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ON A GENERALIZATION OF HAMBURGER'S THEOREM

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Introduction

The relationship between Poisson's summation formula and Hamburger's theorem [2] which is a characterization of Riemann's zetafunction by the functional equation was already mentioned in Ehrenpreis-Kawai [1]. There Poisson's summation formula was obtained by the functional equation of Riemann's zetafunction. This procedure is another proof of Hamburger's theorem. Being interpreted in this way, Hamburger's theorem admits various interesting generalizations, one of which is to derive, from the functional equations of the zetafunctions with Grössencharacters of the Gaussian field, Poisson's summation formula corresponding to its ring of integers [1]. The main purpose of the present paper is to give a generalization of Hamburger's theorem to some zetafunctions with Grössencharacters in algebraic number fields. More precisely, we first define the zetafunctions with Grössencharacters corresponding to a lattice in a vector space, and show that Poisson's summation formula yields the functional equations of them. Next, we derive Poisson's summation formula corresponding to the lattice from the functional equations.

§1. Notations and formulation of the theorem

We denote by $\boldsymbol{R}$ and $\boldsymbol{C}$ the field of real numbers and the field of complex numbers respectively. Let F be an algebraic number field of degree n with signature $[r_1, r_2]$. We can naturally embed F into $\boldsymbol{R}^{r_1} \times \boldsymbol{C}^{r_2}$. Put $V = \boldsymbol{R}^{r_1} \times \boldsymbol{C}^{r_2}$. Then V may be regarded as a commutative ring. For $x = (x^{(1)}, \cdots, x^{(r_1+r_2)}) \in V$, we put $trx = \sum_{p=1}^{r_1+r_2} tr_{\boldsymbol{R}} x^{(p)}$, $Nx = \prod_{p=1}^{r_1+r_2} |N_{\boldsymbol{R}} x^{(p)}|$, $e(x) = \exp(2\pi\sqrt{-1}\, trx)$. We define a lattice in V as a subgroup of V having a basis $\{\alpha_1, \cdots, \alpha_n\}$ independent over $\boldsymbol{R}$. Let L be a lattice in V. Then its dual can be defined as the set L^* consisting of all $x \in V$ such that $e(xy) = 1$ for all $y \in L$. We can show easily that L^*

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is a lattice in V. Let O_F be the ring of integers of F and E_F be the group of units of F. Throughout this paper, we consider only L satisfying $O_F L \subset L$.

Next, if D is a real number satisfying $D \geqq 1$, we define the functions $A^L_{(a,b)}(s)$, $B^{L^*}_{(a,b)}(s)$ as follows:

$$A^L_{(a,b)}(s) = \sum_{(m)} \frac{c_m \lambda_a(m) \bar{\mu}_b(m)}{Nm^s},$$

$$B^{L^*}_{(a,b)}(s) = \sum_{(\nu)} \frac{d_\nu \lambda_a(\nu) \bar{\mu}_b(\nu)}{N\nu^s},$$

for $\operatorname{Re} s > D$, where (m), (ν) runs through $E_F \backslash L - \{0\}$, $E_F \backslash L^* - \{0\}$ and c_m, d_ν are complex numbers. Moreover we define

$$\lambda_a(x) = \prod_{p=r_1+1}^{r_1+r_2} \left(\frac{x^{(p)}}{|x^{(p)}|} \right)^{a_p},$$

$$\mu_b(x) = \prod_{p=1}^{r_1+r_2} |x^{(p)}|^{\sqrt{-1} b_p},$$

where $a = (a_{r_1+1}, \cdots, a_{r_1+r_2})$, $b = (b_1, \cdots, b_{r_1+r_2})$ and a_p $(p = r_1 + 1, \cdots, r_1 + r_2)$ are integers and b_p are real numbers such that $\sum_{p=1}^{r_1+r_2} b_p = 0$. $\bar{\mu}_b$ is complex conjugate to μ_b, and for $\eta \in E_F$, we assume that $c_{\eta m} = c_m$, $d_{\eta\nu} = d_\nu$, $\lambda_a(\eta) = 1$, $\mu_b(\eta) = 1$.

We further assume that $A^L_{(a,b)}(s)$ and $B^{L^*}_{(a,b)}(s)$ both absolutely converge in the domain $\{s \in \boldsymbol{C} \mid \operatorname{Re} s > 1\}$, and can be analytically continued to the whole plane as meromorphic functions having at most simple poles at $s = 1$ if $(a, b) = (0, 0)$ and can be analytically continued to the whole plane as entire functions if $(a, b) \neq (0, 0)$. Moreover we assume that $(s-1)A^L_{(a,b)}(s)$ and $(s-1)B^{L^*}_{(a,b)}(s)$ are entire functions of finite order.

Let $\Phi(x) = \Phi(x^{(1)}, \cdots, x^{(r_1+r_2)})$ be in $\mathscr{S}_V$ which is the Schwartz space over V when we regard V as $\boldsymbol{R}^n$. Moreover we define

$$\Phi^*(y) = \int_V \Phi(x) e(xy) dx,$$

where

$$dx = dx^{(1)} \cdots dx^{(r_1)} |dx^{(r_1+1)} \wedge d\overline{x^{(r_1+1)}}| \cdots |dx^{(r_1+r_2)} \wedge d\overline{x^{(r_1+r_2)}}|,$$
$$x = (x^{(1)}, \cdots, x^{(r_1+r_2)}), \qquad y = (y^{(1)}, \cdots, y^{(r_1+r_2)}) \in V.$$

We put

$$Z^L_{(a,b)}(s) = \sum_{(m)} \frac{\lambda_a(m) \bar{\mu}_b(m)}{Nm^s},$$

where the meaning of (m) is the same as before.

We further put

$$M=\sum_{p=r_1+1}^{r_1+r_2}|a_p|\,,\quad N=\sum_{p=r_1+1}^{r_1+r_2}b_p\,,\quad A(L)=\left(\frac{2^{r_1}\cdot C(L)^2}{(2\pi)^n}\right)^{1/2},$$

$$c(L)=\left|\det\begin{pmatrix}\alpha_1^{(1)}, & \cdots, & \alpha_1^{(r_1+r_2)}, & \overline{\alpha_1^{(r_1+r_2)}}\\ \vdots & & & \vdots\\ \alpha_n^{(1)}, & \cdots, & \alpha_n^{(r_1+r_2)}, & \overline{\alpha_n^{(r_1+r_2)}}\end{pmatrix}\right|.$$

Then our proposition and theorem are stated as follows:

PROPOSITION. *The functions $Z^L_{(a,b)}(s)$ satisfy*

$$\begin{aligned}&A(L)^s\prod_{p=1}^{r_1}\left(\frac{s}{2}+\frac{|a_p|}{2}+\frac{\sqrt{-1}b_p}{2}\right)\prod_{p=1}^{r_1+r_2}\Gamma\left(s+\frac{|a_p|}{2}+\frac{\sqrt{-1}b_p}{2}\right)Z^L_{(a,b)}(s)\\ &=(-\sqrt{-1})^M\cdot 2^{\sqrt{-1}N}A(L^*)^{1-s}\prod_{p=1}^{r_1}\Gamma\left(\frac{1-s}{2}+\frac{|a_p|}{2}-\frac{\sqrt{-1}b_p}{2}\right)\\ &\times\prod_{p=r_1+1}^{r_1+r_2}\Gamma\left(1-s+\frac{|a_p|}{2}-\frac{\sqrt{-1}b_p}{2}\right)Z^{L^*}_{(-a,-b)}(1-s)\end{aligned}\tag{1}$$

for all λ_a, μ_b.

THEOREM. *Suppose that the functions $A^L_{(a,b)}(s)$ and $B^{L^*}_{(a,b)}(s)$ satisfy*

$$\begin{aligned}&A(L)^s\prod_{p=1}^{r_1}\Gamma\left(\frac{s}{2}+\frac{|a_p|}{2}+\frac{\sqrt{-1}b_p}{2}\right)\prod_{p=r_1+1}^{r_1+r_2}\Gamma\left(s+\frac{|a_p|}{2}+\frac{\sqrt{-1}b_p}{2}\right)A^L_{(a,b)}(s)\\ &=(-\sqrt{-1})^M\cdot 2^{\sqrt{-1}N}A(L^*)^{1-s}\prod_{p=1}^{r_1}\left(\frac{1-s}{2}+\frac{|a_p|}{2}-\frac{\sqrt{-1}b_p}{2}\right)\\ &\times\prod_{p=r_1+1}^{r_1+r_2}\Gamma\left(1-s+\frac{|a_p|}{2}-\frac{\sqrt{-1}b_p}{2}\right)B^{L^*}_{(-a,-b)}(1-s)\end{aligned}\tag{2}$$

for all λ_a, μ_b.

Then we have

$$\sum_{m\in L}c_m\Phi^*(m)=c(L)^{-1}\sum_{\nu\in L^*}d_\nu\Phi(\nu)\,.$$

Moreover the coefficients c_m, d_ν are all equal.

§2. Proof of Proposition

As is widely known, Poisson's summation formula is stated as follows:

$$\sum_{m\in L}\Phi^*(m)=c(L)^{-1}\sum_{\nu\in L^*}\Phi(\nu)\,,$$

where $\Phi(x)$ is in $\mathscr{S}_V$. Put

$$\Phi(x) = \exp(-\pi|x^{(1)} + u^{(1)}|^2 t_1) \cdots \exp(-2\pi|x^{(r_1+r_2)} + u^{(r_1+r_2)}|^2 t_{r_1+r_2})$$

in (3), where $t_1, \cdots, t_{r_1+r_2} > 0$, $u^{(1)}, \cdots, u^{(r_1)}$ are real variables and $u^{(r_1+1)}, \cdots, u^{(r_1+r_2)}$ are complex variables. Then we can readily show

$$\begin{aligned}(4)\qquad &\sum_{m\in L} \exp\left(-\pi \sum_{p=1}^{n} |m^{(p)} + u^{(p)}|^2 t_p\right) \\ &= c(L)^{-1}\frac{1}{\sqrt{t_1 \cdots t_n}} \sum_{\nu\in L^*} \exp\left(-\pi \sum_{p=1}^{n} |\nu^{(p)}|^2 \frac{1}{t_p} + 2\pi\sqrt{-1} \sum_{p=1}^{n} \nu^{(p)} u^{(p)}\right),\end{aligned}$$

where

$$u^{(r_2+p)} = \overline{u^{(p)}}\,, \qquad t_{p+r_2} = t_p\,,$$
$$m^{(r_2+p)} = \overline{m^{(p)}}\,, \qquad \nu^{(r_2+p)} = \overline{\nu^{(p)}}$$
$$(p = r_1 + 1, \cdots, r_1 + r_2)\,.$$

Put $u^{(1)} = \cdots = u^{(n)} = 0$ in (4). Then we obtain

$$\begin{aligned}(5)\qquad &\sum_{m\in L} \exp\left(-\pi \sum_{p=1}^{n} |m^{(p)}|^2 t_p\right) \\ &= c(L)^{-1} \sum_{\nu\in L^*} \exp\left(\sum_{p=1}^{n} |\nu^{(p)}|^2 \frac{1}{t_p}\right).\end{aligned}$$

We denote by $\boldsymbol{R}^+$ the group of positive real numbers. We put $G = (\boldsymbol{R}^+)^{r_1+r_2}$ and $\|t\| = \prod_{p=1}^{r_1} t_p \prod_{p=r_1+1}^{r_1+r_2} t_p^2$ for $t = (t_1, \cdots, t_{r_1+r_2}) \in G$. Moreover let G^0 be the subgroup consisting of all $y \in G$ such that $\|y\| = 1$. Then we have a product $G = \boldsymbol{R}^+ \times G^0$, where $\rho = (\rho^{1/n}, \cdots, \rho^{1/n}) \in G$ for $\rho \in \boldsymbol{R}^+$. Let W be the image of the group of units E_F in G, and E be a fundamental domain for W^2 in G^0, where $W^2 = \{x^2 \mid x \in W\}$.

We denote the left hand side of (5) by $\theta(t, L)$. Then we have

$$(6)\qquad \theta(t, L) = c(L)^{-1}\frac{1}{\sqrt{\|t\|}}\theta(t^{-1}, L^*)\,.$$

On the other hand, we see

$$\begin{aligned}&\left(\frac{2^{r_1}}{(2\pi)^n}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} Z^L_{(0,0)}(s) \\ &= \int_G \sum_{(m)} \exp\left(-\pi \sum_{p=1}^{n} |m^{(p)}|^2 t_p\right) \|t\|^{s/2} \frac{dt}{t}\,,\end{aligned}$$

where

$$\frac{dt}{t} = \frac{dt_1}{t_1} \cdots \frac{dt_{r_1+r_2}}{t_{r_1+r_2}},$$

$$= \frac{1}{W_F} \int_0^\infty \int_E (\theta(yc, L) - 1) d^*c y^{s/2} \frac{dy}{y},$$

where d^*c means the appropriate measure on G^0, $t = yc (y \in \boldsymbol{R}^+, c \in G^0)$, and W_F is the number of roots of unity in F

$$= \frac{1}{W_F} \int_1^\infty \int_E (\theta(yc, L) - 1) d^*c y^{s/2} \frac{dy}{y}$$

$$\times \frac{1}{W_F} \int_0^1 \int_E (\theta(yc, L) - 1) d^*c y^{s/2} \frac{dy}{y}.$$

By using (6), we obtain

$$\int_0^1 \int_E (\theta(yc, L) - 1) d^*c y^{s/2} \frac{dy}{y}$$

$$= -\frac{2\mu^*(E)}{s} + \int_0^1 \int_E \theta(yc, L) d^*c y^{s/2} \frac{dy}{y}$$

$$= -\frac{2\mu^*(E)}{s} - \frac{2\mu^*(E)}{(1-s)} c(L)^{-1}$$

$$+ \int_1^\infty \int_E c(L)^{-1} (\theta(yc, L^*) - 1) d^*c y^{(1-s)/2} \frac{dy}{y},$$

where $\mu^*(E)$ denotes the integral of 1 over E with respect to d^*c. Therefore we have

$$A(L)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} Z^L_{(0,0)}(s)$$

$$= A(L^*)^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} Z^{L^*}_{(0,0)}(1-s),$$

using $c(L)c(L^*) = 1$.

We denote by D_a the differential operator given as the product of all $(\partial/\partial u^{(p)})^{a_p} (p = r_1 + 1, \cdots, r_1 + r_2)$ for non-negative a_p and all $(\partial/\partial u^{(p+r_2)})^{-a_p}$ $(p = r_1 + 1, \cdots, r_1 + r_2)$ for negative a_p. Assume $(a_{r_1+1}, \cdots, a_{r_1+r_2}) \neq (0, \cdots, 0)$, and put $u^{(1)} = \cdots = u^{(n)} = 0$ after operating D_a on the both sides of (4), Then we obtain

$$(7) \quad \sum_{m \in L} \lambda'_a(m) \exp\left(-\pi \sum_{p=1}^n |m^{(p)}|^2 t_p\right)$$

$$= c(L)^{-1} \frac{\prod_{p=r_1+1}^{r_1+r_2} t_p^{-|a_p|}}{\sqrt{t_1 \cdots t_n}} (-\sqrt{-1})^M \sum_{\nu \in L^*} \lambda'_a(\nu) \exp\left(-\pi \sum_{p=1}^n |\nu^{(p)}|^2 \frac{1}{t_p}\right)$$

with

$$\lambda'_a(m) = \lambda_a(m)\,|\,m^{(r_1+1)|a_{r_1+1}|} \cdots m^{(r_1+r_2)|a_{r_1+r_2}|}\,| , \qquad \lambda'_a(0) = 0 .$$

We denote the left hand side of (7) by $\theta_a(t, L)$. Then we have

$$(8) \qquad \theta_a(t, L) = c(L)^{-1} \frac{\prod_{p=r_1+1}^{r_1+r_2} t_p^{-|a_p|}}{\sqrt{\|t\|}} (-\sqrt{-1})^M \theta_{-a}(t^{-1}, L^*) .$$

On the other hand, we see

$$(2\pi)^{-\sum_{p=r_1+1}^{r_1+r_2} a_p/2}\, 2^{-\sqrt{-1}\sum_{p=r_1+1}^{r_1+r_2} b_p/2} \left(\frac{2^{r_1}}{(2\pi)^n}\right)^{s/2} \prod_{p=1}^{r_1} \Gamma\left(\frac{s}{2} + \frac{|a_p|}{2} + \frac{\sqrt{-1}\,b_p}{2}\right)$$
$$\times \prod_{p=r_1+1}^{r_1+r_2} \Gamma\left(s + \frac{|a_p|}{2} + \frac{\sqrt{-1}\,b_p}{2}\right) Z^L_{(a,b)}(s)$$
$$= \frac{1}{W_F} \int_G \sum_{(m)} \lambda'_a(m) \exp\left(-\pi \sum_{p=1}^{n} |m^{(p)}|^2 t_p\right) \mu_{b/2}(t) \prod_{p=r_1+1}^{r_1+r_2} t_p^{|a_p|/2} \|t\|^{s/2} \frac{dt}{t}$$

with

$$\mu_{b/2}(t) = \prod_{p=1}^{r_1+r_2} t_p^{\sqrt{-1}\, b_p/2} ,$$
$$= \frac{1}{W_F} \int_0^\infty \int_E \theta_a(yc, L) \mu_{b/2}(yc) \prod_{p=r_1+1}^{r_1+r_2} (yc_p)^{|a_p|/2} d^*c y^{s/2} \frac{dy}{y}$$

with

$$c = (c_1, \cdots, c_{r_1+r_2}) \in G^0 ,$$
$$= \frac{1}{W_F} \int_1^\infty \int_E \theta_a(yc, L) \mu_{b/2}(yc) \prod_{p=r_1+1}^{r_1+r_2} (yc_p)^{|a_p|/2} d^*c y^{s/2} \frac{dy}{y}$$
$$+ \frac{1}{W_F} \int_0^1 \int_E \theta_a(yc, L) \mu_{b/2}(yc) \prod_{p=r_1+1}^{r_1+r_2} (yc_p)^{|a_p|/2} d^*c y^{s/2} \frac{dy}{y} .$$

By using (8), we obtain

$$\int_0^1 \int_E \theta_a(yc, L) \mu_{b/2}(yc) \prod_{p=r_1+1}^{r_1+r_2} (yc_p)^{|a_p|/2} d^*c y^{s/2} \frac{dy}{y}$$
$$= \int_1^\infty \int_E \theta_{-a}(yc, L^*) \prod_{p=r_1+1}^{r_1+r_2} (yc_p^{-1})^{-|a_p|/2} \mu_{-b/2}(yc) d^*c y^{(1-s)/2} \frac{dy}{y}$$
$$\cdot (-\sqrt{-1})^M \cdot c(L)^{-1} .$$

Therefore we have

$$A(L)^s \prod_{p=1}^{r_1} \Gamma\left(\frac{s}{2} + \frac{|a_p|}{2} + \frac{\sqrt{-1}\,b_p}{2}\right) \prod_{p=r_1+1}^{r_1+r_2} \Gamma\left(s + \frac{|a_p|}{2} + \frac{\sqrt{-1}\,b_p}{2}\right) Z^L_{(a,b)}(s)$$

$$= A(L^*)^{1-s} \prod_{p=1}^{r_1} \Gamma\left(\frac{1-s}{2} + \frac{|a_p|}{2} - \frac{\sqrt{-1}\,b_p}{2}\right) \prod_{p=r_1+1}^{r_1+r_2}$$
$$\times \Gamma\left(1 - s + \frac{|a_p|}{2} - \frac{\sqrt{-1}\,b_p}{2}\right) Z^{L^*}_{(-a,-b)}(1-s)(-\sqrt{-1})^{M} \cdot 2^{\sqrt{-1}N} .$$

This completes the proof of the proposition.

§ 3. Proof of Theorem

Let $\Phi(x)$ be in $\mathscr{S}_V$. Put $\Phi_e(x) = \sum_{\varepsilon \in E_F} \Phi(\varepsilon x)$, $|x^{(p)}| = \rho_p$ $(p = 1, \cdots, r_1)$, $x^{(p)} = \rho_p \exp(\sqrt{-1}\theta_p)(p = r_1 + 1, \cdots, r_1 + r_2)$, $\rho = \rho_1 \cdots \rho^2_{r_1+r_2}$. Moreover let $P(\rho)$ be the set of $x = (x^{(1)}, \cdots, x^{(r_1+r_2)}) \in V$ with a common ρ. Then $\varepsilon \in E_F$ operates on $P(\rho)$ by $x \to \varepsilon x$ and $P(\rho)$ has a compact fundamental domain $E_F \backslash P(\rho)$. On the space V we define a measure

$$2^{r_2} \frac{dx^{(1)}}{\rho_1} \cdots \frac{d\rho_{r_1+r_2}}{\rho_{r_1+r_2}} d\theta_{r_1+1} \cdots d\theta_{r_1+r_2} ,$$

and on $P(\rho)$ the induced measure $dP(\rho)$. Then $\Phi_e(x)$ has a multiplicative Fourier expansion

$$\Phi_e(x) = \sum_{\lambda} \Psi_\lambda(\rho)\lambda(x) , \tag{9}$$

where $\lambda(x) = \lambda_{-a}(x)\overline{\mu_{-b}}(x)$, the sum extends over λ satisfying the condition that λ_a, μ_b are well defined. Moreover

$$\Psi_\lambda(\rho) = c \cdot \int_{E_F \backslash P(\rho)} \Phi_e(x)\bar{\lambda}(x) dP(\rho) , \tag{10}$$

where $c = \left(\int_{E_F \backslash P(\rho)} dP(\rho)\right)^{-1}$.

On the other hand, we see

$$\Phi^*_e(x) = \sum_{\lambda} \hat{\Psi}_\lambda(\rho)\bar{\lambda}(x) , \tag{11}$$

where $\Phi^*_e(x) = \sum_{\varepsilon \in E_F} \Phi^*(\varepsilon x)$. We further note

$$\int_V \Psi_\lambda(\rho)\lambda(x)e(xy)dx = \hat{\Psi}_\lambda(\rho')\bar{\lambda}(y) , \tag{12}$$

where $y = (y^{(1)}, \cdots, y^{(r_1+r_2)})$, $|y^{(p)}| = \rho'_p$ $(p = 1, \cdots, r_1)$, $y^{(p)} = \rho'_p \exp(\sqrt{-1}\theta'_p)$ $(p = r_1 + 1, \cdots, r_1 + r_2)$, $\rho' = \rho'_1 \cdots \rho'^2_{r_1+r_2}$. We assume that $r_1 \neq 0$. Otherwise we can put $\rho = \rho_1 \cdots \rho_{r_2}$ and prove Theorem similarly. We put $\rho'_1 = t$, $\rho'_2 = \cdots = \rho'_{r_1+r_2} = 1$, $\theta'_{r_1+1} = \cdots = \theta'_{r_1+r_2} = 0$ in (12), and take the integral of both sides with respect to t from 0 to ∞ after multiplying them by t^{s-1}. Before proceeding further, we recall the following formulas:

$$J_a(z) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} \exp(\sqrt{-1}(a\theta - z\sin\theta))d\theta\,, \tag{13}$$

$$\int_0^\infty x^\mu J_a(xy)(xy)^{1/2}dx = 2^{\mu+1/2}y^{-\mu-1}\frac{\Gamma(\mu/2 + a/2 + 3/4)}{\Gamma(-\mu/2 + a/2 + 1/4)} \quad (y > 0)\,, \tag{14}$$

$$\int_0^\infty x^{s-1}\cos x\,dx = 2^{s-1}\pi^{1/2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\,, \tag{15}$$

where $J_a(z)$ is the Bessel function. Thus by a direct computation using (13), (14) and (15), we have

$$\begin{aligned} M(\hat{\Psi}_\lambda, s) = \left(\frac{2^{r_1}}{(2\pi)^n}\right)^{s-1/2}\sqrt{-1}^M 2^{-\sqrt{-1}N}\prod_{p=1}^{r_1}\frac{\Gamma(s/2 + \sqrt{-1}b_p/2)}{\Gamma((1-s)/2 - \sqrt{-1}b_p/2)} \\ \times \prod_{p=r_1+1}^{r_1+r_2}\frac{\Gamma(s + |a_p|/2 + \sqrt{-1}b_p/2)}{\Gamma(1 - s + |a_p|/2 - \sqrt{-1}b_p/2)}M(\Psi_\lambda, 1-s)\,, \end{aligned} \tag{16}$$

where $M(\Psi_\lambda, s) = \int_0^\infty \Psi_\lambda(\rho)\rho^s d\rho/\rho$. Combining (16) with (2), we finally obtain

$$M(\hat{\Psi}_\lambda, s)A^L_{(a,b)}(s) = c(L)^{-1}M(\Psi_\lambda, 1-s)B^{L^*}_{(-a,-b)}(1-s)\,. \tag{17}$$

Consider, on the other hand, the inverse Mellin transform of the left hand side of (17) along the line $\mathrm{Re}\, s = 1/2$, and shift the integral to the line $\mathrm{Re}\, s = \sigma_1$ ($\sigma_1 > D$) using the Phramén-Lindelöf theorem. Then, provided that $(a, b) \neq (0, 0)$, the result is

$$\sum_{(m)} c_m\hat{\Phi}_\lambda(m)\,, \tag{18}$$

where $\hat{\Phi}_\lambda(m) = \hat{\Psi}_\lambda(Nm)\bar{\lambda}(m)$. Similarly, if we consider the inverse Mellin transform of the right hand side of (17) along the line $\mathrm{Re}\, s = 1/2$, in turn shift the integral to the line $\mathrm{Re}\, s = 1 - \sigma_1$ ($\sigma_1 > D$) provided that $(a, b) \neq (0, 0)$, then we have

$$c(L)^{-1}\sum_{(\nu)} d_\nu\Phi_\lambda(\nu)\,, \tag{19}$$

where $\Phi_\lambda(\nu) = \Psi_\lambda(N\nu)\lambda(\nu)$. Therefore by (18) and (19), we obtain

$$\sum_{(m)} c_m\hat{\Phi}_\lambda(m) = c(L)^{-1}\sum_{(\nu)} d_\nu\Phi_\lambda(\nu)\,. \tag{20}$$

Even if $(a, b) = (0, 0)$, the same arguments lead to

$$(21)\qquad \begin{aligned}&\sum_{(m)} c_m\hat{\Phi}_1(m) - M(\hat{\Psi}_1, 1)A \\ &\quad = c(L)^{-1}\sum_{(\nu)} d_\nu \Phi_1(\nu) - M(\Psi_1, 1)B\cdot c(L)^{-1}\,,\end{aligned}$$

where we denote by A, B the residue of $A^L_{(0,0)}(s)$, $B^{L^*}_{(0,0)}(s)$ at $s=1$ respectively. Hence by (20), (21), and the definition of $\Phi_e(x)$, we obtain

$$(22)\qquad \begin{aligned}&\sum_{m\in L-\{0\}} c_m\Phi^*(m) + M(\Psi_1, 1)B\cdot c(L)^{-1} \\ &\quad = \sum_{\nu\in L^*-\{0\}} d_\nu \Phi(\nu) + M(\hat{\Psi}_1, 1)A\,.\end{aligned}$$

We can simplify the expression $M(\Psi_1, 1)B\cdot c(L)^{-1}$. It follows from (10) that

$$(23)\qquad \begin{aligned}M(\Psi_1, 1) &= \int_0^\infty \Psi_1(\rho)\rho\frac{d\rho}{\rho} \\ &= \int_0^\infty c\cdot\int_{E_F\backslash P(\rho)} \Phi_e(x)dP(\rho)\rho\frac{d\rho}{\rho} \\ &= c\int_0^\infty\int_{P(\rho)} \Phi(x)\,dP(\rho)\frac{d\rho}{\rho} \\ &= c\cdot\Phi^*(0)\,.\end{aligned}$$

Similarly we have

$$(24)\qquad M(\hat{\Psi}_\lambda, 1) = c\cdot\Phi(0)\,.$$

Putting $c_0 = c\cdot B\cdot c(L)^{-1}$, $d_0 = c\cdot A\cdot c(L)$ and using (23) and (24), we obtain

$$(25)\qquad \sum_{m\in L} c_m\Phi^*(m) = c(L)^{-1}\sum_{\nu\in L^*} d_\nu\Phi(\nu)\,.$$

Next we shall show that the coefficients c_m, d_ν in (25) are all equal. To do this we define $\psi_0(x)$ as a function on V which vanishes except in a sufficiently small neighborhood of $x=0$ and satisfy the condition that $\psi_0(0)=1$. For $\ell\in L^*$, we put $\psi_\ell(x)=\psi_0(x-\ell)$, and put $\psi_\ell(x)=\psi_0(x-\ell)$ in (25). Then we have

$$\begin{aligned}c(L)^{-1}d_\ell &= c(L)^{-1}\sum_{\nu\in L^*} d_\nu\psi_\ell(\nu) \\ &= \sum_{m\in L} c_m\psi_\ell^*(m)\,.\end{aligned}$$

On the other hand, we see

$$\begin{aligned}\psi_\ell^*(m) &= \int_V \psi_\ell(x)e(xm)dx \\ &= \int_V \psi_0(x-\ell)e(xm)dx\end{aligned}$$

$$= \int_V \psi_0(t) e(tm) dt\, e(\ell m)\ ,$$
$$= \psi_0^*(m)$$

with $t = x - \ell$, and note $e(lm) = 1$ for all $m \in L$. Therefore we have $d_\ell = d_0$. Similarly, using $\Phi^{**}(x) = \Phi(-x)$, we have $c_m = c_0$ for all $m \in L$. Thus we obtain

$$c_0 \sum_{m \in L} \Phi^*(m) = c(L)^{-1} d_0 \sum_{\nu \in L^*} \Phi(\nu)\ .$$

On the other hand, since $\sum_{m \in L} \Phi^*(m) = c(L)^{-1} \sum_{\nu \in L^*} \Phi(\nu)$ holds, we have $c_0 = d_0$. This completes the proof of Theorem.

Remarks. 1. In the special case where $F =$ the field of rational numbers and $L =$ the ring of rational integers, our theorem reduces to Hamburger's theorem.

2. According to Hecke's theorem (Satz 176 in Hecke [5]), the different $\mathfrak{A}$ of a field F can be expressed as $e\mathfrak{A}^2$, where $e \in F$, $\mathfrak{A}$ is an ideal of F. Hence if F is a totally imaginary field and $L = (\sqrt{e}\,\mathfrak{A})^{-1}$, we obtain $L = L^*$.

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