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STABLE VECTOR BUNDLES ON QUADRIC HYPERSURFACES

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§0.

Barth, Hulek and Maruyama have showed that the moduli of stable rank 2 vector bundles on $\boldsymbol{P}^2$ are nonsingular rational varieties. There are also many examples of stable rank 2 vector bundles on $\boldsymbol{P}^3$. On the other hand, there is essentially only one example of rank 2 bundles on $\boldsymbol{P}^4$, which is constructed by Horrocks and Mumford. We hope the study of rank 2 bundles on hypersurfaces in $\boldsymbol{P}^4$ may give more insight to the study of vector bundles on $\boldsymbol{P}^4$. In this paper, we establish some general properties of stable rank 2 bundles on quadric hypersurfaces. We show the restriction theorem (1.4), (1.6), the existence of the spectrum (2.2), and the vanishing theorem (2.4), are also true for the stable rank 2 reflexive sheaves on quadric hypersurfaces just as in the case when the base variety is $\boldsymbol{P}^n$. Though the methods to prove such results are similar to those we use for projective spaces, there are some technical difficulties. We should also mention that we shall always assume the base field is characteristic 0 and algebraically closed, and we shall use the definition of stability introduced by Mumford and Takemoto.

§1.

We let Q_n be a nonsingular quadric hypersurface in $\boldsymbol{P}^{n+1}$. There is the following incidence correspondence:

$$\text{(1.A)}\qquad \begin{array}{ccc} X & \xrightarrow{q} & Y \subseteq G(1,4) \\ {\scriptstyle p}\downarrow & & \\ Q_3 & & \end{array}$$

where Y is the subvariety in $G(1, 4)$, which corresponds to the set of lines in Q_3. X is the corresponding universal $\boldsymbol{P}^1$-bundle on Y. It is not hard to check that X a conic bundle over Q_3.

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LEMMA 1.1. *Let* $x \in Y$ *and* $L = q^{-1}(x)$. *Then*

$$\Omega^1_{X/Q_3}|_L \cong \mathcal{O}_{P^1}(1) .$$

Proof. There are the following exact sequences:

$$0 \longrightarrow I_L/I_L^2 \longrightarrow \Omega^1_{X/k}|_L \longrightarrow \Omega^1_{L/k} \longrightarrow 0$$

and

$$0 \longrightarrow p^*\Omega^1_{Q_3/k}|_L \longrightarrow \Omega^1_{X/k}|_L \longrightarrow \Omega^1_{X/Q_3}|_L \longrightarrow 0 .$$

Because $I_L/I_L^2 \cong 3\mathcal{O}_L$ and $\Lambda^3\Omega^1 Q_{3/k}|_L \cong \mathcal{O}_{P^1}(-3)$, it is easy to show $\Omega^1_{X/Q_3}|_L \cong \mathcal{O}_L(1)$ by Chern classes computation.

DEFINIITON 1.2 (Mumford-Takemoto). A reflexive sheaf E of rank r on Q_n is stable (respectively, semistable) if for every proper subsheaf F $(1 \leq \operatorname{rank} F \leq \operatorname{rank} E - 1)$,

$$\frac{c_1(F)\cdot c_1(\mathcal{O}_{Q_n}(1))^{n-1}}{\operatorname{rank} F} < \frac{c_1(E)\cdot c_1(\mathcal{O}_{Q_n}(1))^{n-1}}{\operatorname{rank} E}$$

(respectively, $\leq$).

PROPOSITION 1.3. *Let E be a semistable reflexive sheaf of rank r on Q_3. Suppose that E_L, the restriction of E to a general line L, is isomorphic to $\oplus_{j=1}^r \mathcal{O}_{P^1}(a_j)$ with $a_1 \leq a_2 \leq \cdots \leq a_r$. Then $a_{i+1} - a_i \leq 1$ $(1 \leq i \leq r-1)$.*

Proof. The proof is fairly standard ([6]) and we shall merely give a sketch. We shall use the notations in the diagram 1.A. Suppose $a_{i+1} - a_i > 1$. It is easy to construct a sheaf $F \subseteq p^*E$ of rank i such that $F|_L \cong \oplus_{j=1}^i \mathcal{O}(a_j)$ and $p^*E/F|_L = G|_L \cong \oplus_{j=r+1}^r \mathcal{O}(a_j)$.

Let $U \subseteq Q_3$ be the open set where E is locally free. There is a set $V \subseteq X$ such that $F|_V$ and $G|_V$ are locally free. By the universal property of $Gr(E_U)$, there is a morphism $f\colon V \to Gr(E_U)$

$$\begin{array}{ccc} V & \subseteq & X \\ \downarrow{\scriptstyle p} & & \downarrow \\ U & \subseteq & Q_3 \end{array}$$

such that G is the pull back of the universal quotient bundle. Since $f^*\Omega^1_{Gr/U}|_L \cong F \otimes G^*|_L$ and $a_{i+1} - a_i > 1$,

$$df\colon f^*\Omega^1_{Gr/U}|_L \longrightarrow \Omega^1_{v/U}|_L \cong \mathcal{O}_L(1)$$

(Lemma 1.1) is the zero map. So $df = 0$ therefore f factors through p and E will have a subsheaf F', with the property $F'|_L \cong \oplus_{j=1}^{i} \mathcal{O}_L(a_j)$, which contradicts the fact that E is semistable.

PROPOSITION 1.4. *Let E be a semistable vector bundle of rank r on Q_2. Suppose E_H, the restriction of E to a general hyperplane section (conic), is isomorphic to $\oplus_{i=1}^{r} \mathcal{O}_{P^1}(a_i)$ with $a_1 \leq a_2 \leq \cdots \leq a_r$. Then $a_{i+1} - a_i \leq 1$.*

Proof. The proof is similar to the proof of Proposition 1.3. We shall leave it to the readers.

COROLLARY 1.5. *Let E be rank 2 semistable rank 2 vector bundle on Q_2, such that $\Lambda^2 E \cong \mathcal{O}_{Q_2}$ (respectively, $\Lambda^2 E \cong \mathcal{O}_Q(-1)$). Then E_H, the restriction of E to a general hyperplane section (conic), is isomorphic to $\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}$ (respectively, $\mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-1)$).*

Let $[Q_2]$, $[L]$, and $[P]$ (quadric surface, line, and point) be the generators of $H^2(Q_3, Z)$, $H^4(Q_3, Z)$, and $H^6(Q_3, Z)$ respectively. There are the following relationships: $[Q_2]^2 = 2[L]$ and $[Q_2]\cdot[L] = 2[P]$. If F is a coherent sheaf on Q_3, the Chern polynomial of F is of the form $1 + c_1(F)[Q_2]t + c_2(F)[L]t^2 + c_3(F)[P]t^3$. Often we shall simply call $c_i(F) = c_i$ the Chern classes of F.

It is not hard to show the following Riemann Roch formula:

$$\chi(F) = \tfrac{1}{6}(2c_1^3 - 3c_1c_2 + 3c_3) + \tfrac{3}{2}(c_1^2 - c_2) + \tfrac{13}{6}c_1 + \operatorname{rank}(F) .$$

If rank $(F) = 2$, there are the following congruence relationships:

(a) $c_1 = 0$, then $c_2 = c_3 \pmod 2$

(b) $c_1 = -1$, then $c_3 = 0 \pmod 2$.

If E is a rank 2 reflexive sheaf on Q_3, then as in [7] one can check that $c_3(E)$ = number of points where E fails to be locally free.

THEOREM 1.6. *Let E be a rank 2 stable reflexive sheaf on $Q_n (n \geq 3)$. Then E_H, the restriction of E to a general hyperplane, is again stable.*

Proof. By tensoring E by $\mathcal{O}_{Q_n}(t)$ if necessary, we may assume that $\Lambda^2 E$ is either $\mathcal{O}_{Q_n}$ or $\mathcal{O}_{Q_n}(-1)$. We shall work out the details for the case $\Lambda^2 E = \mathcal{O}_{Q_n}$. Since the proof of the other case is similar, we shall omit it.

We consider the following incidence correspondence. First we'll show $h^0(E_H) = 0$.

$$\begin{array}{ccc} P(T_{P^{n+1}}(-1)|_{Q_n}) \cong X & \xrightarrow{q} & P^{n+1*} . \\ \downarrow p & & \\ Q_n & & \end{array}$$

For $n \geq 4$, this will show that E_H is stable. Let a be the integer such that $h^0(E_H(a)) \neq 0$ and $h^0(E_H(a-1)) = 0$. Suppose $a \leq 0$ and we shall derive a contradiction. Because $q_* p^* E(a) \neq 0$ and torsion free, there is the least integer t such that $h^0(q_* p^* E(a) \otimes \mathcal{O}_{P^{n+1}}(t)) \neq 0$. We can construct the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-a, -t) \longrightarrow p^* E \longrightarrow I_Z(a, t) \longrightarrow 0 .$$

(We are using the notation $\mathcal{O}_X(-a, -t)$ for $p^* \mathcal{O}_{Q_n}(-a) \otimes q^* \mathcal{O}_{P^{n+1*}}(-t)$.) By the universal property of $P_{Q_n}(E)$, there is a map $f: X - Z \to P_{Q_n}(E)$ such that $f^* \mathcal{O}_{P(E)}(1) \cong \mathcal{O}_{X-Z}(a, t)$ and $f^* \Omega^1_{P(E)/Q_n} \cong \mathcal{O}_{X-Z}(-2a, -2t)$. According to Propositions 3.1 and 3.2 in [5], these are the following diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^* \Omega^1_{P(E)/Q_n} & \longrightarrow & p^* E \otimes f^* \mathcal{O}_{P(E)}(-1) & \longrightarrow & f^* \mathcal{O}_{P(E)} & \longrightarrow & 0 \\ & & \downarrow df & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega^1_{X/Q_n}|_U & \longrightarrow & p^* T_{P^{n+1}}(-1) \otimes \mathcal{O}_{P(T(-1))}(-1)|_U & \longrightarrow & \mathcal{O}_U & \longrightarrow & 0 \end{array}$$

where $U = X - Z$. Also

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & \mathcal{O}_H \oplus \Omega^1_{P^n}(1)|_H & \longrightarrow & \Omega^1_{P^n}(1)|_H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \psi & & \\ 0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & E_H(a) & \longrightarrow & I_{Z_H}(2a) & \longrightarrow & 0 \end{array}$$

where rank $\psi \geq 1$ and $Z_H = Z \cap H$. However, $\operatorname{Hom}(\Omega^1_{P^n}(1)|_H, I_{Z_H}(2a)) \subseteq \operatorname{Hom}(\Omega^1_{P^n}(1)|_H, \mathcal{O}_H(2a)) = H^0(T_{P^n}(2a-1)|_H)$. If $a < 0$, then $H^0(T_{P^n}(2a-1)|_H) = 0$. So we may assume $a = 0$. Since $H^0(T_{P^n}(-1)) \cong H^0(T_{P^n}(-1)|_H)$, any section of $T_{P^n}(-1)|_H$ is either nowhere vanishing or vanishing at one point. If $n \geq 4$ and $\dim Z_H \geq 1$, then $\psi = 0$ and we have a contradiction. On the other hand, if $Z_H = \phi$, $E_H \cong 2\mathcal{O}_H$. In this case, $H^1(E_H(m)) = 0$ for all m. Therefore $H^1(E(m)) = 0$ for all m. Since $H^1(E(-(1)) = 0$, the restriction map $\operatorname{Hom}(2\mathcal{O}_{Q_n}, E) \to \operatorname{Hom}(2\mathcal{O}_H, E_H)$ is surjective. Hence $H^0(E) \neq 0$, E is not stable. The remaining case is when $n = 3$ and Z_H is a simple point. In this case, $c_2(E) = 1$ and $h^1(E_H(m)) = 0$ for $m \neq -1$. Therefore $H^1(E(m)) = 0$ for $m \leq -2$. From the exact sequence, $0 \to E(t) \to E(t+1) \to E_H(t+1) \to 0$, we find $h^1(E(-1)) \leq 1$. Since $h^0(E) = 0$ and

$h^0(E_H) = 1$, $h^1(E(-1)) \geq 1$. Then $h^1(E(-1)) = 1$ and $h^1(E(m)) = 0$ for $m \geq 0$. Thus $H^0(E(1)) \to H^0(E_H(1))$ is surjective. There is an exact sequence $H^0(E(1)) \otimes \mathcal{O}_{Q_3} \to E(1) \to F \to 0$ with $\dim \operatorname{supp} F \leq 0$. Let C be the zero set of a general section of $E(1)$. Then C has at most finite number of singularities. The extension, $0 \to \mathcal{O}_{Q_3} \to E(1) \to I_C(2) \to 0$, is given by an element in $\operatorname{Ext}^1(I_C(2), \mathcal{O}_{Q_3}) \cong \operatorname{Ext}^2(\mathcal{O}_C(2), \mathcal{O}_{Q_3}) \cong H^0(\omega_C(1))$. Furthermore, C is not contained in any hyperplane, $\deg C = 3$, and $h^1(I_C) = 1$. So C has two connected components and one of the components is a line L. But $h^0(\omega_L(1)) = 0$. This contradicts the fact E is reflexive, hence it can only fail to be locally free at finite number of points. So far we have shown $h^0(E_H) = 0$. It remains to show that E_H is stable when $n = 3$. By Proposition 1.3, we may assume the restriction of E to a general line in either rulings is isomorphic to $\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}$. If there is an injective map $g: \mathcal{O}_H(a, b) \to E_H$ with $a + b \geq 0$ and $h^0(\mathcal{O}_H(a, b)) = 0$, then we may assume $a > 0$ and $b < 0$. Then the restricting g to a general line in the second ruling, we get a nonzero map $\mathcal{O}_{P^1}(a) \to 2\mathcal{O}_{P^1}$ which is a contradiction. So E_H is stable.

§2.

Let E be a rank 2 stable reflexive sheaf on Q_3 with $c_1(E) = 0$, $c_2(E) = c_2$. We shall show that there is a vector bundle $\mathcal{H}$ of rank c_2 on P^1, which we shall call the spectrum of E, such that $h^1(E(m)) = h^0(\mathcal{H}(m + 1))$ for $m \leq -1$. This is analogous to the results of Barth, Elencwajg, and Hartshorne ([3] and [7]) about the spectrums of stable reflexive sheaves of rank 2 on P^3. The methods to prove this result are similar to those given by Hartshorne in [7]. We shall need the following technical lemma.

Let $\mathcal{O}_{Q_2}(1, 0)$ and $\mathcal{O}_{Q_2}(0, 1)$ be the line bundles associated with the rulings. In general, we write $\mathcal{O}_{Q_2}(a, b)$ for $\mathcal{O}_{Q_2}(a, 0) \otimes \mathcal{O}_{Q_2}(0, b)$.

LEMMA 2.1. *Let E be a rank 2 vector bundle on Q_2.*

$$M = \bigoplus_{m \in Z} H^1(E \otimes \mathcal{O}_{P^3}(m)|_{Q_2}) .$$

Suppose $N \subseteq M$ be a graded submodule. Let $n_t = \dim N_t$ where N_t is the graded component of N in degree t.

(1) (a) *If E is semistable and $\Lambda^2 E \cong \mathcal{O}_{Q_2}$ (respectively, $\mathcal{O}_{Q_2}(-1, -1)$) then $n_t \leq n_{t+1}$ for $t \leq -2$ (respectively, -1).*

(b) *If $n_t \neq 0$, then $1 + n_t \leq n_{t+1}$ for $t \leq -3$ (respectively, $t \leq -2$).*

(2) *If E is semistable and $\Lambda^2 E \cong \mathcal{O}_{Q_2}$ (respectively, $\mathcal{O}_{Q_2}(-1, -1)$), and $n_{-1} \neq 0$ (respectively, $0 < n_{-1} < h^1(E(-1))$), then $1 + n_{-2} \leq n_{-1}$ (respectively, $1 + n_{-1} \leq n_0$).*

(3) *Suppose E is semistable and $\Lambda^2 E \cong \mathcal{O}_{Q_2}$ (respectively, $\mathcal{O}_{Q_2}(-1, -1)$). If $n_{t_0} \neq 0$ and $1 + n_{t_0} = n_{t_0+1}$ for some $t_0 \leq -3$ (respectively, $t_0 \leq -2$), then there is a nonzero element $s \in H^0(\mathcal{O}_{Q_2}(1, 0))$ or $H^0(\mathcal{O}_{Q_2}(0, 1))$ such that $x \cdot N_m = 0$ for $m \leq t_0 + 1$ where $x \in H^0(\mathcal{O}_{Q_2}(1, 1))$ and $s \mid x$. Furthermore, if $y \in H^0(\mathcal{O}_{Q_2}(1, 1))$ is not divisible by s, then $N_m \xrightarrow{y} N_{m+1}$ is injective for $m \leq t_0 + 1$.*

Proof. (1) (a) Suppose H is a general hyperplane section of Q_2 defined by an element $x \in H^0(\mathcal{O}_{Q_2}(1))$, such that $E_H \cong \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}$ (Corollary 1.5). There is the exact sequence $0 \to E(t) \to E(t+1) \to E_H(t+1) \to 0$. Since $h^0(E_H(t+1)) = 0$ for $t \leq -2$, the map $N_t \xrightarrow{x} N_{t+1}$ is injective and $n_t \leq n_{t+1}$.

(b) We shall prove (b) by induction on the length of N. If $n_{t+1} < 3 + n_t$, $(t \leq -3)$ then by the bilinear map lemma ([7], Lemma 5.1), there is an element $x \in H^0(\mathcal{O}_{Q_2}(1))$, such that $N_t \xrightarrow{x} N_{t+1}$ is not injective. Let C be the conic defined by x.

Because $N_t \xrightarrow{x} N_{t+1}$ is not injective, $h^0(E_C(t+1)) \neq 0$. There is the exact sequence:

$$\bigoplus_j H^0(E_C(j)) \xrightarrow{\delta} \bigoplus_j H^1(E(j-1)) \xrightarrow{x} \bigoplus H^1(E(j)) .$$

Set $U = \operatorname{Im}(\delta) \cap N(-1)$ and $V = \operatorname{Im}(x|_{N(-1)})$. Then there are the following exact sequences: $0 \to U \to N(-1) \to V \to 0$ and $0 \to V \to N \to N/V \to 0$. We notice that V is also a submodule of N. Furthermore, $U_t = \delta\delta^{-1}(N_t)$ for $t \leq 0$.

Case 1. Assume C is nonsingular. Then $E_C \cong \mathcal{O}_{P^1}(a) \oplus \mathcal{O}_{P^1}(-a)$ with $2t + 2 + a \geq 0$. Also it is easy to check that $\dim U_j + 2 \leq \dim U_{j+1}$ if $U_j \neq 0$ and $j \leq 1$. Now $\dim U_t + \dim V_{t+1} = \dim N_t$ and $\dim U_{t+1} + \dim V_{t+2} = \dim N_{t+1}$. By (a), $\dim V_{t+1} \leq \dim V_{t+2}$. Thus $\dim N_{t+1} \geq 2 + \dim N_t$. (2.1.1.A)

Case 2. Assume $C = L_1 + L_2$ (union of two lines). There is an exact sequence

(2.1.1.B) $$0 \longrightarrow E_{L_1}(t) \longrightarrow E_C(t+1) \longrightarrow E_{L_2}(t+1) \longrightarrow 0 .$$

In this case, it is possible $\delta^{-1}(N_{t+1}) \subseteq H^0(E_{L_1}(t+1))$. So we can only conclude $\dim U_{t+1} \geq 1 + \dim U_t$. Hence $\dim N_{t+1} \geq 1 + \dim N_t$. The proof for the Case, $\Lambda^2 E \cong \mathcal{O}_{Q_2}(-1)$, is similar and we will omit it.

(2) If $n_{-2} + 3 > n_{-1}$, then we can find an element $0 \neq x \in H^0(\mathcal{O}_{Q_2}(1))$ such that $N_{-2} \xrightarrow{x} N_{-1}$ is not injective. Let C be the conic defined by x.

Case 1. Assume C is nonsingular.

Then $E|_C \cong \mathcal{O}_{P^1}(a) \oplus \mathcal{O}_{P^1}(-a)$ with $a \geq 2$, because $h^0(E_C(-1, -1)) \neq 0$. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E(-1) & = & E(-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & \mathcal{O}_{P^1}(-a) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{P^1}(a) & \longrightarrow & E_C & \longrightarrow & \mathcal{O}_{P^1}(-a) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} .$$

Since E is stable, then E' is semistable and $h^0(E) = 0$. Furthermore $\Lambda^2 E' \cong \mathcal{O}_{Q_2}(-1)$. There is the following exact sequence

$$\bigoplus_t H^0(\mathcal{O}_{P^1}(a + 2t)) \xrightarrow{\delta} \bigoplus_t H^1(E(-1 + t)) \xrightarrow{\alpha} \bigoplus_t H^1(E'(t)) .$$

Set $A = (N) \cap \operatorname{Im} \delta$ and $B = \operatorname{Im}(\alpha|_N)$. Then B is a submodule of $H^1(E'(t))$. There are the following exact sequences:

$$0 \longrightarrow A_{-2} \longrightarrow N_{-2} \longrightarrow B_{-1} \longrightarrow 0$$

and

$$0 \longrightarrow A_{-1} \longrightarrow N_{-1} \longrightarrow B_0 \longrightarrow 0 .$$

$\dim N_i = \dim A_i + \dim B_{i+1}$ (for $i = -1$ and -2). Also $\dim A_{-1} \geq 2 + \dim A_{-2}$ and $\dim B_0 \geq \dim B_{-1}$ (Lemma 2.1 (1) (*a*)). Thus $\dim N_{-1} \geq 2 + \dim N_{-2}$.

Case 2. Assume $C = L_1 + L_2$.

Since $h^0(E_C(-1)) \neq 0$, either $E|_C \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-1)$ or we may assume $E_{L_1} = \mathcal{O}_{L_1}(a) \oplus \mathcal{O}_{L_1}(-a)$ with $a \geq 2$. In the case $E_C \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-1)$, the argument given for the case that C is nonsingular will also work here. So let us assume $E_{L_1} = \mathcal{O}_{L_1}(a) \oplus \mathcal{O}_{L_1}(-a)$ with $a \geq 2$. Without losing generality, we may assume $\mathcal{O}_{Q_2}(L_1) \cong \mathcal{O}_{Q_2}(1, 0)$. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E(-1,-1) & = & E(-1,-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E' & \longrightarrow & E(0,-1) & \longrightarrow & \mathcal{O}_{L_1}(-a-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{L_1}(a-1) & \longrightarrow & E_{L_1}(0,-1) & \longrightarrow & \mathcal{O}_{L_1}(-a-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & .
\end{array}$$

There is the exact sequence

$$H^0(\mathcal{O}_{L_1}(a-1+t)) \xrightarrow{\delta'} \bigoplus_t H^1(E^0(-1+t)) \xrightarrow{\alpha'} H^1(E'(t)) .$$

Set $A' = N \cap \operatorname{Im} \delta$ and $B' = \operatorname{Im} \alpha|_N$. Then

$$0 \longrightarrow A'_{-2} \longrightarrow N_{-2} \longrightarrow B'_{-1} \longrightarrow 0$$
$$0 \longrightarrow A'_{-1} \longrightarrow N_{-1} \longrightarrow B'_0 \longrightarrow 0 .$$

Since $E'_H \cong \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-2)$ for a general hyperplane section H, $\dim B'_{-1} \leq \dim B'_0$. Also $\dim A'_{-1} \geq 1 + \dim A'_0$. Thus $\dim N_{-1} \geq 1 + \dim N_{-2}$.

(3) Using the same notations as in the proof of 1 (b), we see $C = L_1 + L_2$ (2.1.1.A).

Also $\dim V_{t_0+2} = 0$ (Lemma 2.1 (1) (b) and (2)). So $N_j \cong U_j$ for $j \leq t_0 + 1$. Then the section $s \in H^0(\mathcal{O}_{Q_2}(1, 0))$ which define L_1 will have the desired property.

THEOREM 2.2. *Let E be a stable reflexive sheaf of rank 2 with $c_1(E) = 0$. (respectively, -1) on Q_3. Then there is a unique set of integers $\{k_i\}_{i=1,2,\dots,c_2}$ (respectively, $\{k_i\}_{i=1,2,\dots,c_2-1}$) called the spectrum of E with the following properties: Let $\mathcal{H} = \oplus_i \mathcal{O}_{P^1}(k_i)$. Then*

(1) (a) $h^1(Q_3, E(\ell)) = h^0(\mathcal{H}(\ell + 1))$ *for* $\ell \leq -1$ (*respectively*, $\ell \leq 0$).

(b) $h^2(E(\ell)) = h^1(\mathcal{H}(\ell + 1))$ *for* $\ell \geq -2$.

(2) (a) *If an integer* $k > 0$ *is in the spectrum, then* $0, 1, \cdots, k$ *are also in the spectrum.*

(b) *If* $k < 0$ *is in the spectrum, then* $-1, -2, \cdots, k$ *are also in the spectrum.*

(3) $\sum k_i = (-c_3 + c_2)/2$ *(respectively,* $-c_3/2 - c_2 + 1$*).*

(4) *If* E *is a vector bundle, then* $\{-k_i - 1\} = \{k_i\}$ *(respectively,* $\{-k_i - 2\} = \{k_i\}$*).*

Proof. (1) Let Q_2 be a general hyperplane section of Q_3 defined by $x \in H^0(\mathcal{O}_{Q_3}(1))$, such that E_{Q_2} is a stable vector bundle (Theorem 1.6). Let $M = \oplus_j H^1(E_{Q_2}(j))$ and $N \subseteq M$ be the image of $\oplus_j H^1(E(j))$ under the restriction map. For $\ell \leq 0$, there is the following exact sequence,

$$0 \longrightarrow H^1(E(\ell - 1)) \longrightarrow H^1(E(\ell)) \longrightarrow N_\ell \longrightarrow 0 .$$

So $\dim N_\ell = h^1(E(\ell)) - h^1(E(\ell - 1))$. We are looking for the set of integers $\{k_i\}$ satisfying (a) and (b). Condition (a) can be expressed as $h^1(E(\ell)) = \sum_{k \geq -\ell-1} (k_i + \ell + 2)$. So $\dim N_\ell = \sum_{k_i \geq -\ell-1} 1 = \#\{k_i \geq -\ell - 1\}$. Since

$$\dim N_\ell - \dim N_{\ell-1} \geq 0 \qquad \text{for } \ell \leq -1$$

(respectively, $\ell \leq 0$) by Lemma 2.1, it is possible to find those integers k_i satisfying condition (a). Furthermore those $k_i \geq 0$ (respectively ≥ -1) are determined by (a). For (b), let R be the kernel of

$$\bigoplus_j H^2(E(j)) \xrightarrow{x} \bigoplus_j H^2(E(j + 1)) .$$

For $\ell \geq -2$, there is the following exact sequence

$$0 \longrightarrow R_{\ell+1} \longrightarrow H^2(E(\ell)) \longrightarrow H^2(E(\ell + 1)) \longrightarrow 0 .$$

So $\dim R_{\ell+1} = h^2(E(\ell)) - h^2(E(\ell + 1))$. Furthermore R is a quotient module of $\oplus_\ell H^1(Q_2, E_{Q_2}(\ell))$. Now condition (b) can be expressed as $h^2(E(\ell)) = \sum_{k_i \leq -\ell-3} -(k_i - \ell - 2)$ for $\ell \geq -2$. $\dim R_{\ell+1} = h^2(E(\ell)) - h^2(E(\ell + 1)) = \sum_{k_i \leq -\ell-3} 1 = \#\{k_i \leq -\ell - 3\}$. Since $\dim R_{\ell+1} - \dim R_{\ell+2} \geq 0$ by the dual formulation of Lemma 2.1 (a), we can find integers $\{k_i\}$ satisfying (b). Those integers $k_i \leq -1$ are uniquely determined by (b). If $c_1(E) = -1$, then we have two determination of $\#\{k_i\} = -1$, we have to check that they agree. (i.e., $\dim N_0 - \dim N_{-1} = \dim R_{-1} - \dim R_0$). But this is clear, because $\dim N_0 + \dim R_0 = \dim N_{-1} + \dim R_{-1} = h^1(E_{Q_2}(-1)) = h^1(E_{Q_2})$. Now $\dim N_{-1} + \dim R_{-1} = \#\{k_i\} = h^1(E_{Q_2}(-1)) = c_2$ (respectively, $c_2 - 1$). So there are c_2 (respectively, $c_2 - 1$) integers in total.

(2) This follows from the fact that if $N_\ell \neq 0$ and $\ell \leq -2$, then $\dim N_{\ell+1} - \dim N_\ell > 0$ (Lemma 2.1.1 (b) and (2)). Similarly by the dual formulation, if $\dim R_{\ell+1} \neq 0$, then $\dim R_\ell - \dim R_{\ell+1} > 0$ for $\ell \geq -1$.

(3) Finally the proof for (3) and (4) are similar to the proof given in Propositions 7.2 and 7.3 in [7]. We shall omit it here.

THEOREM 2.3. *Let E be a stable rank 2 reflexive sheaf with $c_1(E) = 0$ (respectively, $c_1(E) = -1$) on Q_3. Let $\{k_i\}_{i=1,\cdots,c_2}$ (respectively, $\{k_i\}_{i=1,\cdots,c_2-1}$) be its spectrum.*

(a) *If $k > 1$ (respectively, $k \geq 1$) occurs in the spectrum, then $1, \cdots, k - 1$ occur at least twice (respectively, $0, 1, \cdots, k-1$).*

(b) *If $k < -2$ occurs in the spectrum, then $k+1, \cdots, -2$ occur at least twice in the spectrum.*

Proof. (a) Let Q_2 be a general hyperplane section of Q_3 defined by an element $\ell_1 \in H^0(\mathcal{O}_{Q_3}(1))$, such that E_{Q_2} is a stable vector bundle. Let $N \subseteq \oplus_j H^1(E_H(j))$ be the image of $\oplus_j H^1(E(j))$ under the restriction map. If for some t $(1 \leq t \leq k-1)$ occurs only once in the spectrum, then $\dim N_{-t-1} - \dim N_{-t-2} = 1$ by the construction in Theorem 2.2. By Lemma 2.1 (3), there is an element $s \in H^0(\mathcal{O}_{Q_2}(1, 0))$ such that $xN_m = 0$ for $m \leq -t-1$ for any $x \in H^0(\mathcal{O}_{Q_2}(1, 0))$ where $s \mid x$. Let $\ell_2 \in H^0(\mathcal{O}_{Q_3}(1))$, such that ℓ_2 is independent from ℓ_1 and $\ell_2|_{Q_2}$ is divisible by s. Let m_0 be the smallest integer such that $H^1(E(m_0)) \neq 0$. Consider $V = \operatorname{span}\{\ell_1, \ell_2\}$ and the multiplication map $\alpha\colon V \otimes H^1(E(m_0)) \to H^1(E(m_0+1))$. Consider the following diagram:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & & H^1(E(m_0)) \\ \downarrow & & \downarrow{\scriptstyle \ell_1} \\ H^1(E(m_0)) & \xrightarrow{\ell_2} & H^1(E(m_0+1)) \\ \downarrow & & \downarrow \\ N_{m_0} & \xrightarrow{\overline{\ell}_2} & N_{m_0+1} \end{array}$$

Since $\overline{\ell}_2 \cdot N_{m_0} = 0$, ℓ_2 maps $H^1(E(m_0))$ into the image of ℓ_1. So $\dim \operatorname{Im} \alpha = h^1(E(m_0))$. By the bilinear map lemma, ([7], Lemma 5.1) there is an element $\ell_2' \in V$, such that

$$H^1(E(m_0)) \xrightarrow{\ell_2'} H^1(E(m_0+1))$$

is not injective. Let Q_2' be another general hyperplane section of Q_3, such that $E_{Q_2'}$ is stable and $\ell_2' | Q_2'$ defined a nonsingular conic. By the same argument, there is an element $s' \in H^0(\mathcal{O}_{Q_2'}(1, 0))$ such that if $x \in H^0(\mathcal{O}_{Q_2'}(1))$ is divisible by s, then $x \cdot N_m' = 0$ for $m \leq -t - 1$, where N' is the image of $\oplus_j H^1(E(j))$ in $\oplus_j H^1(E_{Q_2'}(j))$. Consider the following diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} \ell_2' & \longrightarrow & H^1(E(m_0)) & \xrightarrow{\ell_2'} & H^1(E(m_0+1)) \\ & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ & & & & N'_{m_0} & \xrightarrow{\overline{\ell_2'}} & N'_{m_0+1} \end{array}$$

Since ℓ_2' defines a nonsingular conic, ℓ_2' is not divisible by s' and $\overline{\ell_2'}: N'_{m_0} \to N'_{m_0+1}$ is injective (Lemma 2.1 (3)). Now $\overline{\ell_2'} \circ \pi_1$ is injective but $\pi_2 \ell_2'$ is not injective, which is a contradiction.

COROLLARY 2.4. *Let E be a stable rank 2 vector bundle on Q_3 with $c_1(E) = 0$ (respectively, $c_1(E) = -1$). Then $h^1(E(\ell)) = 0$ for $\ell \leq (-c_2/4) - 1$ (respectively, $-c_2/4$).*

Proof. Using Theorem 2.2 and Theorem 2.3, we see it will be sufficient to show $S = \{-k-3, -k-2, -k-2, -k-1, -k-1, \cdots, -2, -2, -1, 0, 1, 1, \cdots, k, k, k+1\}$ is not spectrum of a stable rank 2 vector bundle on Q_3. Suppose for contradiction that S is the spectrum of E. Then $c_2(E) = 4k + 4$. Also $h^1(E(-k-2)) = 1$ and $h^1(E(-k-1)) = 4$. So the map $H^0(\mathcal{O}_{Q_3}(1)) \otimes H^1(E(-k-2)) \to H^1(E(-k-1))$ is not injective. Let $x \in H^0(\mathcal{O}_{Q_3}(1))$ such that

$$H^1(E(-k-2)) \xrightarrow{x} H^1(E(-k-1))$$

is not injective. Let Q_2 be the corresponding hyperplane. Since

$$h^0(E_{Q_2}(-k-1)) \neq 0\,, \qquad h^0(E_{Q_2}) \geq 1 + (2k+2) + (k+1)^2\,.$$

On the other hand, $h^1(E(-1)) = (2k+2) + (k+1)^2$ from the spectrum. By considering the exact sequence $H^0(E) \to H^0(E_{Q_2}) \to H^1(E(-1))$, we see $h^0(E) \neq 0$ which is a contradiction.

Remark. The following example will show the bound given in Corollary 2.4 is the best possible. Let Z be two disjoint conic in Q_3. Using Z we can construct the following extension,

$$0 \longrightarrow \mathcal{O}_{Q_3} \longrightarrow F(1) \longrightarrow I_Z(2) \longrightarrow 0$$

where $c_1(F) = 0$ and $c_2(F) = 2$. Let $Y = Y_1 \coprod Y_2$ where Y_1 is the zero set of a general section of $F(m)$ and Y_2 is a curve of the type $(2m-1, 2m-1)$ on a nonsingular quadric. Using Y we can construct the following extension:

$$0 \longrightarrow \mathcal{O} \longrightarrow E(m) \longrightarrow I_Y(2m) \longrightarrow 0$$

where E is a rank 2 bundle with $c_1(E) = 0$ and $c_2(E) = 4m$. Also $h^1(E(-m)) = h^1(I_Y) = 1$.

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