{pr}]{} & M \\ \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \tilde{V} & \xrightarrow[\tilde{\pi}]{} & M \end{array} ,$$

where pr denotes a projection. Thus we have the following diagram:

$$\begin{array}{ccc} L \times \tilde{M} & \xrightarrow[\mathrm{pr}]{} & \tilde{M} \\ \Big\downarrow{\scriptstyle p\circ d} & & \Big\downarrow{\scriptstyle q} \\ V & \xrightarrow[\pi]{} & M \end{array}$$

Now, we introduce a metric on M and then introduce a "bundle-like metric" on V (see [4] for a definition of a bundle-like metric). Furthermore, we consider a metric induced by $p \circ d$ (resp. q) as a metric on $L \times \tilde{M}$ (resp. $\tilde{M}$). Then the metric on $L \times \tilde{M}$ is also a bundle-like metric for pr.

Next, we define a topology on $\mathrm{FOL}^k(V^n)$ as follows. Any $\mathscr{F} \in \mathrm{FOL}^k(V^n)$ (We omit k from $\mathscr{F}^k$) corresponds to a section $f_{\mathscr{F}}: V^n \to E(\gamma_{n-k,k})$ which defines $\mathscr{F}$, where $E(\gamma_{n-k,k})$ is a total space of a bundle associated with the tangent bundle of V^n with Grassmann manifold $G_{n-k,k}$ as fiber. Then we define an ε-neighborhood of $\mathscr{F}, N(\mathscr{F}, \varepsilon)$, to be the set $\{\mathscr{F}' \in \mathrm{FOL}^k(V^n); \|f_{\mathscr{F}} - f_{\mathscr{F}'}\| < \varepsilon\}$, where $\| \quad \|$ denotes a usual norm in the space of sections of $\gamma_{n-k,k}$. It is easy to show that $\mathscr{F}'$ belongs to $N(\mathscr{F}, \varepsilon)$ if and only if $\tilde{\mathscr{F}}'$ belongs to $N(\tilde{\mathscr{F}}, \varepsilon)$, where $\tilde{\mathscr{F}}$ (resp. $\tilde{\mathscr{F}}'$) is a foliation induced by $p \circ d$ from $\mathscr{F}$ (resp. $\mathscr{F}'$). We define a smooth function $f: L \times \tilde{M} \to \boldsymbol{R}$ by $f(y, z) = g(y)$, where $g: L \to \boldsymbol{R}$ is a Morse function without singular points of index 1. Then for any $\tilde{\mathscr{F}}' \in N(\tilde{\mathscr{F}}, \varepsilon)$ (ε is sufficiently small), $f \mid L'$ satisfies all conditions of Proposition B for any leaf L' of $\tilde{\mathscr{F}}'$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} L \times \tilde{M} & \xrightarrow[\tilde{\pi}']{} & \tilde{M} \\ \Big\downarrow{\scriptstyle p\circ d} & & \Big\downarrow{\scriptstyle q'} \\ V & \xrightarrow[\pi']{} & V/\mathscr{F}' \end{array},$$

where $\tilde{\pi}'$ is a fiber map, $V/\mathscr{F}'$ is a closed manifold, q' is a universal covering map and $\tilde{M}$ is considered as $* \times \tilde{M}, * \in L$, since we have $\Gamma_0(f) = * \times \tilde{M}$. Then we may easily check that $\|\mathrm{pr} - \tilde{\pi}'\| < N\varepsilon$ for some integer $N > 0$. Let δ be a small positive number. Then we may choose open neighborhoods $U_i \supset W_i$, $i = 1, \cdots, \ell$, in M, such that 1) U_i (resp. W_i) is an open ball of radius δ, (resp. $\frac{3}{4}\delta$) centered at x_i, 2) $\bigcup_{i=1}^{\ell} W_i = M$, and 3) π is trivial on U_i for each i. We construct a smooth isotopy of V inductively. Let $\tilde{U}_1 (\supset \tilde{W}_1)$ be a connected component of $q^{-1}(U_1)$ $(\supset q^{-1}(W_1))$. By putting $\varepsilon < \delta/8N$, we may check that $(\tilde{\pi}')^{-1}(\tilde{W}_1) \subset L \times \tilde{U}_1$.

Thus there exists a smooth isotopy $h_t(0 \leqq t \leqq 1)$ of $L \times \tilde{M}$ such that 1) $h_0 =$ identity, 2) $h_t | L \times (\tilde{M} - \tilde{U}_1) =$ identity, 3) h_t preserves $\tilde{M}(\equiv * \times M)$ pointwise, and 4) $h_1 | (\tilde{\pi}')^{-1}(\tilde{W}_1)$ is a bundle map. Since $p \circ d | L \times \tilde{U}_1$ is diffeomorphic, the composition $(p \circ d) \circ h_t \circ (p \circ d)^{-1}$ defines an isotopy g_t of V, i.e.,

$$g_t(z) = \begin{cases} (p \circ d) \circ h_t \circ (p \circ d)^{-1}(z) \,, & \text{for } z \in (p \circ d)(L \times \tilde{U}_1) = \pi^{-1}(U_1) \,, \\ z, & \text{otherwise.} \end{cases}$$

Furthermore, this isotopy g_t induces an isotopy of $L \times \tilde{M}$. Next, we construct an isotopy on $\pi^{-1}(U_2)$ and so on. Thus we may define an isotopy of V, isotopic to the identity, which transforms $\mathscr{F}'$ to $\mathscr{F}$. Hence $\mathscr{F}$ is integrably homotopic to $\mathscr{F}'$. Thus we complete the proof.

Proof of Example 3). We may easily see that V^n is a total space of L^{n-1}-bundle over S^1 by Reeb [3]. By the above argument, we have two commutative diagrams:

$$\begin{array}{ccc} \tilde{L} \times \boldsymbol{R} & \xrightarrow[\mathrm{pr}]{} & \boldsymbol{R} \\ \downarrow{\scriptstyle p \circ d} & & \downarrow{\scriptstyle q} \\ V & \xrightarrow[\pi]{} & S^1 \end{array} \,, \qquad \begin{array}{ccc} \tilde{L} \times \boldsymbol{R} & \xrightarrow[\tilde{\pi}']{} & \boldsymbol{R} \\ \downarrow{\scriptstyle p \circ d} & & \downarrow{\scriptstyle q'} \\ V & \xrightarrow[\pi']{} & S^1 \end{array} \,.$$

Since $\|\mathrm{pr} - \tilde{\pi}'\| < N\varepsilon$ for some integer $N > 0$, there exists a diffeomorphism of $\tilde{L} \times \boldsymbol{R}$, isotopic to the identity, which transforms $\tilde{\mathscr{F}}'$ to $\tilde{\mathscr{F}}$ and preserves $* \times \boldsymbol{R}, * \in L$, pointwise. Let $[0, s_0)$ (resp. $[0, s_1)$) is a periodic interval for q (resp. q'). We may suppose that $|s_0 - s_1| < N\varepsilon$. Then we define $\bar{\varphi}\colon [0, s_0) \to [0, s_1)$ as follows:

$$\bar{\varphi}(t) = \begin{cases} t, & \text{for } t \text{ near to } 0 \,, \\ t + (s_1 - s_0) \,, & \text{for } t \text{ near to } s_0 \,, \end{cases}$$

and

$$\|\bar{\varphi} - \mathrm{id}\| < K\varepsilon \,, \qquad \text{for some } K > 0 \,.$$

So we define a diffeomorphism $\varphi\colon S^1 \to S^1$ by $\varphi(s) = \bar{q}' \circ \bar{\varphi} \circ \bar{q}^{-1}(s)$, where $\bar{q}$ (resp. $\bar{q}'$) is a restricted function to $[0, s_0)$ (resp. $[0, s_1)$). Since $\varphi \circ q$ is sufficiently near to $q', \varphi \circ \pi$ is sufficiently near π' in the C^∞-topology of the space of submersions of V^n to S^1. Therefore, there exists a smooth homotopy $\pi_t\colon V \to S^1$ $(0 \leqq t \leqq 1)$ such that for each t, π_t is a submersion and $\pi_0 = \varphi \circ \pi$, $\pi_1 = \pi'$. This homotopy gives an integrable homotopy of $\mathscr{F}$ and $\mathscr{F}'$.

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