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Nagoya Math. J.
Vol. 98 (1985), 139-156

RELATIVE INVARIANTS AND b-FUNCTIONS OF PREHOMOGENEOUS VECTOR SPACES

$(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$

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Introduction

Let G be a connected linear algebraic group, ρ a rational representation of G on a finite-dimensional vector space V, all defined over C.

A polynomial $f(x)$ on V is called a relative invariant, if there exists a rational character $\chi : G \to C^{\times}$ satisfying

$$f(\rho(g)\cdot x) = \chi(g)f(x)\,, \qquad \text{for any } g \in G \text{ and } x \in V.$$

The triplet (G, ρ, V) is called a prehomogeneous vector space (abbrev. P.V.), if there exists a proper algebraic subset S of V such that $V - S$ is a single G-orbit. The algebraic set S is called the singular set of (G, ρ, V) and any point in $V - S$ is called a generic point of (G, ρ, V).

Let $GL(d_1, \cdots, d_r)$ be a parabolic subgroup of the general linear group $GL(n, C)$ defined by (1.1) in Section 1, $\rho : G \to GL(n, C)$ be an n-dimensional representation of G. In this paper, we shall be concerned with the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$, where $\tilde{\rho}_1$ is defined by

$$\rho_1(g, a)x = \rho(g)xa^{-1} \quad ((g, a) \in G \times GL(d_1, \cdots, d_r),\ x \in M(n, C))\,.$$

Assume that $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$ is a P.V. We shall introduce the b-function of $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$, after M. Sato, in Section 3. Theorem 3.1 gives an explicit form of the b-function. In Section 4, we shall be concerned with triplets $\{(G \times B_n, \tilde{\rho}_1, M(n, C))\}$ where G is a semi-simple connected linear algebraic group, B_n is the upper triangular group and ρ is an irreducible representation on an n-dimensional vector space V. We shall determine all prehomogeneous vector space $\{(G \times B_n, \tilde{\rho}_1, M(n, C)\}$, and construct their relative invariants.

Received May 7, 1984.

NOTATIONS. C, $C^{\times}$ and Z_+ are the complex number field, the group of non-zero complex numbers and the set of non-negative integers, respectively. $GL(n, C)$, B_n and B_n^- are the complex general linear group, the complex upper triangular group and the complex lower triangular group.

§ 1. A generalization of castling transform

Let G be a connected linear algebraic group, V an m-dimensional vector space, and ρ a rational representation of G on V, all defined over the complex number field C. By choosing a basis of V, we may identify V with C^m. Let $d_1, \cdots, d_r$ be positive integers and set

$$n = d_1 + \cdots + d_r \text{ and } d^{(i)} = d_1 + \cdots + d_i \quad (1 \leqq i \leqq r).$$

We denote by $GL(d_1, \cdots, d_r)$ the parabolic subgroup of the general linear group $GL(n, C)$ consists of all matrices of the form

$$g = \begin{bmatrix} g_{11} & g_{12} \cdots g_{1r} \\ 0 & g_{22} \cdots g_{2r} \\ \vdots & \vdots \\ 0 & 0 \quad \cdots g_{rr} \end{bmatrix} \tag{1.1}$$

where $g_{ii} \in GL(d_i, C)$ $(1 \leqq i \leqq r)$.

We may identify the vector space $\overset{n}{\oplus} V$ with the vector space $M(m, n, C)$ consists of all m by n matrices and identify the vector space $M(m, n, C)$ with it's dual vector space by the inner product

$$(x, y) = \operatorname{Tr} {}^t y \cdot x \quad (x, y \in M(m, n, C)).$$

Let $\tilde{\rho}_1$, $\tilde{\rho}_2$, $\tilde{\rho}_1^*$ and $\tilde{\rho}_2^*$ denote representations of $G \times GL(d_1, \cdots, d_r)$ on $M(m, n, C)$ defined as follows:

$$\begin{aligned} \tilde{\rho}_1(g, a)x &= \rho(g)xa^{-1} \\ \tilde{\rho}_2(g, a)x &= \rho(g)x^t a \\ \tilde{\rho}_1^*(g, a)x &= {}^t\rho(g)^{-1}x^t a \\ \tilde{\rho}_2^*(g, a)x &= {}^t\rho(g)^{-1}xa^{-1} \end{aligned}$$

where $g \in G$ and $a \in GL(d_1, \cdots, d_r)$.

LEMMA 1.1. *The following conditions are equivalent.*

(i) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ *is a P.V.*

(ii) *The triplet* $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2, M(m, n, C))$ *is a P.V.*

Proof. Put $A = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{1}} & \\ 1 & & \end{bmatrix} \in GL(n, C)$, then it is easy to check that $x_0 \in M(m, n, C)$ is a generic point of the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ if and only if $x_0 \cdot A$ is a generic point of $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_2, M(m, n, C))$. Q.E.D.

LEMMA 1.2. *There exists a one-to-one correspondence between relative invariants of* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ *and* $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2, M(m, n, C))$.

Proof. For a polynomial $f(x)$ on $M(m, n, C)$, define the polynomial $\Phi(f)$ by

$$\Phi(f)(x) = f(x \cdot A)\,. \tag{1.2}$$

Then the mapping $f \mapsto \Phi(f)$ gives a one-to-one correspondence between relative invariants. Q.E.D.

LEMMA 1.3. *When* $m > n$, *the following conditions are equivalent.*

(i) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ *is a P.V.*

(ii) *The triplet* $(G \times GL(m - n, d_r, \cdots, d_2), \tilde{\rho}_2^*, M(m, m - d_1, C))$ *is a P.V.*

Proof. For a matrix x in $M(m, n, C)$, denote by x^i the i-th column vector of x $(1 \leqq i \leqq n)$. Let W denote an algebraic variety whose points are matrices x in $M(m, n, C)$ such that column vectors $x^1, x^2, \cdots$ and x^n are linearly independent. Then the group $G \times GL(d_1, \cdots, d_r)$ acts on W, and $(G \times GL(d_1, \cdots, d_r), W)$ has an open orbit if and only if the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ is a P.V., since the Zariski closure of W is $M(m, n, C)$. Let $\mathrm{Flag}(d_1, \cdots, d_r)$ be the flag variety defined by

$$\mathrm{Flag}(d_1, \cdots, d_r) = \left\{(V_1, V_2, \cdots, V_r);\ \begin{array}{l} V_i \in \mathrm{Grass}_{d_1+\cdots+d_i}(C^m) \text{ and} \\ V_1 \subset V_2 \subset \cdots \subset V_r \end{array}\right\}$$

where $\mathrm{Grass}_d(C^m)$ is the Grassmann variety consists of all d-dimensional subspaces of C^m.

For a matrix x in the variety W, let $\mu(x)$ denote the flag$(V_1, \cdots, V_n)$ in $\mathrm{Flag}(d_1, \cdots, d_r)$ such that V_i is the subspace of C^m spanned by the first $d_1 + \cdots + d_r$ column vectors of the matrix x $(1 \leqq i \leqq r)$. Then the mapping μ: $W \mapsto \mathrm{Flag}(d_1, \cdots, d_r)$ is surjective, $G \times GL(d_1, \cdots, d_r)$ equivalent morphism. Since $GL(d_1, \cdots, d_r)$ acts on $\mathrm{Flag}(d_1, \cdots, d_r)$ trivially and it acts on each fibre homogeneously, the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ is a P.V. if and only if $\mathrm{Flag}(d_1, \cdots, d_r)$ is G-prehomogeneous.

For a flag$(V_1, V_2, \cdots, V_r)$ in $\mathrm{Flag}(d_1, \cdots, d_r)$, let $(\tilde{V}_1, \tilde{V}_2, \cdots, \tilde{V}_r)$ be the flag in $\mathrm{Flag}(m-n, d_r, \cdots, d_2)$ defined by

$$\tilde{V}_i = \{y \in M(m, n, C) | (y, x) = 0 \text{ for any } x \text{ in } V_{r-i+1}\} .$$

Then G acts on the flag variety $\mathrm{Flag}(m-n, d_r, \cdots, d_2)$ contragrediently and $\mathrm{Flag}(m-n, d_r, \cdots, d_2)$ is G-prehomogeneous if and only if $\mathrm{Flag}(d_1, \cdots, d_r)$ is G-prehomogeneous.

Since the triplet $(G \times GL(m-n, d_r, \cdots, d_2), \tilde{\rho}_2^*, M(m, m-d_1, C))$ is a P.V. if and only if the flag variety $\mathrm{Flag}(m-n, d_r, \cdots, d_2)$ is G-prehomogeneous, we obtain our assertion. Q.E.D.

Remark. This construction is a natural generalization of the castling transform in the theory of prehomogeneous vector space [2].

LEMMA 1.4. *When $m > n$, there is a one-to-one correspondence between relative invariants of the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ and relative invariants of the triplet $(G \times GL(m-n, d_r, \cdots, d_2), \tilde{\rho}_2, M(m, m-d_1, C))$.*

Proof. Let $f(x^1, \cdots, x^n)$ be a relative invariant of the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$, where x^i is the i-th column vector of an $m \times n$ matrix x $(1 \leqq i \leqq n)$. For $x = (x^1, \cdots, x^n) \in M(m, n, C)$, put

$$X_{i_1 \cdots i_k} = \det\begin{pmatrix} x^1_{i_1} \cdots x^k_{i_1} \\ \vdots \quad\quad \vdots \\ x^1_{i_k} \cdots x^k_{i_k} \end{pmatrix} \quad (1 \leqq k \leqq n \text{ and } 1 \leqq i_1 < \cdots < i_k \leqq m) .$$

Then by the first main theorem for the group $GL(d_1, \cdots, d_r)$, there exists a polynomial F satisfying

$$\begin{gathered} f(x^1, \cdots, x^n) = F(X_{i_1 \cdots i_{d_1}}, X_{j_1 \cdots j_{d_1+d_2}}, \cdots, X_{k_1 \cdots k_{d_1+\cdots+d_r}}) \\ (1 \leqq i_1, \cdots, i_{d_1}, j_1, \cdots, j_{d_1+d_2}, k_1, \cdots, k_{d_1+\cdots+d_r} \leqq m) , \end{gathered}$$

since $f(x)$ is a relative invariant of the group $GL(d_1, \cdots, d_r)$.

For $x = (x^1, \cdots, x^n)$ in $M(m, n, C)$, let $\omega_k = x^1 \wedge \cdots \wedge x^k$ and, for $\tilde{x} = (\tilde{x}^1, \cdots, \tilde{x}^{m-d_1})$ in $M(m, m-d_1, C)$, let $\tilde{\omega}_k = \tilde{x}^1 \wedge \cdots \wedge \tilde{\omega}^{m-k}$ where $k \in \{d_1, d_1+d_2, \cdots, d_1 + \cdots + d_r\}$.

Then it follows that

$$\omega_k = \sum_{i_1<\cdots<i_k} X_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k},$$

$$\tilde{\omega}_k = \sum_{j_1<\cdots<j_{m-k}} \tilde{X}_{j_1\cdots j_{m-k}} e_{j_1} \wedge \cdots \wedge e_{j_{m-k}} \ (e_i = {}^t(0\cdots \overset{i}{1} \cdots 0),\ 1 \leqq i \leqq m),$$

and we have

$$\sum_{i_1<\cdots<i_k} \operatorname{sgn}\begin{pmatrix} 1\cdots k, k+1\cdots m \\ i_1\cdots i_k, j_1\cdots j_{m-k} \end{pmatrix} X_{i_1\cdots i_k} \tilde{X}_{j_1\cdots j_{m-k}} = \det(x^1, \cdots x^k, \tilde{x}^1, \cdots \tilde{x}^{m-k})$$

where

$$\operatorname{sgn}\begin{pmatrix} 1\cdots k, k+1\cdots m \\ i_1\cdots i_k, j_1\cdots j_{m-k} \end{pmatrix}$$

denotes the signature of the permutation $\begin{pmatrix} 1\cdots k, k+1\cdots m \\ i_1\cdots i_k, j_1\cdots j_{m-k} \end{pmatrix}$ if

$$\{i_1\cdots i_k, j_1\cdots j_{m-k}\} = \{1, 2\cdots m\}$$

and zero, if otherwise.

Thus if we put

$$X'_{i_1\cdots i_k} = \operatorname{sgn}\begin{pmatrix} 1\cdots k, k+1\cdots m \\ i_1\cdots i_k, j_1\cdots j_{m-k} \end{pmatrix} \tilde{X}_{j_1\cdots j_{m-k}},$$

we have

$$\sum_{i_1<\cdots<i_k} X_{i_1\cdots i_k} X'_{i_1\cdots i_k} = \det(x^1, \cdots, x^k, \tilde{x}^1, \cdots, \tilde{x}^{m-k}).$$

We define a polynomial $\tilde{f}(\tilde{x})$ on $M(m, m-d_1, C)$ by

$$\tilde{f}(\tilde{x}) = F(X'_{i_1\cdots i_{d_1}} X'_{j_1\cdots j_{d_1+d_2}} \cdots X'_{k_1\cdots k_{d+\cdots+d_r}}).$$

Then $\tilde{f}$ is a relative invariant of the triplet $(G \times GL(m-n, d_r \cdots d_2), \tilde{\rho}_2^*, M(m, m-d_1. C))$, and the mapping $f \mapsto \tilde{f}$ gives a one-to-one correspondence between relative invariants of them. Q.E.D.

Remark. From the construction, f is irreducible if and only if $\tilde{f}$ is irreducible.

By Lemma 1.1~1.4, we have the following proposition.

PROPOSITION 1.1. *When $m>n$, the following* 4 *conditions are equivalent and there are one-to-one correspondences among relative invariants of them.*

(1) *The triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ is a P.V.*

(2) *The triplet $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2, M(m, n, C))$ is a P.V.*

(3) *The triplet* $(G \times GL(d_2, \cdots, d_r, m-n), \tilde{\rho}_1^*, M(m, m-d_1, C)$ *is a P.V.*

(4) *The triplet* $(G \times GL(m-n, d_r, \cdots, d_2), \tilde{\rho}_2^*, M(m, m-d_1, C))$ *is a P.V.*

The following two propositions are shown in a similar manner.

PROPOSITION 1.2. *Let G be a connected linear algebraic group and ρ a linear representation of G on an n-dimensional vector space. Then the following* 4 *conditions are equivalent and there are one-to-one correspondences among their relative invariants*

(1) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$ *is a P.V.*

(2) *The triplet* $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2, M(n, C))$ *is a P.V.*

(3) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1^*, M(n, C))$ *is a P.V.*

(4) *The triplet* $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2^*, M(n, C))$ *is a P.V.*

PROPOSITION 1.3. *When $m > n$, the following* 4 *conditions are equivalent.*

(1) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ *is a P.V.*

(2) *The triplet* $(G \times GL(d_r, \cdots, d_1), \tilde{\rho}_2, M(m, n, C))$ *is a P.V.*

(3) *The triplet* $(G \times GL(d_1, \cdots, d_r, m-n), \tilde{\rho}_1^*, M(m, C))$ *is a P.V.*

(4) *The triplet* $(G \times GL(m-n, d_r, \cdots, d_1), \tilde{\rho}_2^*, M(m, C))$ *is a P.V.*

COROLLARY. *When $m > n$, the following conditions are equivalent.*

(1) *The triplet* $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(m, n, C))$ *is a P.V.*

(2) *The triplet* $(G \times GL(d_1, \cdots, d_r, m-n), \tilde{\rho}_1, M(m, C)$ *is a P.V.*

Let G be a connected linear algebraic group, ρ a representation on an n-dimensional vector space V. Then, by Proposition 1.1, triplet $(G \times GL(1), \rho \otimes \square, V)$ is a P.V. if and only if $(G \times GL(1, n-1), \tilde{\rho}_1, M(n, C))$ is a P.V. We shall devote ourselves to investigate triples $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$.

§ 2. Relative invariants

A sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of non-negative integers in decreasing order;

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

is called a partition, and the sum $|\lambda| = \lambda_1 + \cdots + \lambda_n$ is called the weight of λ.

For a partition λ, we denote by V_λ the vector space consists of all polynomials $f(x)$ on the vector space $M(n, C)$ such that $f(x)$ satisfies; for any matrix t in the group B_n,

$$f(x\cdot t) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n} f(x)$$

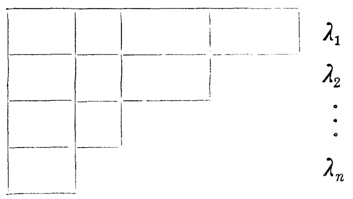
where

$$t = \begin{pmatrix} t_1 & & * \\ & \ddots & \\ & & t_n \end{pmatrix} \in B_n .$$

Let $f(x)$ be a polynomial on $M(n, C)$, and for any element g in $GL(n, C)$, set;

$$g\cdot f(x) = f(g^{-1}x) .$$

Then by the mapping $f \mapsto g\cdot f$, the vector space V_λ can be considered as a $GL(n, C)$-module. As is well known, V_λ is a irreducible $GL(n, C)$-module corresponding to the Young diagram $Y(\lambda)$:



We set:

$$X'_{i_1\cdots i_d} = \operatorname{sgn}\begin{pmatrix} 1, \cdots, d, d+1, \cdots, n \\ i_1, \cdots, i_d, j_1, \cdots, j_{n-d} \end{pmatrix} \det\begin{pmatrix} X_{j_1}^{d+1} \cdots X_{j_1}^{n} \\ \vdots \qquad \vdots \\ X_{j_{n-d}}^{d+1} \cdots X_{j_{n-d}}^{n} \end{pmatrix} .$$

Let $f(x)$ be a relative invariant of the triplet $(G \times GL(d_1, \cdots, d_r)$, $M(n, C)$. Then $f(x)$ has the form

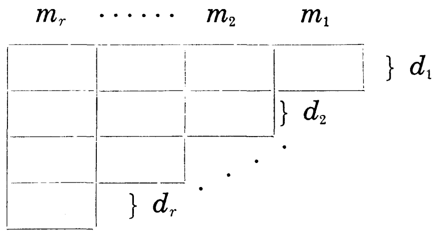
$$f(x) = F(\cdots, X_{i_1\cdots i_{d(\nu)}}, \cdots)(\det x)^{m_r} ,$$

where F is a homogeneous polynomial in $X_{i_1\cdots i_{d(\nu)}}$ $(1 \leqq i_1, \cdots, i_d(\nu) \leqq n$, $1 \leqq \nu < r)$ and m_r is a non-negative integer.

Denoting by m_ν the homogeneous degree of F with respect to $X_{i_1\cdots i_{d(\nu)}}$, for each ν, define a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ as follows:

$$\begin{aligned} \lambda_1 = \cdots = \lambda_{d(1)} &= m_1 + \cdots + m_r , \\ \lambda_{d(1)+1} = \cdots = \lambda_{d(2)} &= m_2 + \cdots + m_r , \\ \cdots\cdots&, \\ \lambda_{d(r-1)+1} = \cdots = \lambda_n &= m_r . \end{aligned}$$

Then the relative invariant $f(x)$ is contained in the vector space V_λ. Thus to a relative invariant $f(x)$, there corresponds the unique Young diagram $Y(\lambda)$:



Let

$$f^*(x) = (\det x)^{m_r} F(X_{i_1\cdots i_{d(1)}}, X_{j_1\cdots j_{d(2)}}, \cdots, X_{k_1\cdots k_{d(r-1)}}) \tag{2.1}$$

be a relative invariant of the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$, and define the polynomial $f^*(x)$ by

$$f^*(x) = (\det x)^{m_r} F(X'_{i_1\cdots i_{d(1)}}, X'_{j_1\cdots j_{d(2)}}, \cdots, X'_{k_1\cdots k_{d(r-1)}}) . \tag{2.2}$$

Then $f^*(x)$ is a relative invariant of the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1^*, M(n, C))$. Since the mapping $f \mapsto f^*$ is one-to-one, we have the following proposition.

PROPOSITION 2.1. *The mapping $f \mapsto f^*$ gives a one-to-one correspondence between the relative invariants of the triplets $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$ and $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1^*, M(n, C))$.*

In general, let a triplet (G, ρ, V) be a P.V., S the singular set, $S_1, \cdots, S_k$ the irreducible components of S with codimension one and $P_1, \cdots, P_k$ irreducible polynomials defining $S_1, \cdots, S_k$, respectively. It is known that the polynomials $P_1, \cdots, P_k$ are algebraically independent relative invariants and any relative invariant $P(x)$ is of the form.

$$P(x) = c \cdot P_1(x)^{l_1} \cdots P_k(x)^{l_k} \ (c \in C,\ (l_1, \cdots, l_k) \in Z_+^{k+1}) .$$

Polynomials $P_1, \cdots, P_k$ are determined up to constant factors, and the set $\{P_1, \cdots, P_k\}$ is called a complete system of irreducible relative invariants of (G, ρ, V).

Let $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$ be a P.V., $\{P_0, \cdots, P_k\}$ a complete system of irreducible relative invariants of $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$. Then, by Proposition 1.2, the triplet $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1^*, M(n, C))$ is a P.V. It is easy to verify that the set $\{P_0^*, \cdots, P_k^*\}$ is a complete system of irreducible relative invariants of $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1^*, M(n, C))$. Denote by $\chi_0, \cdots, \chi_k$ the rational characters of the group $G \times GL(d_1, \cdots, d_r)$ corresponding to $P_0, \cdots, P_k$, respectively.

Since $\det x$ is an irreducible relative invariant of $(G \times GL(d_1, \cdots, d_r)$. $\tilde{\rho}_1, M(n, C))$, from now on, we set:

$$P_0(x) = P_0^*(x) = \det x .$$

We denote by $X_\rho(G \times GL(d_1, \cdots, d_r))$ the group of rational characters corresponding to relative invariants of $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$. The group $X_\rho(G \times GL(d_1, \cdots, d_r))$ is a free abelian group of rank $k+1$ generated by $\chi_0, \cdots, \chi_k$.

Denote by $Y(\lambda^{(0)}), \cdots, Y(\lambda^{(k)})$ the Young diagrams corresponding to relative invariants $P_0, \cdots, P_k$, respectively. Since each P_i is irreducible, the partition $\lambda^{(i)}$ is of the form

$$\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n-1}^{(i)}, 0) \quad (1 \leq i \leq k) .$$

The Young diagram $Y(\lambda^{(0)})$ is given by

$$Y(\lambda^{(0)}) = \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\} n .$$

Denote by $\chi_0^*, \cdots, \chi_k^*$ the rational characters defined as follows:

$$\chi_0^* = \chi_0 \text{ and } \chi_i^* = \chi_i^{-1} \cdot \chi_0^{\lambda_1^{(i)}} \quad (1 \leq i \leq k) .$$

From the construction of the mapping $f \mapsto f^*$, it follows that:

$$P_i^*(\tilde{\rho}_1^*(g) \cdot x) = \chi_i^*(g)^{-1} P_i^*(x) \quad (0 \leq i \leq k)$$

where $g \in G \times GL(d_1, \cdots, d_r)$.

For a character χ in $X_\rho(G \times GL(d_1, \cdots, d_r))$, let

$$\delta(\chi) = (\delta(\chi)_0, \cdots, \delta(\chi)_k) , \quad \text{and} \quad \delta^*(\chi) = (\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k)$$

be the elements in Z^{k+1} such that

$$\chi = \prod_{i=0}^{k} \chi_i^{\delta(\chi)_i} = \prod_{i=0}^{k} \chi_i^{*\delta^*(\chi)_i} .$$

From (2.2), we have

$$\delta(\chi)_0 = \delta^*(\chi)_0 + \sum_{i=1}^{k} \lambda_1^{(i)} \delta^*(\chi)_i \tag{2.3}$$

and

$$\delta(\chi)_i = - \delta^*(\chi)_i \quad (1 \leq i \leq k). \tag{2.4}$$

§ 3. The b-functions

For a rational character χ in $X_\rho(G \times GL(d_1, \cdots, d_r))$, set

$$P(x)^\chi = \prod_{i=0}^{k} P_i(x)^{\delta(\chi)_i} \quad \text{and} \quad P^*(x)^\chi = \prod_{i=0}^{k} P_i^*(x)^{\delta^*(\chi)_i}.$$

If $\delta^*(\chi)_i \geq 0$ for all i (i.e., $P^*(x)^\chi$ is a polynomial), we can introduce a partial differential operator $P^*(\mathrm{grad})^\chi$ in $C[\partial/\partial x_{ij}]$ such that

$$P^*(\mathrm{grad})^\chi \exp(x, x^*) = P^*(x) \exp(x, x^*).$$

Similarly, if $P(x)^\chi$ is a polynomial, we can introduce $P(\mathrm{grad})^\chi$ in $C[\partial/\partial x_{ij}^*]$ such that

$$P(\mathrm{grad})^\chi \exp(x, x^*) = P(x) \exp(x, x^*).$$

For $s = (s_0, \cdots, s_k) \in C^{k+1}$, set

$$P^s = \prod_{i=0}^{k} P_i^{s_i} \quad \text{and} \quad P^{*s} = \prod_{i=0}^{k} P_i^{*s_i}.$$

We consider P^s (resp. P^{*s}) as a function on the universal covering space of $M(n, C) - S$ (resp. $M(n, C) - S^*$).

LEMMA 3.1. (i) *If $\delta^*(\chi)_i \geq 0$ for all i, there exists a polynomial $b_\chi(s)$ in $s = (s_0, \cdots, s_k)$ which satisfies, for all $s \in C^{k+1}$,*

$$P^*(\mathrm{grad})^\chi \cdot P^s(x) = b_\chi(s) P^{s-\delta(\chi)}. \tag{2.5}$$

(ii) *If $\delta(\chi)_i \geq 0$ for all i, there exists a polynomial $b_\chi^*(s)$ in $s = (s_0, \cdots, s_k)$ which satisfies, for all $s \in C^{k+1}$,*

$$P(\mathrm{grad})^\chi \cdot P^{*s}(x) = b_\chi^*(s) P^{*s-\delta^*(\chi)}.$$

Proof. Denoting by $F(x)$ the left hand side of (2.5), we have:

$$F(\tilde{\rho}_1(g) \cdot x) = \chi^{-1}(g) \chi(g)^s F(x)$$

and

$$P(\tilde{\rho}_1(g) \cdot x)^{s-\delta(\chi)} = \chi^{-1}(g) \chi(g)^s P(x)^{s-\delta(\chi)} \quad (g \in G \times GL(d_1, \cdots, d_r)).$$

This shows that $P^{-s+\delta(\chi)}F(x)$ is an absolute invariant, and hence must be a constant $b_\chi(s)$ depending only upon s and χ. It is clear that $b_\chi(s)$ is a polynomial in s. The proof of the part (ii) is similar. Q.E.D.

From the definitions of $b_\chi(s)$ and $b_\chi^*(s)$, it follows that:

(i) If χ and ψ are characters in $X_\rho(G \times GL(d_1, \cdots, d_r))$ such that $\delta^*(\chi)_i \geq 0$ and $\delta^*(\chi)_i \geq 0$ $(0 \leq i \leq k)$, then

$$b_{\chi\psi}(s) = b_\chi(s) b_\psi(s + \delta(\chi)) .$$

(ii) If χ and ψ are characters in $X_\rho(G \times GL(d_1, \cdots, d_r))$ such that $\delta(\chi)_i \geq 0$ and $\delta(\chi)_i \geq 0$ $(0 \leq i \leq k)$, then

$$b_{\chi\psi}^*(s) = b_\chi^*(s) b_\psi^*(s + \delta^*(\chi)) .$$

By the co-cycle properties of $b_\chi(s)$ and $b_\chi^*(s)$, $b_\chi(s)$ and $b_\chi^*(s)$ can be defined for arbitrary character χ in $X_\rho(G \times GL(d_1, \cdots, d_r))$.

Let $\lambda^{(1)}, \cdots, \lambda^{(k)}$ be the partitions corresponding to $P_1, \cdots, P_k$, respectively. For $s = (s_0, s_1, \cdots, s_k) \in C^{k+1}$, put

$$\begin{aligned} \gamma(s) = {} & \Gamma(s_0 + s_1\lambda_1^{(1)} + \cdots + s_k\lambda_1^{(k)} + n) \\ & \times \Gamma(s_0 + s_1\lambda_2^{(1)} + \cdots + s_k\lambda_2^{(k)} + n - 1) \\ & \quad \cdots\cdot \\ & \times \Gamma(s_0 + s_1\lambda_{n-1}^{(1)} + \cdots + s_k\lambda_{n-1}^{(k)} + 2) \\ & \times \Gamma(s_0 + 1) , \end{aligned}$$

and

$$\begin{aligned} \gamma^*(s) = {} & \Gamma(s_0 + s_1\lambda_1^{(1)} + \cdots + s_k\lambda_1^{(k)} + n) \\ & \times \Gamma(s_0 + s_1(\lambda_1^{(1)} - \lambda_{n-1}^{(1)}) + \cdots + s_k(\lambda_1^{(k)} - \lambda_{n-1}^{(k)}) + n - 1) \\ & \quad \cdots\cdot \\ & \times \Gamma(s_0 + s_1(\lambda_1^{(1)} - \lambda_2^{(1)}) + \cdots + s_k(\lambda_1^{(k)} - \lambda_2^{(k)}) + 2) \\ & \times \Gamma(s_0 + 1) . \end{aligned}$$

Now we can state the main theorem of the present paper.

THEOREM 3.1. *Let $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$ be a P.V., $\{P_0, \cdots, P_k\}$ a complete system of relative invariants and $\lambda^{(1)}, \cdots, \lambda^{(r)}$ the partitions corresponding to $P_1, \cdots, P_r$. Then the b-functions $b_\chi(s)$ and $b_\chi^*(s)$ are given by*

$$b_\chi(s) = \frac{\gamma(s)}{\gamma(s - \delta(\chi))} \tag{3.1}$$

and

$$b_\lambda^*(s) = \frac{\gamma^*(s)}{\gamma^*(s - \delta^*(\chi))}. \tag{3.2}$$

Proof. Let B_n^- denote the group consisting of lower triangular n by n matrices, and ρ the representation on the vector space C^n defined by

$$\rho(g)v = g\cdot v \ (g \in B_n^-, \ v = {}^t(v_1, \cdots, v_n) \in C^n).$$

Then the triplet $(B_n^- \times B_n, \tilde{\rho}_1, M(n, C))$ is a P.V., and relative invariants and the b-function are known ([2], p. 150). In this case, Theorem 3.1 is true and we shall reduce the problem to this case.

For a polynomial $f(x)$ in V, we denote by $f^*(x)$ the polynomial defined in (2.2).

For any partition $\lambda = (\lambda_1, \cdots, \lambda_n)$, denoting by π_λ the projection

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \longrightarrow V_\lambda,$$

we shall introduce two $GL(n, C)$-homomorphisms θ_1 and θ_2

$$\Theta_i;\ V_\lambda \otimes V_{\lambda'} \longrightarrow V_{\lambda+\lambda'} \quad (i = 1, 2),$$

where $\lambda + \lambda'$ denotes the partition $(\lambda_1 + \lambda_1', \lambda_2 + \lambda_2', \cdots, \lambda_n + \lambda_n')$.

For any f in V_λ and f' in $V_{\lambda'}$, $\Theta_i(f \otimes f')$ are defined as follows;

$$\Theta_1(f(x) \otimes f'(x)) = f(x)\cdot f'(x)$$

and

$$\Theta_2(f(x) \otimes f'(x)) = \pi_{\lambda_1+\lambda_n}((\det x)^{\lambda_1} f^*(\mathrm{grad}) f'(x)).$$

The decomposition of the $SL(n, C)$-module $V_\lambda \otimes V_{\lambda'}$ into irreducible components contains $V_{\lambda+\lambda'}$ with multiplicity one. The Schur's lemma says that Θ_1 and Θ_2 must be agree up to a constant.

On the other hand, a complete system of relative invariants of $(B_n^- \times B_n, \tilde{\rho}_1, M(n, C))$ is given by $\{\Delta_1(x), \cdots, \Delta_n(x)\}$ where

$$\Delta_i = \det\begin{pmatrix} x_1^1 \cdots x_1^i \\ \vdots \quad\quad \vdots \\ x_i^1 \cdots x_i^i \end{pmatrix} \quad (1 \le i \le n).$$

For relative invariant polynomials f and f' of the P.V. $(G \times GL(d_1, \cdots, d_r)$, $\tilde{\rho}_1$, $M(n, C)$, let λ and λ' denote the corresponding partitions, respectively.

Put, for $i = 1, 2, \cdots, n$,

$$m_i = \lambda_i - \lambda_{i+1} \quad (\text{with } \lambda_{n+1} = 0)$$

and

$$m_i' = \lambda_i' - \lambda_{i+1}' \quad (\text{with } \lambda_{n+1}' = 0)\,.$$

Then $\prod_{i=1}^{n} \Delta_i^{m_i}(x)$ and $\prod_{i=1}^{n} \Delta_i^{m_i'}(x)$ are contained in V_λ and $V_{\lambda'}$, respectively. Therefore we have

$$\frac{f^*(\text{grad})\cdot f'(x)}{f(x)\cdot f'(x)} = \frac{\prod_{i=1}^{n} \Delta_i^*(\text{grad})^{m_i}\cdot \prod_{i=1}^{n} \Delta_i(x)^{m_i'}}{\prod_{i=1}^{n} \Delta_i(x)^{m_i+m_i'}}$$

Thus we can reduce the proof to the case of the P.V. $(B_n^- \times B_n, \tilde{\rho}_1, M(n, C))$, and we obtain (3.1). (3.2) is shown in a similar manner. Q.E.D.

COROLLARY. *Let (G, ρ, V) be a P.V., $\{P_1, \cdots, P_k\}$ a complete system of relative invariants of (G, ρ, V), and $d_1, \cdots, d_k$ degrees of $P_1, \cdots, P_k$, respectively. Then $(G \times GL(1, n-1), \tilde{\rho}_1, M(n, C))$, $n = \dim V$, is a P.V., and the b-functions $b_\chi(s)$ and $b_{\chi^*}(s)$ of $(G \times GL(1, n-1), \tilde{\rho}_1, M(n, C)$ are given by*

$$b_\chi(s) = \frac{\gamma(s)}{\gamma(s-\delta(\chi))} \quad \textit{and} \quad b_\chi^*(s) = \frac{\gamma^*(s)}{\gamma^*(s-\delta^*(\chi))}$$

where

$$\gamma(s) = \Gamma(s_0 + d_1s_1 + \cdots + d_ks_k + n)\Gamma(s_0 + n - 1)\cdots\Gamma(s_0 + 1)$$

and

$$\gamma^*(s) = \Gamma(s_0 + d_1s_1 + \cdots + d_ks_k + n)\Gamma(s_0 + d_1s_1 + \cdots + d_ks_k + n - 1)$$
$$\cdots\Gamma(s_0 + d_1s_1 + \cdots + d_ks_k + 2)\Gamma(s_0 + 1)\,.$$

§ 4. Prehomogeneous vector spaces $(G \times B_n, \tilde{\rho}_1, M(n, C))$

Let G be a connected linear semi-simple algebraic group ($G \neq \{e\}$), $\rho : G \to GL(V)$ an n-dimensional irreducible representation, all defined over C. Let $(G \times B_n, \tilde{\rho}_1, M(n, C))$ be a P.V. Then, $\dim(G \times B_n) \geq \dim M(n, C)$, and hence we have:

$$\dim G \geq \frac{1}{2}n(n-1)\,.$$

Since G is semi-simple, we may assume that a triplet (G, ρ, V) is of the form:

$$G = G_1 \times G_2 \times \cdots \times G_k\,,$$
$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k\,,$$

and

$$V = V(d_1) \otimes V(d_2) \otimes \cdots \otimes V(d_k) \quad \text{with} \quad d_1 \geq d_2 \geq \cdots \geq d_k \geq 2\,,$$

where each G_i is a connected simple algebraic group, ρ_i is an irreducible representations of G_i on the d_i-dimensional C-vector space $V(d_i)$ $(1 \leq i \leq k)$. Therefore if $(G \times B_n, \tilde{\rho}_1, M(n, C))$ is a P.V. we have

$$\sum_{i=1}^{k} \dim G_i \geq \frac{1}{2} d_1 \cdots d_k \quad (d_1 \cdots d_k - 1)\,. \tag{4.1}$$

LEMMA 4.1. *Assume that a triplet* $(G \times B_n, \tilde{\rho}_1, M(n, C))$ *is a P.V. Then we have* $k = 1$ *or* $(G, \rho, V) = (SL(2) \times SL(2), \square \otimes \square, V(4))$.

Proof. The image $\rho_i(G_i)$ of the simple algebraic group G_i is contained in $SL(d_i)$. By (4.1), we have

$$\sum_{i=1}^{k} (d_i^2 - 1) \geq \frac{1}{2} d_1 \cdots d_k \quad (d_1 \cdots d_k - 1)\,.$$

If $k \geq 3$, this inequality implies that

$$1 > d_1^2(2^{2k-3} - 2^{k-2} - k) + k\,.$$

It is easy to show that $2^{2k-3} > 2^{k-2} + k$ for $k \geq 3$. This is impossible and hence we have $k \leq 2$. When $k = 2$, we have $d_1^2 + d_2^2 > \frac{1}{2} d_1 d_2 (d_1 d_2 - 1)$. This inequality implies that $d_1 = d_2 = 2$.

The following lemma can be easily proved.

LEMMA 4.2. *Let G be a semi-simple linear algebraic group and $\rho : G \to GL(V)$ be an irreducible representation on an n-dimensional vector space $V(n)$ satisfying*

$$\dim G \geq \frac{1}{2} n(n-1)\,.$$

Then $(G, \rho, V(n))$ is one of:

(1) $(SL(n), \square, V(n))$
(2) $(SL(2), \square\square, V(3))$
(3) $(SL(4), \begin{smallmatrix}\square\\\square\end{smallmatrix}, V(6))$
(4) $(SL(2) \times SL(2), \square \otimes \square, V(4))$

(5) $(Sp(m), \square, V(n))$ $(n = 2m)$

(6) $(SO(n), \square, V(n))$ $(n \geq 3)$

(7) $(Sp(2), \begin{smallmatrix}\square\\\square\end{smallmatrix}, V(5))$

(for classical group, we write the Young diagram corresponding to ρ).

The image of the representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $Sp(2)$ is $SO(5)$ and the kernel is $\{\pm 1\}$. Therefore we may identify the triplet $(Sp(2), \begin{smallmatrix}\square\\\square\end{smallmatrix}, V(5))$ with the triplet $(SO(5), \square, V(5))$. Similarly, we identify the triplets (2)$\sim$(4) with the triplet (6), $n = 3, 6, 4$, respectively.

Case 1. $(G, \rho, V) = (SL(n), \square, V(n))$.

The triplet $(SL(n), \tilde{\square}, M(n, C))$ is a trivial P.V. (See [2], and the singular set S is given by

$$S = \{X \in M(n, C);\ \det X = 0\}.$$

The Young diagram corresponding to the relative invariant $\det X$ is

$$\left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\} n.$$

By Theorem 1, the b-function is given by

$$b(s) = s(s+1)\cdots(s+n-1).$$

Remark. The b-function of $\det X$ is well known (See [2])

Case 2. $(G, \rho, V) = (SO(n), \square, V(n))$.

The triplet $(SO(n), \tilde{\square}, M(n, C))$ is a P.V., and the b-function of it is known (See [2]).

We shall determine the b-function by Theorem 3.1.

For a $x = [x^1, \cdots, x^n] \in M(n, C)$, put

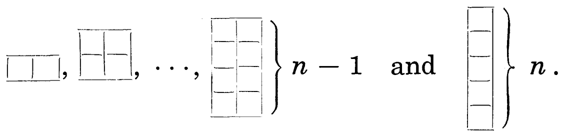
$$P_i(x) = \det\begin{pmatrix}(x^1, x^1), & \cdots, & (x^1, x^n) \\ \vdots & & \vdots \\ (x^n, x^1), & \cdots, & (x^n, x^n)\end{pmatrix} \quad (1 \leq i \leq n-1)$$

where $(x^k, x^l) = {}^t x^k \cdot x^l$, and $P_0(x) = \det x$.

Then the singular set S is $S_0 \cup \cdots \cup S_{n-1}$ with

$$S_i = \{x \in M(n, C);\ P_i(x) = 0\} \quad (0 \leq i \leq n-1).$$

Thus the Young diagrams corresponding to relative invariants $P_1, \cdots, P_{n-1}$ and P_0 are, respectively,



By Theorem 1, we have:

$$b_\chi(s) = \gamma(s)/\gamma(s - \delta\chi)$$

where

$$\begin{aligned}\gamma(s) &= \Gamma(s_0 + 2s_1 + \cdots + 2s_{n-1} + n) \\ &\quad \times \Gamma(s_0 + 2s_2 + \cdots + 2s_{n-1} + n - 1) \cdots \Gamma(s_0 + 1).\end{aligned}$$

Case 3. $(G, \rho, V) = (Sp(m), \square\ V(n))$ $(n = 2m)$

Denote by $[x, y]$ the skew symmetric bilinear form on $V(n) \times V(n)$ defined as follows.

$$\begin{aligned}&[x, y] = x_1 y_1' - x_1' y_1 + \cdots + x_m y_m' - x_m' y_m \\ &\quad \text{with } x = {}^t(x_1, x_1', \cdots, x_m, x_m') \quad \text{and} \quad y = {}^t(y_1, y_1', \cdots, y_m, y_m').\end{aligned}$$

For $x = (x^1, \cdots, x^n) \in M(n, C)$. Put

$$P_k(x) = \mathrm{Pff}([x^i, x^j])_{\substack{1 \leq i \leq 2k \\ 1 \leq j \leq 2k}} \quad (1 \leq k \leq m-1),$$

and

$$P_0(x) = \det(x)$$

where Pff denotes the Pfaffian.

It is easy to show that, if a point x_0 of $M(n, C)$ satisfies $\prod_{i=0}^{m-1} P_i(x) \neq 0$, there exists a $(g, T) \in Sp(m) \times B_n$ such that $g x_0 T^{-1} = 1_n$. Therefore the triplet $(Sp(m), \square, M(n, C))$ is a P.V., and the singular set S is

$$\begin{aligned}&S = S_0 \cup S_1 \cup \cdots \cup S_m \\ &\quad \text{with} \quad S_i = \{X \in M(n, C) | P_i(x) = 0\} \quad (0 \leq i \leq m).\end{aligned}$$

The Young diagrams corresponding to relative invariants $P_1, \cdots, P_{m-1}$ and P_0 are, respectively,

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\,,\ \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\,,\ \cdots,\ \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\} 2m-2 \quad \text{and} \quad \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right\} 2m\,.$$

Put

$$\begin{aligned}\gamma(s) = {} & \Gamma(s_0 + s_1 + \cdots + s_{m-1} + n)\Gamma(s_0 + s_1 + \cdots + s_{m-1} + n - 1)\\ & \times \Gamma(s_0 + s_2 + \cdots + s_{m-1} + n - 2)\Gamma(s_0 + s_2 + \cdots + s_{m-1} + n - 3)\\ & \times \Gamma(s_0 + s_{m-1} + 4)\Gamma(s_0 + s_{m-1} + 3)\\ & \times \Gamma(s_0 + 2)\cdot\Gamma(s_0 + 1)\,.\end{aligned}$$

Then, by Theorem 1, the b-function $b_\chi(s)$ is given by

$$b_\chi(s) = \gamma(s)/\gamma(s - \delta(\chi))\,.$$

Now we obtain the following theorem.

THEOREM 4.1. *Let $(G \times B_n, \tilde{\rho}_1, M(n, C))$ be a P.V., where G* is *a semi-simple connected linear algebraic group, B_n the group consists of all $n \times n$ complex triangular matrix, $\rho : G \to GL(V)$ an irreducible representation on an n-dimensional vector space, all defined over C. Let $\{P_0, \cdots, P_k\}$ be a complete set of irreducible relative invariants of $(G \times B_n, \tilde{\rho}_1, M(n, C))$. Then (G, ρ, V) is one of the following P.V.'s,*

(1) $(SL(n), \square, V(n))$.
  (i) $k = 0$
  (ii) $\gamma(s) = \Gamma(s_0 + n)\cdots\Gamma(s_0 + 1)$
  (iii) $\gamma^*(s) = \Gamma(s_0 + n)\cdots\Gamma(s_0 + 1)$.

(2) $(Sp(m), \square, V(n))$ $(n = 2m)$.
  (i) $k = m - 1$
  (ii)
$$\begin{aligned}\gamma(s) = {} & \Gamma(s_0 + s_1 + \cdots + s_{m-1} - n)\Gamma(s_0 + s_1 + \cdots + s_{m-1} + n - 1)\\ & \times \Gamma(s_0 + s_2 + \cdots + s_{m-2} + n - 2)\\ & \times \Gamma(s_0 + s_2 + \cdots + s_{m-1} + n - 3)\\ & \cdots\\ & \times \Gamma(s_0 + s_{m-1} + 4)\Gamma(s_0 + s_{m-1} + 3)\Gamma(s_0 + 2)\Gamma(s_0 + 1)\,.\end{aligned}$$
  (iii)
$$\begin{aligned}\gamma^*(s) = {} & \Gamma(s_0 + s_1 + \cdots + s_{m-1} + n)\Gamma(s_0 + s_1 + \cdots + s_{m-1} + n - 1)\\ & \times \Gamma(s_0 + s_1 + \cdots + s_{m-2} + n - 3)\\ & \times \Gamma(s_0 + s_1 + \cdots + s_{m-2} + n - 4)\\ & \cdots\\ & \times \Gamma(s_0 + s_1 + 4)\Gamma(s_0 + s_1 + 3)\Gamma(s_0 + 2)\Gamma(s_0 + 1)\,.\end{aligned}$$

(3) $(SO(n), \square, V(n))$.

(i) $k = n - 1$

(ii) $$\begin{aligned}\gamma(s) = \Gamma(s_0 + 2s_1 + &\cdots + 2s_{n-1} + n)\\ &\times \Gamma(s_0 + 2s_2 + \cdots + 2s_{n-1} + n - 1)\\ &\times \Gamma(s_0 + 2s_3 + \cdots + 2s_{n-1} + n - 2)\\ &\times \Gamma(s_0 + 2s_4 + \cdots + 2s_{n-1} + n - 3)\\ &\quad\cdots\\ &\times \Gamma(s_0 + 2s_{n-1} + 2)\Gamma(s_0 + 1)\,.\end{aligned}$$

(iii) $$\begin{aligned}\gamma^*(s) = \Gamma(s_0 + 2s_1 + &\cdots + 2s_{n-1} + n)\\ &\times \Gamma(s_0 + 2s_1 + \cdots + 2s_{n-2} + n - 1)\\ &\times \Gamma(s_0 + 2s_1 + \cdots + 2s_{n-3} + n - 2)\\ &\times \Gamma(s_0 + 2s_1 + \cdots + 2s_{n-4} + n - 3)\\ &\quad\cdots\\ &\times \Gamma(s_0 + 2s_1 + 2)\Gamma(s_0 + 1)\,.\end{aligned}$$

In the next article, we shall be concerned with zeta-functions associated with prehomogeneous vector spaces $(G \times GL(d_1, \cdots, d_r), \tilde{\rho}_1, M(n, C))$.

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