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STABLE VECTOR BUNDLES ON ALGEBRAIC SURFACES II

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This paper is a continuation of "Stable vector bundles on algebraic surfaces" [10]. For simplicity we deal with non-singular projective varieties over the field of complex numbers. Let W be a variety whose fundamental group is solvable, let H be an ample line bundle on W, and let $f: V \to W$ be an unramified covering. Then we show in section 1 that if E is an f^*H-stable vector bundle on V then f_*E is a direct sum of H-stable vector bundles. In particular f_*L is a direct sum of simple vector bundles if L is a line bundle on V. This result is a corollary of the following: Let A be a finite solvable group of automorphisms of a variety V. Suppose A acts freely on V. Let W be the quotient of V by A and let f be the natural morphism $V \to W$. Then the direct image of an f^*H-stable vector bundle on V by f is a direct sum of H-stable vector bundles, and the inverse image of an H-stable vector bundle on W by f is a direct sum of f^*H-stable vector bundles. In section 2 we prove the independence of H in the definition of the H-stability. Namely, let S be a relatively minimal surface, and let E be a vector bundle of rank two on S with $c_1^2(E) \geqq 4c_2(E)$. Then E is H-stable if and only if E is H'-stable, where H and H' are ample line bundles on S. We have proven this in our previous paper [10] in case $c_1^2(E) > 4c_2(E)$ without the assumption of relative minimality of S, and we obtained several results about H-stable vector bundles E with $c_1^2(E) = 4c_2(E)$ [10]. For instance, an H-stable vector bundle with $c_1^2 = 4c_2$ of rank two on an abelian surface is the direct image of a line bundle under an isogeny of a special type. And an H-stable vector bundle with $c_1^2 = 4c_2$ of rank two on a geometrically ruled surface is the vector bundle induced from a stable vector bundle on the base curve tensored with a line bundle on the surface. In connection with these results, we show in section 4 that on an elliptic bundle the vector bundle induced from a stable vector bundle of rank

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two on the base curve is H-stable (in case $c_1^2 = 4c_2$). For its proof we give in section 3 the necessary and sufficiently condition about the ampleness of a line bundle on an elliptic bundle: A line bundle L is ample if and only if $(L^2) > 0$ and $(L, C) > 0$ where C is a fibre. In section 5 we study H-stable vector bundles on an elliptic bundle of a special type i.e. a hyperelliptic surface. Let $A(S)$ be the set of all H-stable vector bundles of rank two with $c_1^2 = 4c_2$ on a hyperelliptic surface S, let $B(S)$ be the set of all indecomposable vector bundles of rank two on S each of which is the direct image of a line bundle under an unramified covering, and let $C(S)$ be the set of all vector bundles on S each of which is the tensor product of a line bundle on S and the vector bundle induced from a stable vector bundle of rank two on the base curve. Then the following holds: 1) $A(S) \supset B(S) \supset C(S)$. 2) If S is basic, then $A(S) \neq C(S)$. 3) If $4K_S = 0$, then $A(S) = B(S)$. 4) If $4K_S \neq 0$, then 4. a) in case $3K_S = 0$ with S basic, we have $B(S) = C(S)$, and 4. b) in case $3K_S \neq 0$ we have $B(S) \neq C(S)$, but $B(S) = C(S)$ under a suitable restriction about vector bundles. Here K_S is the canonical line bundle on S.

1. The direct image of a line bundle under an unramified covering

All the varieties considered below will be assumed to be over an algebraically closed field k, non-singular, projective and irreducible. And all the sheaves will be assumed to be coherent.

DEFINITION. A torsion-free sheaf F of finite rank on a variety V is *quasi locally free* if $\operatorname{depth}_S F \geqq 2$ for any closed subvariety S of V with $\operatorname{codim}_V S \geqq 2$.

Remark 1). Let F and S be as above. We put $U = V - S$. Then the restriction homomorphism $H^0(V, F) \to H^0(U, F)$ is bijective. Hence for any coherent sheaf G on $V, H^0(V, \operatorname{Hom}(G, F)) \xrightarrow{\sim} H^0(U, \operatorname{Hom}(G, F))$.

Remark 2). Let $f: W \to V$ be an unramified covering and let F be quasi locally free on V. Then f^*F is also quasi locally free.

LEMMA (1.1). *Let F be a torsion-free sheaf of finite rank on V. Then there exists a quasi locally free sheaf G such that $F \subset G$ and* $\operatorname{codim}(\operatorname{Supp}(G/F)) \geqq 2$.

Proof. We put $U = \{x \in V \mid F$ is locally free at $x\}$ and $Y = V - U$.

Y is closed in V and $\operatorname{codim}_V Y \geqq 2$. Let $i: U \to V$ be the natural morphism. We put $G = i_* i^* F$. It is clear that $F \subset G$ and codim (Supp (G/F)) $\geqq 2$. Since $G = i_* i^* G$ we have $H^0_Y(G) = H^1_Y(G) = 0$, hence $\operatorname{depth}_Y G \geqq 2$. This proves the lemma by the definition of U.

Remark. If F is a coherent subsheaf of a vector bundle E, then above G is a coherent subsheaf of E.

Let H be an ample line bundle on a variety V. For any non-torsion coherent sheaf F, we put $x(F, H) = d(F, H)/r(F)$ where $d(F, H) = (\operatorname{Inv}(F), H^{s-1})$ and $r(F) = \operatorname{rank} F$ ($s = \dim V$).

DEFINITION. A non-torsion coherent sheaf F on V is *H-stable* (resp. *H-semi-stable*) if every non-torsion coherent subsheaf G of F with $r(G) < r(F)$ we have $x(G, H) < x(F, H)$ (resp. $x(G, H) \leqq x(F, H)$).

This is a generalization of the definition of H-stability (resp. H-semi-stability) given in [10] for vector bundles.

PROPOSITION (1.2). *Let E and F be H-stable quasi locally free sheaves of finite rank with $r(E) = r(F)$ and $d(E, H) = d(F, H)$. If $f: E \to F$ is a non-zero homomorphism, then f is an isomorphism.*

Proof. Put $G =$ Image of f. By definition we have $x(E, H) \leqq x(G, H) \leqq x(F, H)$. By assumption we readily have $r(E) = r(G)$, hence f is injective, i.e. $E \xrightarrow{\sim} G$. We put Supp $(F/E) = S$. Since F/E is torsion and $x(F/E, H) = 0$, we have $\operatorname{codim} S \geqq 2$. Since f is an isomorphism on $V - S$, f is an isomorphism on V by Remark 1).

COROLLARY (1.3). *Let E be an H-stable quasi locally free sheaf of finite rank. Then E is simple. i.e. $H^0(V, \operatorname{Hom}(E, E)) = k$.*

Remark. Prop. (1.2) and Cor. (1.3) are generalizations of Prop. (1.7) and Cor. (1.8) in [10].

LEMMA (1.4). *Let E_1 and E_2 be H-semi-stable vector bundles on V with $x(E_1, H) = x(E_2, H)$. Then an extension E of E_2 by E_1 is H-semi-stable with $x(E, H) = X(E_i, H)$.*

Proof. Suppose E is not H-semi-stable. There exists a coherent subsheaf F of E such that $x(F, H) > x(E, H)$. Let f be the natural homomorphism $F \to E \to E_2$. Since $x(F, H) > x(E, H) = x(E_1, H)$, f is non-zero by the H-semi-stability of E_1. We put $F_1 = \operatorname{kernel}(f)$ and $F_2 =$

image(f). Case 1) If $x(F_1, H) \geqq x(F, H)$, then $x(F_1, H) > x(E_1, H)$ which contradicts the H-semi-stability of E_1 since $F_1 \subset E_1$. Case 2) If $x(F_1, H) < x(F, H)$, then $x(F_2, H) > x(F, H) > x(E, H) = x(E_2, H)$ which contradicts the H-semi-stability of E_2. q.e.d.

Let A be a finite group of automorphisms of V. Suppose A acts freely on V. Let W be the quotient of V by A and let f be the natural morphism $V \to W$ which is an unramified covering by assumption. These notations remain fixed in this section. We remark that for any non-torsion coherent sheaf F on W $x(f^*F, f^*H) = \deg f \cdot x(F, H)$. Hence if f^*F is f^*H-semi-stable, then F is H-semi-stable.

PROPOSITION (1.5). *Let E be an f^*H-semi-stable vector bundle on V where H is an ample line bundle on W. Then $f_*(E)$ is H-semi-stable.*

Proof. Since $f^*f_*(E) = \bigoplus_{a \in A} a^*(E)$, $f^*f_*(E)$ is f^*H-semi-stable by Lemma (1.4). We have the desired result by the above remark.

LEMMA (1.6). *Assume A is a cyclic group of prime order l which is different from the characteristic of k. Let E be an f^*H-stable vector bundle on V for an ample line bundle H on W. If E is not isomorphic to f^*E_1 for any vector bundle E_1 on W, then $f_*(E)$ is H-stable.*

Proof. Suppose f_*E is not H-stable. Since f_*E is H-semi-stable by Prop. (1.5), there exists a quasi locally free subsheaf F of f_*E such that $x(F, H) = x(f_*E, H)$ with $r(F) < r(f_*E)$. We may assume F is H-stable by taking such F with the smallest $r(F)$. Since $f^*f_*E = \bigoplus_{a \in A} a^*E$ and $a^*f^*F = f^*F$ for any $a \in A$, we have a non-zero homomorphism $\alpha: f^*F \to E$. f^*F is f^*H-semi-stable quasi locally free because $x(f^*F, f^*H) = x(f^*f_*E, f^*H)$ and f^*f_*E is f^*H-semi-stable. We put $G_0 = f^*F$ and $\alpha_0 = \alpha$. We fix a generator a of A. Inductively we define f^*H-semi-stable coherent subsheaves G_i of f^*F with $r(G_i) = r(F) - ir(E)$ and $x(G_i, f^*H) = x(f^*F, f^*H)$ such that the natural homomorphism $\alpha_i: G_i \to f^*F \to (a^*)^i(E)$ is non-zero and the kernel of α_i is G_{i+1} $(i = 0, 1, 2, \cdots, l-1)$. Assume $G_0, G_1, \cdots, G_i$ are already defined. First we show $G_{i+1} \neq 0$. By Lemma (1.1), there exists a quasi locally free subsheaf G'_i of f^*F such that $G_i \subset G'_i$ and codim (Supp $(G'_i/G_i)) \geqq 2$. It is clear that G'_i is f^*H-semi-stable. If $G_{i+1} = 0$ i.e. α_i is injective, then the natural homomorphism $\alpha'_i: G'_i \to f^*F \to (a^*)^i(E)$ is also injective modulo codim $\geqq 2$. Since $x(G'_i, f^*H)$

$= x((a^*)^i(E), f^*H)$ and $(a^*)^i(E)$ is f^*H-stable, hence G'_i is f^*H-stable with $r(G'_i) = r((a^*)^iE)$, the cokernel of α'_i is torsion and codim Supp (Cokernel of $\alpha'_i) \geqq 2$. Hence by Remark 1), α'_i is an isomorphism. Thus f^*F contains $(a^*)^i(E)$ as a direct summund. Since $a^*f^*F = f^*F$, f^*F contains $\bigoplus_{i=0}^{l-1}(a^*)^i(E)$ which contradicts $r(f^*F) < l \cdot r(E)$. This proves $G_{i+1} \neq 0$. We remark the following fact before we show α_{i+1} is non-zero. Let E_0 and F_0 be quasi locally free sheaves of finite rank on a variety V such that E_0 is H-semi-stable and F_0 is H-stable for an ample line bundle H on V with $x(E_0, H) = x(F_0, H)$. Then any non-zero homomorphism from E_0 to F_0 is surjective modulo a closed subvariety of codim $\geqq 2$. We have the following diagram

$$\begin{array}{ccc} 0 \longrightarrow G_1 \longrightarrow G_0 = f^*F & \xrightarrow{\alpha_0} & E \\ \| & & \\ (a^*)^{i+1}f^*F & \longrightarrow & (a^*)^{i+1}(E) \end{array}$$

where the two arrows on the right are surjective modulo codim $\geqq 2$. Note that $a^*E \neq E$ by assumption. Hence the natural homomorphism $G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero and the natural homomorphism $G_2 \to G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero since $\alpha_1 : G_1 \to a^*E$ is surjective modulo codim $\geqq 2$. Continuing in this fashion, we see that the natural homomorphism $\alpha_{i+1} : G_{i+1} \to G_i \to \cdots \to G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero. Since $G_{i+1} = \ker(\alpha_i : G_i \to (a^*)^iE)$, G_i is f^*H-semi-stable, $(a^*)^i(E)$ is f^*H-stable, $x(G_i, f^*H) = x((a^*)^iE, f^*H)$ and α_i is surjective modulo codim $\geqq 2$, we have $x(G_{i+1}, f^*H) = x(f^*F, f^*H)$. Hence G_{i+1} is f^*H-semi-stable, and $r(G_{i+1}) = r(G_i) - r(E)$. Thus we have $G_0, G_1, \cdots, G_{l-1}$. Since α_{l-1} is non-zero, we have $r(G_{l-1}) \geqq r(E)$ for the same reason, hence $r(F) \geqq lr(E)$. This contradicts $r(F) < r(f_*E)$. q.e.d.

If $E = f^*E_1$ for a vector bundle E_1 of W in Lemma (1.6), then $f_*E = \bigoplus_i E_1 \otimes N_i$ since $f_*(\mathcal{O}_V) = \bigoplus_i N_i$ with N_i line bundles on W such that $f^*N_i \simeq \mathcal{O}_V$. It is clear that E_1 is H-stable since $f^*E_1 = E$.

PROPOSITION (1.7). *Assume A is solvable and the order of A is prime to the characteristic of k. If E is an f^*H-stable vector bundle on V for an ample line bundle H on W, then $f_*E = \oplus E_i$ where E_i is an H-stable vector bundle on W with $x(f_*E, H) = x(E_i, H)$.*

Proof. It is clear by Lemma (1.6) and the remark above, by successive reduction to the case of prime order.

COROLLARY (1.8). *Assume A is solvable and the order of A is prime to the characteristic. Let L be a line bundle on V. If f_*L is indecomposable, then f_*L is H-stable for any ample line bundle H on W.*

PROPOSITION (1.9). *Assume A is solvable and A has order prime to the characteristic. If E is an H-semi-stable vector bundle on W for an ample line bundle H on W, then f^*E is f^*H-semi-stable.*

Proof. We may assume A is cyclic. We have $f_*(\mathcal{O}_V) = \oplus_i N_i$ with N_i line bundles on W. Since $f_*f^*E = \oplus_i E \otimes N_i$ and $x(E \otimes N_i, H) = x(E, H)$ by Lemma (1.4), f_*f^*E is H-semi-stable. Hence for any coherent subsheaf F of f^*E, $x(f_*F, H) \leqq x(f_*f^*E, H)$. We have the cartesian diagram

$$\begin{array}{ccc} V \times A & \xrightarrow{\sigma} & V \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{f} & W \end{array}$$

where σ is the action of A on V and p_1 is the projection. We put $t =$ order of A. Since f is flat,

$$\begin{aligned}(\mathrm{Inv}\,(f_*F), H) &= \frac{1}{t}\,(\mathrm{Inv}\,(f^*f_*F), f^*H) = \frac{1}{t}\,(\mathrm{Inv}\,(\sigma_*p_1^*F), f^*H) \\ &= \frac{1}{t}\left(\mathrm{Inv}\Big(\bigoplus_{a \in A} a^*F\Big), f^*H\right) = (\mathrm{Inv}\,(F), f^*H)\ .\end{aligned}$$

It is clear that $d(f_*f^*E, H) = d(f^*E, f^*H)$. q.e.d.

PROPOSITION (1.10). *Let A be as above. If E is an H-stable vector bundle on W for an ample line bundle H on W, then $f^*E = \oplus E_i$ where E_i is an f^*H-stable vector bundle on V with $x(f^*E, f^*H) = x(E_i, f^*H)$.*

Proof. We may assume A to be a cyclic group of prime order l. As above we have $f_*(\mathcal{O}_V) = \oplus N_i$. If f^*E is f^*H-stable, we are done. Suppose f^*E is not f^*H-stable. By Prop. (1.9) f^*E is f^*H-semi-stable, hence there exists a quasi locally free subsheaf F of f^*E such that $x(F, f^*H) = x(f^*E, f^*H)$ and $1 \leqq r(F) < r(E)$. We may assume F to be f^*H-stable by choosing such F with the smallest $r(F)$. Since $f_*(F)$ is the subsheaf of the H-semi-stable bundle f_*f^*E (cf. Prop. (1.7)) and $x(f_*F, H) = x(f_*f^*E, H)$, we have $f_*(F)$ is H-semi-stable. On the other hand $f_*(F) \otimes N_i = f_*(F)$ and $f_*(F) \subset \oplus E \otimes N_i = f_*f^*E$, hence there exists a non-zero homomorphism $\beta : f_*(F) \to E$. Then $x(E, H) = x(f_* f^* E, H) = x(f_* F, H) \leqq x$ (Image of β, H). By the H-stability of E we thus $r(E) = r$ (Image of β) $\leqq r(f_* F)$ i.e. $r(E) \leqq lr(F)$. We have $a^* F \neq F$ for any $a \neq 1 \in A$. Indeed if $a^* F = F$ for some $a \neq 1$, hence for all $a \in A$ since A is cyclic of prime order, then $F = f^* F_1$ for some coherent subsheaf F_1 of E, and $x(F_1, H) = x(E, H)$ which contradicts the H-stability of E. We fix a generator a of A. Inductively we define f^*H-semi-stable coherent sheaves G_i with $x(F, H) = x(G_i, f^* H)$ and codim Supp (the torsion part of G_i) $\geqq 2$, and inclusions $\alpha_i : F \to a^* G_i$ $(i = 0, 1, \cdots, l-1)$ such that we have exact sequences

$$0 \longrightarrow F \xrightarrow{\alpha_i} a^* G_i \xrightarrow{\beta_i} G_{i+1} \longrightarrow 0$$

and $\alpha_i = a^*(\beta_{i-1}) a^{*2}(\beta_{i-2}) \cdots a^{*i}(\beta_0) \cdot \alpha_0$. We put $G_0 = f^* E$ and $\alpha_0 =$ the canonical inclusion. Assume $G_0, G_1 \cdots, G_i$ are already defined. Since $x(a^* G_i, f^* H) = x(F, f^* H)$ and $a^* G_i$ is $f^* H$-semi-stable, G_{i+1} is $f^* H$-semi-stable and codim Supp (the torsion part of G_{i+1}) $\geqq 2$. It is sufficient to show α_{i+1} is an inclusion. If $\alpha_{i+1} = 0$, then $a^{*2}(\beta_{i-1}) \cdots a^{*i+1}(\beta_0) \cdot \alpha_0 = 0$ because

$$0 \longrightarrow a^* F \longrightarrow a^{*2} G_i \xrightarrow{a^*(\beta_i)} a^* G_{i+1} \longrightarrow 0 \text{ (exact)} ,$$

$a^{*2}(\beta_{i-1}) \cdots a^{*i+1}(\beta_0) \cdot \alpha_0 : F \to a^{*2} G_i$ and $F \neq a^* F$. Continuing in this fashion, we have a contradiction. Thus α_{i+1} is non-zero, hence it is an inclusion because F is $f^* H$-stable and $a^* G_{i+1}$ is $f^* H$-semi-stable. Therefore we have $G_0, G_1, \cdots, G_{l-1}$. Since $r(G_{l-1}) = r(E) - (l-1) r(F) \leqq r(F)$ for the same reason as above, we have codim Supp (Coker of α_{l-1}) $\geqslant 2$. By Remark 1) there exists a homomorphism $\delta : f^* E \to \alpha_{l-1}(F) \cong F$ such that $\delta \cdot \alpha_0 = \alpha_{l-1}$.

$$\begin{array}{llll}
0 \longrightarrow F \xrightarrow{\alpha_0} & f^* E & & \\
\qquad\quad \vdots & \quad \wr\| & & \\
\qquad\quad \vdots & a^{*l-1} f^* E & \xrightarrow{\quad a^*(\beta_{l-2}) \cdots a^{*l-2}(\beta_0) \quad} & a^* G_{l-1} \longrightarrow 0 \\
\qquad\quad \vdots & & \quad \searrow^{\exists \delta} & \quad \cup \\
\qquad\quad \text{L} - - - & \alpha_{l-1} & - - - - - - - \to & \alpha_{l-1}(F) = F
\end{array}$$

Hence the inclusion $F \to f^* E$ splits, and so $f^* E$ contains $\bigoplus_{i=0}^{l-1} a^{*i} F$ as a direct summund. By comparing the ranks, we have $f^* E = \bigoplus_{i=0}^{l-1} a^{*i} F$.

q.e.d.

COROLLARY (1.11). *Let W be a non-singular projective variety over the complex number field. Assume that the fundamental group of W is solvable. Let H be an ample line bundle on W and let $f: V \to W$ be an unramified covering. Then*

1) *If E is an H-stable vector bundle on W, then $f^*E = \oplus E_i$ where E_i is an f^*H-stable vector bundle on V with $x(E_i, f^*H) = x(f^*E, f^*H)$.*

2) *If E is an f^*H-stable vector bundle on V, then $f_*E = \oplus E_i$ where E_i is an H-stable vector bundle on W with $x(E_i, H) = x(f_*E, H)$.*

Proof. There exist a non-singular projective variety U, a finite group A of automorphisms of U, and a normal subgroup B of A such that W is the quotient of U by A and V is the quotient of U by B. By assumption, A and B are solvable. Hence we have Cor. (1.11) by Prop. (1.7) and Prop. (1.10).

Remark. If W is an abelian variety, then the above corollary holds in arbitrary characteristic.

Let $f: V \to W$ be an unramified covering such that $f_*(\mathcal{O}_V)$ contains a non-trivial line bundle J as a direct summund. Then $f^*J = \mathcal{O}_V$ and we have $f_*E \simeq f_*E \otimes J$ for a vector bundle E on V. Conversely,

LEMMA (1.12). *Let E be a simple vector bundle on V. Assume $r(E)$ is prime to the characteristic of k. If E is isomorphic to $E \otimes J$ for a non-trivial line bundle J on V, then there exist a non-trivial cyclic unramified covering $f: U \to V$ and a simple vector bundle E_1 on U such that $f_*(E_1) = E$.*

Proof. Obviously we have $J^{\otimes r} = \mathcal{O}_V$ with $r = \text{rank of } E$. Let d be the smallest positive integer such that $J^{\otimes d} = \mathcal{O}_V$. Then a locally free sheaf $B = \bigoplus_{i=0}^{d-1} J^{\otimes i}$ can be considered as an $\mathcal{O}_V$-algebra by defining the multiplication by $J^{\otimes d} = \mathcal{O}_V$. $f: U = \operatorname{Spec}(B) \to V$ is an unramified covering of degree d. $H^0(f^* \operatorname{End}(E)) = H^0(f_*f^* \operatorname{End}(E)) = \bigoplus_{i=0}^{d-1} H^0(\operatorname{Hom}(E, E \otimes J^{\otimes i}))$. Hence $H^0(\operatorname{End}(f^*E)) = k[X]/(X^d - 1)$ as k-algebras, which is a direct sum of k since k is algebraically closed. Therefore $f^*E = \bigoplus_{i=1}^{d} E_i$. Thus $E^{\oplus d} = f_*f^*E = \bigoplus_{i=1}^{d} f_*E_i$, hence $E = f_*E_i$ for any i. On the other hand, $d = \dim_k H^0(\operatorname{End}(f^*E)) = \sum_{1 \leq i,j \leq d} \dim_k H^0(\operatorname{Hom}(E_i, E_j))$, hence in particular E_i is simple for any i.

COROLLARY (1.13). *Let E be a simple vector bundle on V of prime* *rank which is different from the characteristic. The following conditions are equivalent to each other*

1) *There exists a non-trivial line bundle J on V such that E is isomorphic to $E \otimes J$.*

2) *There exist an unramified covering $f: U \to V$ and a line bundle L on U such that E is isomorphic to $f_*(L)$. In this case, E is H-stable for any ample line bundle H on V.*

Remark. Let E be isomorphic to $E \otimes J$ for a line bundle J. Then $\mathrm{End}(E)$ contains J as a direct summand. Conversely, let E be a vector bundle such that $\mathrm{End}(E)$ contains J as a direct summand. Moreover, assume E is H-stable for an ample line bundle H. Then E is isomorphic to $E \otimes J$. Indeed, let E be as above. $H^0(\mathrm{Hom}(E, E \otimes J^{-1})) \neq 0$, so $d(J^{-1}, H) \geqq 0$ by the H-stability of E. Since $\mathrm{End}(E)$ is self-dual, it also contains J^{-1} as a direct summand, hence similarly $d(J, H) \geqq 0$. Thus $d(J, H) = 0$. Therefore $E \simeq E \otimes J$ by Prop. (1.2).

2. H-stability on minimal models

PROPOSITION (2.1). *Let S be a relatively minimal non-singular projective surface over the complex number field, and let E be a vector bundle of rank two on S with $N(E) = c_1^2(E) - 4c_2(E) \geqq 0$. If E is H-stable for some ample line bundle H on S, then E is H'-stable for any ample line bundle H' on S.*

Remark. When $N(E)$ is positive, we have shown this in Prop. (2.7) of [10] without the assumption of relative minimality of S. We do not know whether this holds in case $N(E) = 0$ without the assumption of relative minimality of S.

First we show the following lemma:

LEMMA (2.2). *Let S be a non-singular projective surface which satisfies one of the following conditions, and E be a vector bundle of rank two on S with $N(E) \geqq 0$. If E is H-stable for some ample line bundle H on S, then E is H'-stable for any ample line bundle H' on S.*

(1) *The Euler-Poincaré characteristic $\chi(\mathcal{O}_S)$ of $\mathcal{O}_S$ is positive*

(2) *Let H be an ample line bundle on S, K the canonical line bundle on S and L a line bundle on S. If $(L, K) = 0$, $(L^2) = 0$, $H^1(L^{-1}) \neq 0$ and $(L, H) > 0$, then $(L, H') > 0$ for any ample line bundle H' on S.*

Proof of Lemma (2.2). By definition, E is H-stable if and only if we have $(L_2 \otimes L_1^{-1}, H) > 0$ for any morphism $f: T \to S$ obtained by successive dilatations and any extension $0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$ where L_1 and L_2 are line bundles on T and M is a positive exceptional line bundle on T. Put $L = L_2 \otimes L_1^{-1}$. Since $N(E) = (L^2) + 4(M^2) \geqq 0$, we have $(L^2) \geqq -4(M^2) \geqq 0$. Now $H^2(L^{\otimes n}) = 0$ for sufficiently large n because $(L, H) > 0$. By Riemann-Roch theorem,

$$\begin{aligned}\chi(L^{\otimes n}) &= h^0(L^{\otimes n}) - h^1(L^{\otimes n}) + h^2(L^{\otimes n}) \\ &= \frac{1}{2}n^2(L^2) - \frac{1}{2}n(L, K) + \chi(\mathcal{O}_S)\,,\end{aligned}$$

where $h^i(L^{\otimes n}) = \dim_k H^i(S, L^{\otimes n})$.

Case 1. $(L^2) > 0$. Then we have the desired result because $H^0(L^{\otimes n}) \neq 0$ for sufficiently large n.

Case 2. $(L^2) = 0$. Then $(M^2) = 0$ and hence $M = \mathcal{O}_S$. By tensoring with L_1^{-1}, we obtain the extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \otimes L_1^{-1} \longrightarrow L \longrightarrow 0\,.$$

a) $(L, K) < 0$. Then we have the desired result because $H^0(L^{\otimes n}) \neq 0$ for sufficiently large n.

b) $(L, K) = 0$. Then $\chi(L^{\otimes n}) = \chi(\mathcal{O}_S)$. If $\chi(\mathcal{O}_S) > 0$, then $H^0(L^{\otimes n}) \neq 0$ for sufficiently large n. On the other hand the above extension is non-trivial. Hence $H^1(L^{-1}) \neq 0$ and we have the desired result by assumption.

c) $(L, K) > 0$. Then $\chi(L^{\otimes -n}) > 0$ for sufficiently large n. Assume $(L, H') \leqq 0$ for some ample line bundle H' on S. If $(L, H') = 0$, then L is numerically equivalent to $\mathcal{O}_S$ since $(L^2) = 0$. This contradicts $(L, H) > 0$. On the other hand, if $(L, H') < 0$, then for the same reason $H^0(L^{\otimes -n}) \neq 0$ for sufficiently large n. This contradicts $(L, H) > 0$.

Remark. If a line bundle L on S with $\chi(\mathcal{O}_S) = 0$ satisfies the condition (2) of Lemma (2.2), then $0 \neq h^2(L^{-1}) = h^0(L \otimes K)$ i.e. $L \otimes K = \mathcal{O}_S(D)$ for some positive divisor D.

Proof of Prop. (2.1). By the classification theorems of surfaces and our previous result (Th. 3.7, Prop. 4.1 and Prop. 5.1 of [10]), we may assume S is (a) an elliptic surface, (b) a $K3$ surface or (c) a surface of general type. (c) If S is of general type, then $c_2(S) > 0$ by Van de Ven's result [11]. On the other hand $c_1^2(S) > 0$ and hence $\chi(\mathcal{O}_S) > 0$. (b) If S

is a $K3$ surface, then $\chi(\mathcal{O}_S) > 0$. In cases (b) and (c), we have the desired result by Lemma (2.2). Therefore we may assume S is an elliptic surface. Then $\chi(\mathcal{O}_S) = (1/12)c_2(S)$. On the other hand, $c_2(S)$ is equal to the topological Euler-Poincaré characteristic $\chi^t(S)$ of S. Let $p: S \to \varDelta$ be a morphism from S to a non-singular projective curve $\varDelta$ such that the inverse image $p^{-1}(u) = C_u$ of any general point $u \in \varDelta$ is an elliptic curve. Then we have an equality $\chi^t(S) = \chi^t(\varDelta)\chi^t(F) + \Sigma(\chi^t(F_b) - \chi^t(F)) = \Sigma\chi^t(F_b)$ where $\chi^t(*)$ is the topological Euler-Poincaré characteristic and F is any general fibre of p and F_b is a singular fibre. Since S has no exceptional curve of the first kind, $\chi^t(F_b) \geqq \chi^t(F) = 0$ [8] p. 60 and hence $\chi^t(S) \geqq 0$. Hence $\chi(\mathcal{O}_S) \geqq 0$. By Lemma (2.2) we may assume $\chi(\mathcal{O}_S) = 0$ and hence $\chi^t(F_b) = \chi^t(F)$. Thus $F_b = m_b C_b$ where C_b is an elliptic curve [8] p. 60. On the other hand, $K = p^*(K_\varDelta \otimes I) \otimes \mathcal{O}_S(\Sigma(m_b - 1)C_b)$ where I is a line bundle on $\varDelta$ with $\deg I = 0$ and $K_\varDelta$ is the canonical line bundle on $\varDelta$ [3]. Assume a line bundle L on S satisfies the condition of Lemma (2.2) (2). Then by the above Remark, $L \otimes K = \mathcal{O}_S(D)$ for some positive divisor D. Now $0 = (D, K) = (2g - 2 + \Sigma(1 - 1/m_b))(D, F)$ where g is the genus of $\varDelta$. If $D = 0$, then $L = K^{-1}$ and so $(L, H) = -(2g - 2 + \Sigma(1 - 1/m_b))(F, H)$. The sign of (L, H) is thus independent of an ample line bundle H. It remains to treat the case $D \neq 0$. If $p_*(D) \neq \varDelta$, then D is a linear combination of general fibres and C_b. We can thus write L as $p^*(M) \otimes \mathcal{O}_S(\Sigma n_b C_b)$ where M is a line bundle on $\varDelta$ with $\deg M = m$. Since $(L, H) = (m + \Sigma(n_b/m_b))(F, H)$, we have the desired result for the same reason as above. Therefore we may assume $p_*(D) = \varDelta$, hence $(F, D) \neq 0$. Since $0 = (D, K) = (2g - 2 + \Sigma(1 - 1/m_b))(D, F)$, we have $2g - 2 + \Sigma(1 - 1/m_b) = 0$. This equality holds only in $g = 1$ or 0. In case $g = 1$, i.e., $\varDelta$ is an elliptic curve, S has no singular fibre by this equality and hence K is numerically equivalent to $\mathcal{O}_S$. Thus L is numerically equivalent to the non-zero positive divisor D. Therefore we have the desired result. In case $g = 0$, i.e., $\varDelta$ is the projective line, we have $\Sigma(1 - 1/m_b) = 2$. Hence by Suwa's result [9], S can be expressed as an elliptic surface $p': S \to \varDelta'$ where $\varDelta'$ is an elliptic curve, and S has no singular fibre. We have already treated this case above.

3. A criterion for ampleness on an elliptic bundle

As a preparation for the next section we study in this section ampleness and cohomologies of a line bundle on an elliptic bundle.

DEFINITION (Suwa [9]). A non-singular projective surface S is said to be an *elliptic bundle* if there exists a proper, smooth morphism p of S onto a non-singular curve Δ such that the inverse image $p^{-1}(u) = C_u$ of any point $u \in \Delta$ is an elliptic curve.

Note that all the fibres are isomorphic to the same elliptic curve.

Let S be an elliptic bundle. Then we have the following exact sequence

$$0 \longrightarrow p^*(\Omega_\Delta^1) \longrightarrow \Omega_S^1 \longrightarrow \Omega_{S/\Delta}^1 \longrightarrow 0 .$$

On the other hand $\Omega_{S/\Delta}^1 = p^*(I)$ for some line bundle I on Δ since $\Omega_{S/\Delta}^1 \otimes \mathcal{O}_{C_u} \cong \Omega_{C_u}^1 \cong \mathcal{O}_{C_u}$ for any point $u \in \Delta$. Hence we have $\chi(\mathcal{O}_S) = 0$ since $c_2(\Omega_S^1) = (p^*\Omega_\Delta^1, \Omega_{S/\Delta}^1) = 0$, and $K_S = p^*(I \otimes K_\Delta)$ where K_Δ (resp. K_S) is the canonical line bundle of Δ (resp. S). It is well-known that $I = p_*(\Omega_{S/\Delta}^1)$ and $R^1p_*(\mathcal{O}_S)$ are dual to each other.

LEMMA (3.1) (Kodaira). *$I^{\otimes n} = \mathcal{O}_\Delta$ for some positive integer n. (In particular* $\deg I = 0$.)

Proof. By Riemann-Roch theorem, $p_*(ch(\mathcal{O}_S)\cdot T(S)) = ch(p\,!(\mathcal{O}_S))\cdot T(\Delta)$. Since $p\,!(\mathcal{O}_S) = \Sigma(-1)^i R^i p_*(\mathcal{O}_S) = \mathcal{O}_\Delta + I$, the right hand side is I. On the other hand, the left hand side is 0 since $T(S) = 1 + (1/2)p^*(I \otimes K_\Delta)$. Therefore $I^{\otimes n} = \mathcal{O}_S$ for some $n > 0$.

In case Δ is an elliptic curve, $K_S^{\otimes n} = \mathcal{O}_S$ for some $n > 0$. Indeed it is well-known that $n = 12$.

From now on S is an elliptic bundle defined over the complex number field.

PROPOSITION (3.2). *Let S be an elliptic bundle with a fibre C and let L be a line bundle on S. Then L is ample if and only if $(L^2) > 0$ and $(L, C) > 0$.*

Remark (3.3). Let D be an irreducible curve on S which is not a fibre. Then $(D^2) \geqq 0$. Indeed, $2g(D) - 2 \geqq (D, C)(2g(\Delta) - 2)$ by Hurwitz' theorem. On the other hand $2g(D) - 2 = (D + K, D) = (D^2) + (2g(\Delta) - 2)(D, C)$.

Remark (3.4). Let D be a positive divisor on S with $(D^2) > 0$. Then D is ample. Indeed D contains an irreducible curve which is not a fibre. Hence $(D, C) > 0$ and D is ample by Prop. (3.2).

Proof of Prop. (3.2). The necessity is clear. By Nakai's criterion for ampleness, it is enough for sufficiency to show that (L, D) is positive for any irreducible curve D different from the fibre on S. By Hodge index theorem we have (cf. [2], p. 208)

$$\begin{vmatrix} (L^2) & (L, D) & (L, C) \\ (L, D) & (D^2) & (D, C) \\ (L, C) & (D, C) & (C^2) \end{vmatrix} \geqq 0 .$$

Put $(L^2) = a$, $(L, C) = b$ and $(D, C) = n$. Here a, b and n are positive. Now $2nb(L, D) \geqq b^2(D^2) + an^2 \geqq an^2 > 0$. Therefore $(L, D) > 0$.

We now give results about the cohomology of a line bundle L on S. We have the following spectral sequence

$$H^j(\Delta, R^i p_*(L)) \Longrightarrow H^{i+j}(S, L) .$$

Since $R^i p_*(L) = 0$ for $i \geqq 2$,

$$H^0(S, L) = H^0(\Delta, p_*(L)) , \qquad H^2(S, L) = H^1(\Delta, R^1 p_*(L))$$

and $0 \to H^0(\Delta, R^1 p_*(L)) \to H^1(S, L) \to H^1(\Delta, p_*(L)) \to 0$ (exact). Put $h^i(L) = \dim_k H^i(S, L)$.

By Riemann-Roch theorem,

$$h^0(L) - h^1(L) + h^2(L) = \tfrac{1}{2}(L^2) - (g - 1)(L, C)$$

where g is the genus of Δ.

LEMMA (3.5). 1) *If* $(L, C) = n > 0$, *then* $R^1 p_*(L) = 0$ *and* $p_*(L)$ *is a locally free sheaf of rank* n. *Hence* $h^2(L) = 0$.

2) *If* $(L, C) = 0$, *then* $(L^2) \leqq 0$. *If* L *is, moreover, not isomorphic to* $p^*(M)$ *for any line bundle* M *on* Δ, *then* $p_*(L) = 0$, $h^0(L) = h^2(L) = 0$ *and* $h^1(L) = -(1/2)(L^2)$.

3) *If* $(L^2) < 0$, *then* $h^0(L) = 0$.

4) *If* $(L^2) > 0$, *then either* L *or* L^{-1} *is ample.*

5) *If* $(L, C) = n > 0$ *and* $(L^2) = 0$, *then* $h^0(L) \leqq n + 1$.

Proof. 1) By the base change theorem [5] p. 53, we have $R^1 p_*(L) \otimes k(u) \simeq H^1(C_u, L | C_u) = 0$ for any point $u \in \Delta$, hence $R^1 p_*(L) = 0$. By the base change theorem $p_*(L) \otimes k(u) \simeq H^0(C_u, L | C_u)$ for any point $u \in \Delta$. On the other hand $h^0(L | C_u) = n$ for any point $u \in \Delta$. Therefore $p_*(L)$ is locally free of rank n.

2) Put $Y = \{u \in \Delta : L|C_u$ is trivial on $C_u\}$ and $U = \Delta - Y$. Then Y is closed in Δ. Assume U is not empty. The proof of 1) shows that $R^1p_*(L) = 0$ on U, hence $p_*(L) \otimes k(u) \simeq H^0(C_u, L|C_u) = 0$ for any point $u \in U$. Thus $p_*(L) = 0$ on U. On the other hand $p_*(L)$ is torsion free since $p_*(L) \subset p_*(L \otimes H)$ for a very ample line bundle H on S and $p_*(L \otimes H)$ is locally free by 1). Therefore $p_*(L) = 0$ and so $h^0(L) = 0$. Now $h^2(L) = h^1(R^1p_*(L)) = 0$ since $\mathrm{Supp}\,(R^1p_*(L)) = Y$. If U is empty, i.e. $\Delta = Y$, then $L = p^*(M)$ for some line bundle M on Δ, hence $(L^2) = 0$.

3) It follows from Remark (3.3).

4) It follows from 2) and Prop. (3.2).

We omit the proof of 5) because we don't use this result in this paper.

We can get a little more detailed result for an elliptic bundle of a special type i.e. a hyperelliptic surface.

DEFINITION (Suwa [8]). A *hyperelliptic surface* is an elliptic bundle over an elliptic curve whose first Betti number is equal to 2.

THEOREM (Suwa [8]). *Any hyperelliptic surface S can be expressed as the quotient space of an abelian surface A by the group generated by an automorphism a of A. (Here $a^d = 1$ if and only if $K_S^{\otimes d} = \mathcal{O}_S$.)*

LEMMA (3.6). *Let S be a hyperelliptic surface and let L be a line bundle on S.*

1) *If L is ample, then $h^1(L) = h^2(L) = 0$.*

2) *If L^{-1} is ample, then $h^1(L) = h^0(L) = 0$.*

3) *If $(L^2) < 0$, then $h^0(L) = h^2(L) = 0$.*

4) *If $(L, C) = 0$, then $(L^2) = 0$. If moreover L is not isomorphic to $p^*(M)$ for any line bundle M on Δ, then $R^ip_*(L) = 0$ for any $i \geqq 0$.*

Proof. Let $f: A \to S$ be the natural morphism in the above theorem. Since f is finite,

$$H^i(A, f^*(L)) = H^i(S, f_*f^*(L)) = \bigoplus_{j=0}^{d-1} H^i(S, L \otimes K^{\otimes j}) \supset H^i(S, L)\,.$$

Hence 1), 2) and 3) are clear by the cohomology theorem about line bundles on abelian varieties. Let H be an ample line bundle on S. We now prove 4). L is homologous to $H^{\otimes r} \otimes \mathcal{O}_S(sC)$ where r and s are rational numbers, since the second Betti number is equal to 2. Hence $(L^2) = r^2(H^2) + 2rs(H, C) = r^2(H^2) \geqq 0$ since $0 = (L, C) = r(H, C)$. Hence we obtain $(L^2) = 0$ by Lemma (3.5) 2). Assume L is not isomorphic to $p^*(M)$ for any line bundle M on $\varDelta$. It follows from the proof of Lemma (3.5) 2) that $0 = -(1/2)(L^2) = h^1(L) = h^0(R^1p_*(L))$ is equal to the number of points contained in Y where $Y = \{u \in \varDelta : L|C_u$ is trivial on $C_u\}$. Hence Y is empty. q.e.d.

4. Stable vector bundles on an elliptic bundle

In [9] we studied H-stable vector bundles of rank two on geometrically ruled surfaces, $\boldsymbol{P}^2$, and abelian surfaces. In this section and the next section, we continue the study for elliptic bundles and especially hyperelliptic surfaces.

Let $p: S \to \varDelta$ be a proper, smooth morphism from a surface S onto a non-singular curve $\varDelta$, and let E be a stable vector bundle of rank two on $\varDelta$. We proved p^*E is H-stable for any ample line bundle H when the fibre is the projective line [9]. We now prove this when the fibre is an elliptic curve, i.e. S is an elliptic bundle. We do not know whether this holds when the genus of the fibre is more than 1.

PROPOSITION (4.1). *Let $p: S \to \varDelta$ be an elliptic bundle over the complex number field and E a stable vector bundle on $\varDelta$ of rank two. Then p^*E is H-stable for any ample line bundle H. (Here $N(p^*E) = 0$.)*

Proof. Looking closely at proofs of Prop. (2.1) and Lemma (2.2), we see that it is sufficient to show the following: for any morphism $f: T \to S$ obtained by successive dilatations and any quotient line bundle $f^*L \otimes M^{-1}$ of f^*p^*E (where L is a line bundle on S and M is a positive exceptional line bundle), there exists an ample line bundle H on S such that $(1/2)d(p^*E, H) < d(L, H)$. We may assume $\deg c_1(E) = m > 0$. Restricting the given exact sequence $f^*p^*E \to f^*L \otimes M^{-1} \to 0$ to C, we have the exact sequence $\mathcal{O}_C \oplus \mathcal{O}_C \to L \otimes \mathcal{O}_C \to 0$. Hence $(L, C) \geqq 0$. On the other hand, $0 = N(p^*E) = (L^{\otimes 2} \otimes p^*(c_1(E)^{-1}))^2 + 4(M^2)$, hence $(L^2) \geqq m(L, C)$, since $(M^2) \leqq 0$. If $(L, C) > 0$, then $(L^2) > 0$ and hence L is ample by Prop. (3.2). In this case we may assume $H = L$, hence $d(L, L) - (1/2)d(p^*E, L) = (L^2) - (1/2)m(L, C) \geqq (m/2)(L, C) > 0$ and we are done. If $(L, C) = 0$, then $(L^2) \leqq 0$ by Lemma (3.5) and hence $(L^2) = 0$. We thus have $(M^2) = 0$ and so $M = \mathcal{O}_S$. Therefore the above exact sequence is of the following form: $0 \to L_1 \to p^*E \to L \to 0$, where L_1 is a line bundle on S. If $L \neq p^*(L_2)$ for any line bundle L_2 on $\varDelta$, then the same property holds for L_1 and moreover $(L_1, C) = 0$. Thus $p_*(L) = p_*(L_1) = 0$ by Lemma

(3.5). Hence $p_*p^*E = E = 0$, a contradiction. So there exist line bundles L_2 and L_3 on Δ such that $p^*L_2 = L$ and $p^*L_3 = L_1$. And we have an exact sequence $0 \to p^*L_3 \to p^*E \to p^*L_2 \to 0$. By applying p_* we have an exact sequence $0 \to L_3 \to E \to L_2 \to 0$, hence $(1/2)\deg E < \deg L_2$ by the stability of E. Hence $(1/2)d(p^*E, H) < d(p^*L_2, H)$ for any ample line bundle H.

q.e.d.

PROPOSITION (4.2). *Let S be an elliptic bundle, and let E be an H-stable vector bundle of rank two on S for some ample line bundle H. Assume $c_1(E)$ is numerically equivalent to $p^*(L)$ for some line bundle L on Δ. Then $N(E) \leqq 0$.*

Proof. Suppose there exists E with $N(E) > 0$ which satisfies the above condition. We may assume $\deg L$ is sufficiently large. Since $4\chi(E) = N(E)$ and $d(E^* \otimes K, H) < 0$, we have $H^0(E) \neq 0$ by Lemma (2.1) of [9]. Thus there exist a morphism $f: T \to S$ obtained by successive dilatations and an extension of line bundles:

$$0 \longrightarrow f^*(\mathcal{O}_S(D)) \otimes M^{-1} \longrightarrow f^*E \longrightarrow f^*L_1 \otimes M^{-1} \longrightarrow 0$$

where M is a positive exceptional line bundle on T, D is a positive divisor on S and L_1 is a line bundle on S. Assume D contains an irreducible curve D_1 which is not a fibre. Since $N(E) > 0$, E is $p^*(L_2^{\otimes n}) \otimes H$-stable for any $n > 0$ where L_2 is a line bundle on Δ with $\deg L_2 = 1$. Hence we have inequalities

$$0 \leqq (D, p^*(L_2^{\otimes n}) \otimes H) < \tfrac{1}{2}d(E, p^*(L_2^{\otimes n}) \otimes H)\ ,$$

i.e. $0 \leqq n(D_1, C) + (D_1, H) < (1/2)d(E, H)$ for any $n > 0$. Hence $(D_1, C) = 0$, i.e. $D_1 = 0$ which contradicts our assumption. Therefore all irreducible components of D are fibres and hence $N(E) = 4(M^2) \leqq 0$. This contradicts $N(E) > 0$.

DEFINITION. An elliptic bundle $p: S \to \Delta$ is said to be *basic* if there exists a section o of p, i.e. $p \cdot o = 1_\Delta$.

The following Lemma is well-known.

LEMMA (4.3). *Let S be a basic elliptic bundle with a fibre C, and let L be a line bundle on S. Put $(L, C) = n$. Then there exist a section s of p and a line bundle M on Δ such that*

$$L \simeq p^*(M) \otimes \mathcal{O}_S(s(\Delta) + (n-1)o(\Delta))\ .$$

Proof. Tensoring $\mathcal{O}_S((-n+1)o(\Delta))$, we may assume $(L, C) = 1$, hence there exists a unique point P_u on C_u such that $L|_{C_u} \cong \mathcal{O}_{C_u}(P_u)$ for every $u \in \Delta$. We can take a section s such that $s(u) = P_u$. Since $L|_{C_u} \simeq \mathcal{O}_{C_u}(s(\Delta))$ for every point $u \in \Delta$, we have $L \cong \mathcal{O}_S(s(\Delta)) \otimes p^*(M)$ for some line bundle M.

PROPOSITION (4.4). *Let S be a basic elliptic bundle with a fibre C, and let E be a vector bundle of rank two on S with $(c_1(E), C)$ odd. If E is H-stable for some ample line bundle H on S, then $N(E) \leqq 4g$, where g is the genus of Δ.*

Proof. Note first that $N(E) =$ even. Indeed, $(s(\Delta), s(\Delta)) = 0$ for every section s of p. [3]. Hence for a line bundle $L, (L^2) = 2n \deg(M) + 2(n-1)(o(\Delta), s(\Delta))$ by Lemma (4.3). Suppose there exists a vector bundle E with $N(E) \geqq 4g+2$ which satisfies the above conditions. Since $(c_1(E^*) \otimes \mathcal{O}_S(no(\Delta)), C) = (c_1(E^*), C) + 2n$, we may assume $(c_1(E^*), C) = 1$. Thus $c_1(E)$ is numerically equivalent to $mC - s(\Delta)$ for some section s of p by Lemma (4.3). Since

$$2g - 2 + \tfrac{1}{2}N(E) - 4g + 3 \geqq 2$$

and

$$c_1(E \otimes \mathcal{O}_S(kC)) = c_1(E) \otimes \mathcal{O}_S(2kC) ,$$

we may assume $2g - 2 + (1/2)N(E) > m \geqq 4g - 3$. On the other hand, the Euler-Poincaré characteristic $\chi(E)$ of E is $-(1/2)m + g - 1 + (1/4)N(E) > 0$ and $(c_1(E^* \otimes K), \mathcal{O}_S(s(\Delta)) \otimes \mathcal{O}_S(C)) = 4g - 3 - m \leqq 0$. Thus $H^0(E) \neq 0$ by Lemma (2.1) of [9], since $\mathcal{O}_S(s(\Delta) + C)$ is ample. Hence there exist a morphism $f: T \to S$ obtained by successive dilatations and an extension of line bundles on T

$$0 \longrightarrow f^*(\mathcal{O}_S(D)) \otimes M \longrightarrow f^*E \longrightarrow f^*(L) \otimes M^{-1} \longrightarrow 0$$

where M is a positive exceptional line bundle on T, D is a positive divisor on S and L is a line bundle on S. By the H-stability of E, we have inequalities

$$0 \leqq (D, nC + s(\Delta)) < \tfrac{1}{2}(mC - s(\Delta), nC + s(\Delta))$$

i.e. $0 < (1/2)(-n + m)$ for any $n > 0$, a contradiction.

5. Stable vector bundles on a hyperelliptic surface

PROPOSITION (5.1). *Let S be a non-singular projective surface over the complex number field with $K_S^{\otimes n} = \mathcal{O}_S$ for some positive integer n where K_S is the canonical line bundle on S, and let E be a simple vector bundle of rank two on S. Then $N(E) = c_1^2(E) - 4c_2(E) \leqq 0$.*

Proof. By Riemann-Roch theorem, $N(E) + 4\chi(\mathcal{O}_S) = h^0(\mathrm{End}(E)) - h^1(\mathrm{End}(E)) + h^2(\mathrm{End}(E)) \leqq 1 - h^1(\mathcal{O}_S) + h^0(\mathrm{Hom}(E, E \otimes K_S))$ since the canonical injection $\mathcal{O}_S \to \mathrm{End}(E)$ splits. On the other hand, by the classification theory of surfaces, S is one of the following:

(1) Enriques surface $K_S^{\otimes 2} = \mathcal{O}_S$

(2) regular surface with $K_S = \mathcal{O}_S$

(3) two-dimensional abelian variety

(4) hyperelliptic surface.

Hence Prop. (5.1) is clear in cases (2) and (3). If E is H-stable for some ample line bundle H on S in case (1), $h^0(\mathrm{Hom}(E, E \otimes K_S)) \leqq 1$ by Prop. (1.2). And if E is not H-stable and simple, we may assume there exist a morphism $f: T \to S$ obtained by successive dilatations and an extension of line bundles on $T: 0 \to M \to f^*E \to M^{-1} \otimes f^*L \to 0$ where L is a line bundle with $(L, H) \leqq 0$ and M is a positive exceptional line bundle on T. If $H^0(\mathrm{Hom}(E, E \otimes K_S)) \neq 0$, then there exists a non-zero homomorphism $x: E \to E \otimes K_S$. Hence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{y} & f^*E & \longrightarrow & M^{-1} \otimes f^*L & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle x \neq 0} & & & & \\ 0 & \longrightarrow & M \otimes f^*K_S & \longrightarrow & f^*(E \otimes K_S) & \xrightarrow[z]{} & M^{-1} \otimes f^*(L \otimes K_S) & \longrightarrow & 0 \,. \end{array}$$

If $zxy = 0$, then the natural homomorphism $M^{-1} \otimes f^*L \to M^{-1} \otimes f^*(L \otimes K_S)$ is non-zero and hence $H^0(K_S) \neq 0$ which contradicts our assumption. Therefore $zxy \neq 0$ and so $0 \neq H^0(M^{-2} \otimes f^*(L \otimes K_S)) \subset H^0(L \otimes K_S)$. Thus $L \otimes K_S = \mathcal{O}_S$ and hence $M = \mathcal{O}_S$ since $(L \otimes K_S, H) \leqq 0$. The above extension is thus of the following form: $0 \to \mathcal{O}_S \to E \to K_S^{-1} \to 0$ a contradiction, since $H^1(K_S) = 0$. If E is H-stable for some ample line bundle H on S in case (4), then $h^0(\mathrm{Hom}(E, E \otimes K_S)) \leqq 1$, hence $N(E) \leqq 0$ because $N(E)$ is even. Therefore it is sufficient to show the following:

PROPOSITION (5.2). *Let S be a hyperelliptic surface, and let E be a vector bundle of rank two on S with $N(E) \geqq 0$. Then E is simple if and* *only if E is either H-stable for some ample line bundle H or of the form $E_0 \otimes L$, where we have a non-trivial extension $0 \to \mathcal{O}_S \to E_0 \to K_S^{-1} \to 0$.*

Remark. Such E_0 is unique up to isomorphism since $H^1(K_S) = \boldsymbol{C}$.

Proof. E_0 is simple by Oda's lemma. Assume E is simple but not H-stable for some ample line bundle H. Tensoring a suitable line bundle on S, we may assume there exist a morphism $f: T \to S$ and an extension of line bundles on $T: 0 \to M \to f^*E \to f^*L \otimes M^{-1} \to 0$ where L is a line bundle on S with $(L, H) \leqq 0$, $h^0(L) = h^0(L^{-1}) = 0$ and M is a positive exceptional line bundle on T by Prop. (2.9) of [9]. Since $N(E) \geqq 0$, we have $(L^2) \geqq 0$. But $(L^2) = 0$ since $(L, H) \leqq 0$ and $h^0(L^{-1}) = 0$. Hence $M = \mathcal{O}_S$ and the above extension is of the following form: $0 \to \mathcal{O}_S \to E \to L \to 0$. Since $0 = \chi(L^{-1}) = -h^1(L^{-1}) + h^2(L^{-1})$ and E is simple, we have $0 \neq h^1(L^{-1}) = h^2(L^{-1}) = h^0(L \otimes K_S)$. Thus $L \otimes K_S \simeq \mathcal{O}_S$ because $(L \otimes K_S, H) \leqq 0$. q.e.d.

From now on we assume S is a hyperelliptic surface and E is a vector bundle on S of rank two. On S one of the following holds [9]:

(I) $K_S^{\otimes 2} = \mathcal{O}_S$ $(K_S \neq \mathcal{O}_S)$	(II) $K_S^{\otimes 3} = \mathcal{O}_S$ $(K_S \neq \mathcal{O}_S)$
(III) $K_S^{\otimes 4} = \mathcal{O}_S$ $(K_S^{\otimes 2} \neq \mathcal{O}_S)$	(IV) $K_S^{\otimes 6} = \mathcal{O}_S$ $(K_S^{\otimes 2}, K_S^{\otimes 3} \neq \mathcal{O}_S)$

PROPOSITION (5.3). *Assume $K_S^{\otimes 4} = \mathcal{O}_S$ (i.e. case (I) or (III)). If E is stable with $N(E) = 0$, then there is a non-trivial line bundle J such that E is isomorphic to $E \otimes J$.*

Proof. Put $\operatorname{End}(E) = \mathcal{O}_S \oplus E_1$. By Suwa's theorem which we mentioned in § 3, there is an unramified covering $f: A \to S$ of degree d, where $d = 2$ in case (I), and $d = 4$ in case (III), and A is an abelian surface. Since $h^0(\operatorname{End}(f^*E)) = \sum_{i=0}^{d-1} h^0(\operatorname{Hom}(E, E \otimes K_S^{\otimes i}))$, we may assume f^*E is simple because f^*E is not simple if and only if E is isomorphic to $E \otimes K_S^{\otimes j}$, where $j = 1$ in case (I) and $j = 2$ in case (III). By Morikawa and Oda [6] there exist an isogeny $g: B \to A$ and a line bundle L on B such that $g_*L = f^*E$. Hence $f^*E_1 = L_1 \oplus L_2 \oplus L_3$ for some line bundles L_i on A $(i = 1, 2, 3)$ because $\operatorname{End}(f^*E) = \operatorname{End}(g_*L) = \oplus L'$ where L' runs over all the line bundles L' on A such that $g^*L' = T_a^*L \otimes L^{-1}$ for some point a in $\ker(g)$ [6]. Hence

$$f_*f^*E_1 = \bigoplus_{i=0}^{d-1} E_1 \otimes K_S^{\otimes i} = f_*L_1 \oplus f_*L_2 \oplus f_*L_3$$

and the rank of f_*L_i is d. Thus E_1 is decomposable, i.e. $E_1 = J \oplus E_2$ for some line bundle J on S, which is non-trivial since E is simple. Hence E is isomorphic to $E \otimes J$ by the last Remark in section 1.

From now on we assume S is a basic hyperelliptic surface with the typical fibre C. Fix a global section o of p. For every point $u \in \Delta$ $p^{-1}(u) = C_u$ is an elliptic curve with the zero point which is the intersection point of C_u and $o(\Delta)$, and C_u is isomorphic to the same curve C. Then there exists an isogeny $\Delta' \to \Delta$ such that the fibre product $\Delta' \times_\Delta S$ is isomorphic to $\Delta' \times C$ [9], i.e. the diagram is cartesian

$$\begin{array}{ccc} \Delta' \times C & \longrightarrow & S \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ \Delta' & \longrightarrow & \Delta\,. \end{array}$$

Let $H^0(\Delta, S)$ be the set of all sections of p. Now we have a homomorphism of abelian groups $f: H^0(\Delta, S) \to H^0(\Delta', \Delta' \times C)$. On the other hand, for any section s different from 0, $s(\Delta)$ is numerically equivalent to $o(\Delta)$ since the base number of S is 2, hence the intersection of $s(\Delta)$ and $o(\Delta)$ is empty. Thus the image of $\Delta' \xrightarrow{s'} \Delta' \times C \xrightarrow{pr_2} C$ does not contain the zero point of C, and it defines one point of C. Hence f factors: $H^0(\Delta, S) \to C(k) \subset H^0(\Delta', \Delta' \times C)$. By means of Suwa's result [9] and the concrete representation of S by Kodaira [3], we can calculate $H^0(\Delta, S)$ easily.

LEMMA (5.4).
$$H^0(\Delta, S) = \begin{cases} Z/2Z + Z/2Z & \textit{in case I} \\ Z/3Z & \textit{in case II} \\ Z/2Z & \textit{in case III} \\ 0 & \textit{in case IV}\,. \end{cases}$$

By Lemma (4.3) and Lemma (5.4), we get the following:

PROPOSITION (5.5). *If S is a basic hyperelliptic surface with $K_S^{\otimes 4} \neq \mathcal{O}_S$, then we have a canonical isomorphism* $\mathrm{Pic}(\Delta)_2 \simeq \mathrm{Pic}(S)_2$.

Here we define $\mathrm{Pic}(V)_2 = \{L \in \mathrm{Pic}(V) : L^{\otimes 2} = \mathcal{O}_V\}$ for a variety V. And we define $\mathrm{Pic}'(S/\Delta) = \{L \in \mathrm{Pic}(S) : o^*(L) \simeq \mathcal{O}_\Delta\}$, where o is a global section of p. It is well-known that we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\Delta) \longrightarrow \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}'(S/\Delta) \longrightarrow 0\,.$$

By Lemma (4.3), we can express $\mathrm{Pic}'(S/\Delta)$ by $H^0(\Delta, S)$ and $I^{-1} = R^1p_*(\mathcal{O}_S)$

$= \mathcal{O}_{o(\Delta)}(o(\Delta))$. Let $\tilde{\Delta}$ be an unramified covering of Δ. Put $\tilde{S} =$ the fibre product $S \times_{\Delta} \tilde{\Delta}$. If S and $\tilde{S}$ are in the same class, then $\mathrm{Pic}'(S/\Delta) \simeq \mathrm{Pic}'(\tilde{S}/\tilde{\Delta})$ by Lemma (5.4).

Remark. Let $f: T \to S$ be an unramified covering of degree two. Then T and S are in the same class if and only if $J \not\simeq K_S^{\otimes i}$ where $f_*(\mathcal{O}_T) = \mathcal{O}_S \oplus J$.

PROPOSITION (5.6). *Let S be a basic hyperelliptic surface with $K_S^{\otimes 4} \neq \mathcal{O}_S$ and E a stable vector bundle of rank two on S with $N(E) = 0$. If E is isomorphic to $E \otimes J$ for some non-trivial line bundle J with $J \not\simeq K_S^{\otimes i}$, then there exist a stable vector bundle F on Δ and a line bundle M on S such that E is isomorphic to $p^*(F) \otimes M$.*

Remark. 1) We have $J^{\otimes 2} = \mathcal{O}_S$ since $r(E) = 2$, hence the above condition $J \not\simeq K_S^{\otimes i}$ always holds in case II and is equivalent to $J \not\simeq K_S^{\otimes 2}$ in case IV. 2) A hyperelliptic surface in case IV is always basic [9].

Proof. Since J is contained in $\mathrm{Pic}(S)_2$, so $J = p^*(J_1)$ for some $J_1 \in \mathrm{Pic}(\Delta)_2$ by Prop. (5.5). Put $\Delta' = \mathrm{Spec}(\mathcal{O}_\Delta \oplus J_1)$. Then $g: \Delta' \to \Delta$ is an unramified covering of degree two and so is $f: S' = \mathrm{Spec}(\mathcal{O}_S \oplus J) \to S$. We have the cartesian diagram

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta' & \xrightarrow{g} & \Delta \end{array} .$$

By Cor. (1.12), there exists a line bundle L_1 on S' such that $E = f_*(L_1)$. On the other hand, by our assumption, S and S' are in the same class. Hence we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(\Delta) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}'(S/\Delta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| \wr & & \\ 0 & \longrightarrow & \mathrm{Pic}(\Delta') & \longrightarrow & \mathrm{Pic}(S') & \longrightarrow & \mathrm{Pic}'(S'/\Delta') & \longrightarrow & 0 \end{array} .$$

Therefore $L_1 = f^*(L_2) \otimes p'^*(L_3)$ for some line bundle L_2 on S and L_3 on Δ. Hence $E \otimes L_2^{-1} = f_* p'^*(L_3) = p^* g_*(L_3)$, and $g_*(L_3)$ is stable on Δ by Cor. (1.8). Thus we have the desired result.

EXAMPLE (5.7). There is an H-stable vector bundle E of rank two on a basic hyperelliptic surface with $N(E) = 0$ which is not of the form in Prop. (5.6). Let S be a basic hyperelliptic surface with a section o of p and the fibre C. Put $H = \mathcal{O}_S(o(\Delta)) \otimes p^*(M)$ and $L = \mathcal{O}_S(o(\Delta)) \otimes p^*(I)$, where M is a line bundle on Δ with $\deg M = 1$ and $I = R^1p_*(\mathcal{O}_S)$. We have an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow p^*(I^{-1}) \longrightarrow \mathcal{O}_{o(\Delta)} \otimes p^*(I^{-1}) = I^{-1} \longrightarrow 0 .$$

Since $(L, C) = 1$ and hence $p_*(L^{-1}) = 0$ by the base change theorem, we have

$$0 \longrightarrow I^{-1} \longrightarrow I^{-1} \longrightarrow R^1p_*(L^{-1}) \longrightarrow R^1p_*(p^*(I^{-1})) \longrightarrow 0 .$$

Thus $R^1p_*(L^{-1}) = \mathcal{O}_\Delta$ since $R^1p_*p^*I^{-1} = R^1p_*(\mathcal{O}_S) \otimes I^{-1} = \mathcal{O}_\Delta$. Hence $H^1(L^{-1}) = H^0(R^1p_*L^{-1}) = H^0(\mathcal{O}_\Delta) = \boldsymbol{C}$ by Leray spectral sequence. And we have a non-trivial extension: $0 \to \mathcal{O}_S \to E \to L \to 0$. Setting $H = \mathcal{O}_S(o(\Delta) + C)$, we see that E is H-stable with $N(E) = 0$ since $(c_1(E), H) = 1$ by Lemma (3.8) of [10], and $c_1(E) = L$ is numerically equivalent to $o(\Delta)$. Thus E is not isomorphic to $p^*(F) \otimes N$ where F is stable on Δ and N is a line bundle on S.

Let S be a hyperelliptic surface. Let $A(S)$ be the set of stable vector bundles E of rank two on S with $N(E) = 0$. Let $B(S)$ be the set of simple vector bundles of rank two which can be expressed as $f_*(L)$ where $f: T \to S$ is an unramified covering, L is a line bundle on T, and a^*L is numerically equivalent to L but not isomorphic to L. Here a is an automorphism of T and $S = T/\langle a \rangle$. Let $C(S)$ be the set of vector bundles on S which can be expressed as $p^*F \otimes M$ where F is a stable vector bundle of rank two on Δ and M is a line bundle on S.

PROPOSITION (5.8). i) $A(S) \supset B(S) \supset C(S)$.

ii) *If S is basic, then* $A(S) \neq C(S)$.

iii) *In cases* I *and* III, *we have* $A(S) = B(S)$.

iv) *In case* II *and S is basic, we have* $B(S) = C(S)$.

v) *In case* IV, $B(S) \cap \{E : E \neq E \otimes K_S^{\otimes 3}\} = C(S) \cap \{E : E \otimes K_S^{\otimes 3} \neq E\}$ *and* $B(S) \neq C(S)$.

Proof. i) The first inclusion follows from Cor. (1.8). The second inclusion follows from either Oda's result [5], or Atiyah's result [1] and Lemma (1.12). ii) follows from Example (5.7). iii) follows from Prop. (5.3) and Cor. (1.13). iv) follows from Prop. (5.6). v) The first statement follows from Prop. (5.6). Put $S' = \mathrm{Spec}\,(\mathcal{O}_S \oplus K_S^{\otimes 3})$. Then we have the cartesian diagram

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta' & \longrightarrow & \Delta \,. \end{array}$$

Let L be $\mathcal{O}_{S'}(s'(\Delta') - o'(\Delta'))$ such that $o' \neq s' \in H^0(\Delta', S')$. We see that f_*L is contained in $B(S)$, but is not contained in $C(S)$. Indeed, if $f_*L \in C(S)$, then $f_*L = p^*(E) \otimes L_1$ where E is a stable vector bundle on Δ and L_1 is a line bundle on S. Since $p^*(I_\Delta) \cong K_S$ and $f_*L \otimes K_S^{\otimes 3} \simeq f_*L$, we have $E \otimes I_\Delta^{\otimes 3} \simeq E$, hence we see easily that $L \simeq p'^*(L_2) \otimes f^*(L_1)$, a contradiction. It is clear that $f_*L \in B(S)$ since L is numerically equivalent to 0.

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