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Nagoya Math. J.
Vol. 92 (1983), 21-37

ON THE HOLONOMY LIE ALGEBRA AND THE NILPOTENT COMPLETION OF THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF HYPERSURFACES

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§1. Introduction

The purpose of this paper is to establish the following isomorphism of Lie algebras.

MAIN THEOREM. *Let X be the complement of a hypersurface S in the complex projective space $\boldsymbol{P}^N$. Then the tower of nilpotent complex Lie algebras associated with the fundamental group $\pi_1(X, *)$ and the holonomy Lie algebra $\mathfrak{g}_S$ attached to S are isomorphic. In particular, if S is the union of hyperplanes $\bigcup_{j=1}^{m+1} S_j$ in $\boldsymbol{P}^N$, the nilpotent completion of $\pi_1(X, *)$ is isomorphic to the nilpotent completion of*

$$\mathrm{Lib}\,(X_1, X_2, \cdots, X_{m+1})/\mathscr{A}$$

where we denote by $\mathrm{Lib}\,(X_1, X_2, \cdots, X_{m+1})$ *a free Lie algebra generated by* $X_1, X_2, \cdots, X_{m+1}$ *over* $\boldsymbol{C}$, *and* $\mathscr{A}$ *is the homogeneous ideal generated by the following elements*

I) $\sum_{j=1}^{m+1} X_j$,

II) $[X_{\nu_j}, X_{\nu_1} + \cdots + X_{\nu_p}]$, $\quad 1 \leqq j \leqq p$

where the hyperplanes $S_{\nu_1}, \cdots, S_{\nu_p}$ *satisfy* $H \cap S_{\nu_1} \cap \cdots \cap S_{\nu_p} \neq \phi$ *for a generic plane* H *and* $H \cap S_{\nu_1} \cap \cdots \cap S_{\nu_p} \cap S_k = \phi$ *if* $k \notin \{\nu_1, \cdots, \nu_p\}$.

For a smooth manifold we have a surjective homomorphism from the tower of the nilpotent completion of the holonomy Lie algebra to the tower of the nilpotent complex Lie algebras associated with the fundamental group (cf. [C]). Our main theorem guarantees that this map is an isomorphism in the case of the complement of hypersurfaces (cf. [A] Theorem 2).

In Section 2 we review the notion of holonomy Lie algebras and

Received September 29, 1981.

minimal algebras and we formulate the Sullivan's theorem of fundamental groups. The key lemma to prove our main theorem is the vanishing of certain Massey products. We shall discuss this in a more general situation in Section 4. In Section 5, we interpretate our main theorem by means of Poincaré-Koszul series. I would like to thank Professor Kazuhiko Aomoto for giving me helpful suggestions.

§ 2. Holonomy Lie algebras and minimal algebras

Let X be a simplicial complex. Let $[\omega_1], \cdots, [\omega_m]$ be a basis of $H^1(X; \boldsymbol{C})$. Let $H^1(X; \boldsymbol{C}) \wedge H^1(X; \boldsymbol{C})$ be the vector subspace of $H^2(X; \boldsymbol{C})$ generated by the decomposable elements, i.e., the elements $x \in H^2(X; \boldsymbol{C})$ which can be written in the form $\sum \alpha_{i,j}[\omega_i] \cup [\omega_j]$. Let $[v_1], \cdots, [v_s]$ be a basis of $H^1(X; \boldsymbol{C}) \wedge H^1(X; \boldsymbol{C})$. We write $[\omega_i] \cup [\omega_j]$ as a linear combination of $[v_1], \cdots, [v_s]$

$$[\omega_i] \cup [\omega_j] = \sum_{k=1}^{s} c_k^{i,j}[v_k] .$$

Let $\mathrm{Lib}(X_1, X_2, \cdots, X_m)$ be a free Lie algebra generated by $X_1, \cdots, X_m$ over $\boldsymbol{C}$. Let $\mathcal{N}$ be the homogeneous ideal of $\mathrm{Lib}(X_1, \cdots, X_m)$ generated by following elements

$$\sum_{i,j} c_k^{i,j}[X_i, X_j] , \qquad 1 \leqq k \leqq s .$$

Following Chen [C], we form the quotient Lie algebra

$$\mathfrak{g} = \mathrm{Lib}(X_1, X_2, \cdots, X_m)/\mathcal{N} .$$

It can be shown that this Lie algebra does not depend of the choice of the bases. We call the obtained Lie algebra the holonomy Lie algebra of X. In particular, if X is the complement of a hypersurface S in $\boldsymbol{P}^N$, we call this Lie algebra *the holonomy Lie algebra attached to* S and we denote it by $\mathfrak{g}_S$.

The following proposition gives an important example of a holonomy Lie algebra.

PROPOSITION 2.1. *Let X be $\boldsymbol{P}^N$ minus a finite number of hyperplanes $S_1, S_2, \cdots, S_{m+1}$. The holonomy Lie algebra attached to the configuration S is described in the following way. Let* $\mathrm{Lib}(X_1, X_2, \cdots, X_{m+1})$ *be a free Lie algebra generated by $X_1, X_2, \cdots, X_{m+1}$ over $\boldsymbol{C}$. Let $\mathcal{N}$ be the homogeneous ideal generated by the following elements*

I) $\sum_{j=1}^{m+1} X_j$,

II) $[X_{\nu_j}, X_{\nu_1} + \cdots + X_{\nu_p}]$, $\quad 1 \leqq j \leqq p$

where the corresponding hyperplanes $S_{\nu_1}, \cdots, S_{\nu_p}$ *meet each other at one point in a generic plane* H *and* $S_{\nu_1} \cap \cdots \cap S_{\nu_p} \cap S_k \cap H = \phi$ *if* $k \notin \{\nu_1, \cdots, \nu_p\}$. *The holonomy Lie algebra attached to the configuration* $\mathfrak{g}_S$ *is isomorphic to the Lie algebra* $\mathrm{Lib}(X_1, \cdots, X_{m+1})/\mathcal{N}$.

Proof. We denote by H_j the hyperplane in C^N defined by $H_j = S_j \cap (P^N - S_{m+1})$ for $1 \leqq j \leqq m$. We have the family of hyperplanes $\{H_j\}_{1 \leqq j \leqq m}$ in C^N. We put $X = C^N - \bigcup_{j=1}^m H_j$. Let f_j be a linear defining equation of H_j. We denote by ω_j the differential form $d \log f_j$. It is known that the cohomology ring $H^*(X; C)$ is generated by $[\omega_j]$, $1 \leqq j \leqq m$ ([B]). In particular, $H^2(X; C)$ is generated by $[\omega_i \wedge \omega_j]$. We can choose a basis of $H^2(X; C)$ in the following way. To each family of indices $\nu_1, \cdots, \nu_p$ defined in the statement of the propositions, we associate the family of elements of $H^2(X; C)$

$$\mathscr{V}_{\nu_1 \nu_k} = [\omega_{\nu_1} \wedge \omega_{\nu_k}] , \qquad 1 \leqq k \leqq p .$$

By using the relations

$$\omega_{\nu_1} \wedge \omega_{\nu_j} + \omega_{\nu_j} \wedge \omega_{\nu_k} + \omega_{\nu_k} \wedge \omega_{\nu_1} = 0$$

it can be proved that these elements form a basis of $H^2(X; C)$. Let $c_{p,q}^{i,j}$ be a number defined by

$$[\omega_i \wedge \omega_j] = \sum_{p,q} c_{p,q}^{i,j} \mathscr{V}_{p,q} .$$

Then, we have

$$c_{\nu_1 \nu_k}^{i,j} = \begin{cases} 1 & (i \neq \nu_k,\ j = \nu_k) \\ -1 & (i = \nu_k,\ j \neq \nu_k) \\ 0 & (\text{otherwise}) . \end{cases}$$

Thus we get the holonomy Lie algebra $\mathrm{Lib}(X_1, \cdots, X_m)/\mathcal{N}'$ where $\mathcal{N}'$ is the homogeneous ideal generated by

$$[X_{\nu_j}, X_{\nu_1} + \cdots + X_{\nu_p}] \qquad 1 \leqq j \leqq p$$

where $H_{\nu_1}, \cdots, H_{\nu_p}$ meet each other in a generic plane L in C^N and $H_{\nu_1} \cap \cdots \cap H_{\nu_p} \cap H_k \cap L = \phi$ if $k \notin \{\nu_1, \cdots, \nu_p\}$. We can prove that this Lie algebra is isomorphic to the Lie algebra $\mathrm{Lib}(X_1, \cdots, X_{m+1})/\mathcal{N}$ in the statement of the proposition by using the fact that

$$[X_1 + \cdots + X_m, X_{\nu_1}] - [X_{\nu_1} + \cdots + X_{\nu_p}, X_{\nu_1}]$$

is an element of $\mathcal{N}'$, which completes the proof.

(2.2) NOTATION. We denote by

$$\mathrm{Lib}(X_1, \cdots, X_m) = \Gamma_0 \mathrm{Lib}(X_1, \cdots, X_m) \supset \Gamma_1 \mathrm{Lib}(X_1, \cdots, X_m)$$
$$\cdots \supset \Gamma_j \mathrm{Lib}(X_1, \cdots, X_m) \supset \cdots$$

the lower central series of $\mathrm{Lib}(X_1, \cdots, X_m)$ defined by

$$\Gamma_{n+1} \mathrm{Lib}(X_1, \cdots, X_m) = [\mathrm{Lib}(X_1, \cdots, X_m), \Gamma_n \mathrm{Lib}(X_1, \cdots, X_m)] .$$

Let us review briefly Sullivan's theorem of the nilpotent completion of fundamental groups ([M]). In the followings, we deal with differential graded algebras over a field k ($k = \boldsymbol{Q}, \boldsymbol{R}$ or $\boldsymbol{C}$). We denote by A^j the degree j part of a differential graded algebra (d.g.a.) A.

(2.3) NOTATION. Let V be a graded vector space. We denote by $\Lambda(V)$ the free graded-commutative algebra generated by V. If V is homogeneous of degree r, $\Lambda(V)$ (also denoted by $\Lambda_r(V)$) is the symmetric algebra when r is even and is the exterior algebra when r is odd.

(2.4) DEFINITION. By a (*nilpotent*) *Hirsch extension* of a differential graded algebra A, we mean an inclusion $A \subset B$ of d.g.a. such that B is isomorphic to $A \otimes \Lambda_j(V)$ as a graded algebra and the differential of B sends V into A^{j+1}.

(2.5) DEFINITION. We shall say that a d.g.a. M is a *minimal algebra* if the following conditions are satisfied.

a) $M^0 = k$,

b) There exists an increasing filtration

$$k = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_j \subset \cdots$$

such that M_j is a subalgebra of M, $M_j \subset M_{j+1}$ is a Hirsch extension for each j and $\bigcup_{j=0}^{\infty} M_j = M$.

c) The differential of M, d is decomposable, i.e.,

$$dM \subset M_+ \wedge M_+$$

where M_+ is the augmentation ideal of M defined by $\oplus_{j \geqq 1} M^j$.

(2.6) DEFINITION. Let A be a d.g.a. over k. An *i-minimal model* of A is a d.g.a. $\mathcal{M}(i)$ and a homomorphism of d.g.a. $\rho: \mathcal{M}(i) \to A$ such that

a) $\mathscr{M}(i)$ is a minimal algebra,

b) $\mathscr{M}(i)$ is generated by the elements of degree i, and

c) $\rho^*: H^\cdot(\mathscr{M}(i)) \to H^\cdot(A)$ is an isomorphism in degree i and is injective in degree $= i + 1$.

By a theorem of Sullivan ([M]), $\mathscr{M}(i)$ exists and is unique up to isomorphism for a connected d.g.a. A. Let X be a smooth manifold. Let $\mathscr{M}_X(i)$ be the i-minimal model of the algebra of $\boldsymbol{C}$-valued C^∞-differential forms on X. We shall call the algebra $\mathscr{M}_X(i)$ the i-minimal model of X.

(2.7) Let $\boldsymbol{C} = \mathscr{M}_X(1)_0 \subset \mathscr{M}_X(1)_1 \subset \cdots \subset \mathscr{M}_X(1)_j \subset \cdots$ be the increasing filtration of $\mathscr{M}_X(1)$ defined inductively in the following way. Let $\mathscr{M}_X(1)_1$ be the subalgebra of $\mathscr{M}_X(1)$ generated by x such that $dx = 0$ and $\deg x = 1$. We define $\mathscr{M}_X(1)_{n+1}$ to be the subalgebra of $\mathscr{M}_X(1)$ generated by $x \in \mathscr{M}_X(1)$ such that $dx \in \mathscr{M}_X(1)_n$ $(n \geqq 1)$ and $\deg x = 1$. We shall call the above filtration *the canonical series* of the 1-minimal model $\mathscr{M}_X(1)$.

(2.8) Let $\mathscr{M}_X(1)_j = \oplus_{k \geqq 0} \mathscr{M}_X(1)_j^k$ be the decomposition of $\mathscr{M}_X(1)_j$ by its degree. Then we have the following increasing sequence of vector spaces

$$\boldsymbol{C} = \mathscr{M}_X(1)_0^1 \subset \mathscr{M}_X(1)_1^1 \subset \cdots \subset \mathscr{M}_X(1)_j^1 \subset \cdots .$$

Let $\mathscr{L}_j$ denote the dual vector space $\operatorname{Hom}(\mathscr{M}_X(1)_j^1; \boldsymbol{C})$. We have the following sequence

$$0 \longleftarrow \mathscr{L}_1 \longleftarrow \mathscr{L}_2 \longleftarrow \cdots \longleftarrow \mathscr{L}_j \longleftarrow \cdots .$$

Let $\{\omega_\alpha\}$ be a basis of $\mathscr{M}_X(1)_j^1$ and let $\{\omega_\alpha^*\}$ be the dual basis. When $d\omega_r$ is written in the form

$$\sum_{p,q} \gamma_{p,q}^r \omega_p \wedge \omega_q$$

in $\mathscr{M}_X(1)$, we define $[\omega_p^*, \omega_q^*]$ to be $\sum_r \gamma_{p,q}^r \omega_r^*$. In this way the complex vector spaces $\mathscr{L}_j$ have the structure of nilpotent Lie algebras.

(2.9) Let G be a group. We denote by

$$G = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_j \supset \cdots$$

its lower central series. By using the central extension

$$0 \longrightarrow \Gamma_n/\Gamma_{n+1} \longrightarrow G/\Gamma_{n+1} \longrightarrow G/\Gamma_n \longrightarrow 1$$

we can inductively define the complex Lie group $G/\Gamma_n \otimes \boldsymbol{C}$ ([FGM], [S2]). We shall call the sequence

$$0 \longleftarrow G/\Gamma_1 \otimes \boldsymbol{C} \longleftarrow \cdots \longleftarrow G/\Gamma_n \otimes \boldsymbol{C} \longleftarrow \cdots$$

the tower of the nilpotent completion of G over $\boldsymbol{C}$. Sullivan's theorem can be formulated in the following way.

THEOREM 2.10 (Sullivan [M], [S2]). *Let*

$$\cdots \longleftarrow \mathscr{L}_j \longleftarrow \mathscr{L}_{j+1} \longleftarrow \cdots$$

be the tower of nilpotent Lie algebras constructed from the 1*-minimal model* $\mathscr{M}_X(1)$ (see (2.8)). *We have the following isomorphisms via exponential maps*

$$\begin{array}{ccccc} \cdots \longleftarrow & \mathscr{L}_j & \longleftarrow & \mathscr{L}_{j+1} & \longleftarrow \cdots \\ & \downarrow \wr\!\! \mid\!\mid & & \downarrow \wr\!\! \mid\!\mid & \\ \cdots \longleftarrow & \pi_1(X)/\Gamma_j \otimes \boldsymbol{C} & \longleftarrow & \pi_1(X)/\Gamma_{j+1} \otimes \boldsymbol{C} & \longleftarrow \cdots . \end{array}$$

By taking a projective limit, we can conclude that $\mathscr{L} = \varprojlim \mathscr{L}_j$ *is the Lie algebra associated with the nilpotent completion of* $\pi_1(X)$ *over* $\boldsymbol{C}$.

(2.11) DEFINITION. In view of (2.10), we shall call the sequence of Lie algebras

$$\cdots \longleftarrow \mathscr{L}_j \longleftarrow \mathscr{L}_{j+1} \longleftarrow \cdots$$

defined in (2.8) *the tower of nilpotent complex Lie algebras associated with* $\pi_1(X)$.

§ 3. The vanishing of Massey products

Let W be a smooth projective variety and let D be a divisor with normal crossings. We denote by X the complement $W - D$. We shall use the notations of [M]. Let D^p be the subvariety of D consisting of points X such that $\operatorname{mult}_x D \geqq p$. We denote by $\tilde{D}^p \to D^p$ its normalization. We denote the ambient variety W by D^0.

Let A_X be the differential graded algebra over $\boldsymbol{Q}$ whose degree n part is defined by

$$A_X^n = \bigoplus_p H^{n-p}(\tilde{D}^p; \varepsilon_{\boldsymbol{Q}}^p) \tag{3.1}$$

and the differential $d\colon A_X^n \to A_X^{n+1}$ is defined by the Gysin homomorphism. We shall use the following results of Deligne and Morgan ([M]) to prove the vanishing of certain Massey products.

THEOREM 3.2 (Deligne, Morgan [M]). i) *There exists an increasing filtration* $\mathscr{W}$ *in* A_X, *which is called a weight filtration, and a decreasing* *filtration $\mathscr{F}$ in $A_X \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ such that the induced filtration in $V = H^{n-p}(\tilde{D}^p; \varepsilon^p_{\boldsymbol{C}})$ is $(n+p)$-opposed to its complex conjugate, i.e.,*

$$\mathscr{F}^p(\mathrm{Gr}^{\mathscr{W}}_n(V)) \oplus \overline{\mathscr{F}^{q+1-p}}(\mathrm{Gr}^{\mathscr{W}}_n(V)) = \mathrm{Gr}^{\mathscr{W}}_n(V) \tag{3.3}$$

where $q = n + p$.

ii) *Let $\mathscr{M}_X(j)_{\boldsymbol{Q}}$ be the j-minimal model of X over $\boldsymbol{Q}$. Then, $\mathscr{M}_X(j)_{\boldsymbol{Q}}$ is isomorphic to the j-minimal model of A_X.*

iii) *There exists an increasing filtration $\tilde{\mathscr{W}}$ in $\mathscr{M}_X(j)_{\boldsymbol{Q}}$ and a decreasing filtration $\tilde{\mathscr{F}}$ in $\mathscr{M}_X(j)$ such that*

(a) *The differential d and the product Λ are strictly compatible with $\tilde{\mathscr{W}}$ and $\tilde{\mathscr{F}}$ in the sense of* [M].

(b) *The filtration $\tilde{\mathscr{W}}$ and $\tilde{\mathscr{F}}$ induce the mixed Hodge structure on $H^{\cdot}(\mathscr{M}_X(j))$ which is compatible with the mixed Hodge structure on $H^{\cdot}(X)$ with respect to the homomorphism ρ^* defined in* (2.6).

(3.4) Definition. Following Kraines [Kr], let us define the Massey product on the first cohomology. Let $\gamma_1, \cdots, \gamma_p$ be the elements of $H^1(X; \boldsymbol{R})$. We shall say that $\{\gamma_1, \cdots, \gamma_p\}$ is a *Massey system* if there exists a collection of 1-forms $S = \{m_{i,j};\ 1 \leqq i \leqq j \leqq p,\ j - i < p - 1\}$ which satisfies the following conditions M1) and M2).

M1) $m_{i,i}$ is closed and its cohomology class $[m_{i,i}]$ is γ_i.
for $i = 1, \cdots, p$ and

M2) $dm_{i,j} = \sum_{k=i}^{j-1} m_{i,k} \wedge m_{k+1,j}$ if $i < j$.
Then $\sum_{k=1}^{p-1} m_{1,k} \wedge m_{k+1,p}$ turns out to be a closed form. We denote its cohomology class by $\langle \gamma_1, \cdots, \gamma_p \rangle_S$, and we call it the *Massey product* of $\gamma_1, \cdots, \gamma_p$ with respect to the system S. In general, this cohomology class depends on the defining system. We shall say that the Massey product $\langle \gamma_1, \cdots, \gamma_p \rangle$ is zero if $\langle \gamma_1, \cdots, \gamma_p \rangle_S = 0$ for any defining system S.

Proposition 3.5. *Let W be a smooth projective variety with $H^1(W; \boldsymbol{C}) = 0$ and let D be a divisor with normal crossings. Let X be $W - D$. We have the following decomposition of the* 1-*minimal model.*

i) $\mathscr{M}_X(1) = \underset{p \geqq 0}{\mathscr{M}_X}(1)^{p,p}$

where $\mathscr{M}_X(1)^{p,p}$ is the (p,p)-part of $\mathscr{M}_X(1)$ in the category of mixed Hodge structure. We have a complex $(\mathscr{M}_X(1)^{p,p}, d)$ and we have

ii) $H^2(\mathscr{M}_X(1)^{p,p}) = 0$ *if* $p \geqq 3$.

Proof. Let us preserve the notations of (3.2). Let $\alpha\colon H^0(\tilde{D}^1; \boldsymbol{C}) \to H^2(W; \boldsymbol{C})$ be the Gysin homomorphism. We have the following isomorphism

$$H^1(X; C) \cong H^1(W; C) \oplus \operatorname{Ker} \alpha .$$

In our situation, we have $H^1(X; C) \cong \operatorname{Ker} \alpha$. By means of the part i) of the theorem (3.2), it turns out that the cohomology $H^1(X; C)$ contains only degree (1, 1) part in the mixed Hodge structure. Let

$$C = \mathscr{M}_X(1)_0 \subset \mathscr{M}_X(1)_1 \subset \cdots \subset \mathscr{M}_X(1)_j \subset \cdots$$

be the canonical series of the 1-minimal model of X defined in (2.8). We have $\mathscr{M}_X(1)_1 \cong \Lambda(H^1(X; C))$. We can make $\mathscr{M}_X(1)_1$ into a bigraded algebra in such a way that the elements of $H^1(X; C)$ have the bidegree (1.1) and the bidegrees are compatible with the exterior products. Inductively we can assign the bidegrees to the elements of $\mathscr{M}_X(1)$ in such a way that the differential d preserves the bidegrees and the bidegrees are compatible with the exterior products. In this way we have the following decomposition of the 1-minimal model

$$\mathscr{M}_X(1) = \bigoplus_{p \geqq 0} \mathscr{M}_X(1)^{p,p} .$$

This decomposition is related to the filtration $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{W}}$ in (3.2) in the following manner.

$$\tilde{\mathscr{F}}^p(\mathscr{M}_X(1)) = \bigoplus_{j \leqq p} \mathscr{M}_X(1)^{j,j} ,$$
$$\tilde{\mathscr{W}}_q(\mathscr{M}_X(1)) = \bigoplus_{2j \geqq q} \mathscr{M}_X(1)^{j,j}$$

which completes the proof of i). Since the differential d is strictly compatible with the filtrations, d preserves bidegrees and we have the complex $(\mathscr{M}_X(1)^{p,p}, d)$. For the part ii), let us observe that the homomorphism $\rho: \mathscr{M}_X(1) \to A_X$ induces a bidegree preserving homomorphism $\rho^*: H^2(\mathscr{M}_X(1)) \to H^2(X; C)$ from the part iii) of the theorem (3.2). From the definition of the 1-minimal model $\mathscr{M}_X(1)$, homomorphism $\rho^*: H^2(\mathscr{M}_X(1)) \to H^2(X; C)$ is injective. We have the following decomposition of $H^2(X; C)$ in the mixed Hodge structure ([D], [M]).

(3.6) $$H^2(X; C) = \bigoplus_{2 \leqq p+q \leqq 4} H^{p,q}(X; C) .$$

Let x be an element of $\mathscr{M}_X(1)^{p,p}$ such that $dx = 0$ and $p \geqq 3$. By means of the decomposition (3.6), we have $\rho^*(x) = 0$ in $H^2(X; C)$. From the injectivity of ρ^*, it follows that x must be a coboundary in $\mathscr{M}_X(1)^{p,p}$.

Under the same hypothesis as in (3.5), we have the following corollary.

COROLLARY 3.7. *For any Massey system* $\{\gamma_1, \cdots, \gamma_p\}$, $(\gamma_i \in H^1(X; C))$ *the p-tuple Massey products* $\langle \gamma_1, \cdots, \gamma_p \rangle$ *are null if* $p \geqq 3$.

Proof. Let $\{\gamma_1, \cdots, \gamma_p\}$ be a Massey system with a defining system of 1-forms

$$S = \{m_{i,j};\ 1 \leqq i \leqq j \leqq p,\ j - i < p - 1\}\ .$$

We have $\bar{m}_{i,j}$ in $\mathscr{M}_X(1)$ such that $\rho(\bar{m}_{i,j}) = m_{i,j}$. We can prove inductively that bidegree $\bar{m}_{i,j} = (j - i + 1, j - i + 1)$. If $p \geqq 3$, we have $\langle \gamma_1, \cdots, \gamma_q \rangle_S = 0$ by (3.5). We can use the same agrument for any defining system S, which completes the proof.

§ 4. Proof of the main theorem

For the proof of our main theorem we need several lemmas. Let W be a smooth projective variety and let D be a divisor with normal crossings. As in Section 3, we assume that $H^1(W; C) = 0$.

(4.1) NOTATIONS. Let A be the complex vector space $H_1(X; C)$. We have an isomorphism $H_1(X; C) \cong \operatorname{Coker} \alpha^* \oplus H^1(W; C)$ where α^* denotes the dual of the Gysin homomorphism $\alpha: H^0(\tilde{D}^1; C) \to H^2(W; C)$. The vector space A has a natural bigrading

$$A = A_{-1,0} \oplus A_{0,-1} \oplus A_{-1,-1}$$

dual to the bigrading of the first cohomology in the mixed Hodge structure. Let Lib (A) denote the free Lie algebra generated by A. By using the bigrading in A, we can make Lib (A) into a bigraded Lie algebra in such a way that the brackets products are compatible with the bidegrees.

Let $0 \leftarrow \mathscr{L}_1 \leftarrow \mathscr{L}_2 \leftarrow \cdots \leftarrow \mathscr{L}_j \leftarrow \cdots$ be the tower of nilpotent Lie algebras associated to the fundamental group $\pi_1(X)$ in the sense of (2.11). The Lie algebra $\mathscr{L}_1$ is an abelian Lie algebra. Let $\alpha_1: A \to \mathscr{L}_1$ be the identity homomorphism. By lifting this homomorphism we have a homomorphism from the free Lie algebra Lib (A) to the tower of complex nilpotent Lie algebras associated with the fundamental group. Let us note that $A = A_{-1,-1}$ from the hypothesis $H^1(W; C) = 0$. Let $\mathscr{J}_n$ be Ker (Lib $(A) \to \mathscr{L}_n$) and let J_n denote the vector space defined by $\mathscr{J}_n/[\text{Lib}\,(A), \mathscr{J}_n]$. We have the following isomorphisms

$$H^2(\mathscr{M}_X(1)) \cong H^2(\mathscr{L}_n) \cong \operatorname{Hom}(J_n, C)\ .$$

The Lie algebra $\mathscr{J}_n$ has the bigrading $\mathscr{J}_n = \oplus_{p \geqq 0} \mathscr{J}_n^{-p,-p}$ induced from the bigrading of Lib (A) defined in (4.1). Let $C=\mathcal{M}_X(1)_0\subset\cdots\subset\mathcal{M}_X(1)_j\subset\cdots$ be the canonical series of the 1-minimal model of X in the sense of (2.7). Let V_n be the vector space which is used to construct the Hirsch extension $\mathcal{M}_X(1)_{n-1}\subset\mathcal{M}_X(1)_n$, i.e., $\mathcal{M}_X(1)_n$ is isomorphic to $\mathcal{M}_X(1)_{n-1}\otimes\Lambda(V_n)$ as C-algebras.

LEMMA 4.2. *Let* $\mathcal{M}_X(1)=\oplus_{p\geq 0}\mathcal{M}_X(1)^{p,p}$ *be the bigrading defined in* (3.5). *This bigrading induces the bigrading in* V_n *of the following form*

$$V_n=\bigoplus_{p\geqq n}V_n^{p,p}\,.$$

Proof. We prove by induction with respect to n. In the case $n=1$, the vector space V_1 is isomorphic to $A=\operatorname{Ker}(H^0(\tilde{D}^1)\to H^2(W))$, therefore we have $V_1=V_1^{1,1}$. Let us assume that V_k is decomposed $\oplus_{p\geqq k}V_k^{p,p}$ for $k\leqq n$. We want to prove $V_{n+1}=\oplus_{p\geqq n+1}V_{n+1}^{p,p}$. Let $x\in V_{n+1}$ be a homogeneous element. Let us assume that bidegree $x\leqq(n,n)$. Since the differential d preserves bidegrees, we have bidegree $dx\leqq(n,n)$. By the definition of the canonical series of the 1-minimal model (2.7), dx must be written in the form

$$dx=\sum c_{i,j}y_i\wedge y_j$$

with some $c_{i,j}\in C$ and $y_i,y_j\in\mathcal{M}_X(1)_n$. By the hypothesis of the induction y_i must be contained in $\mathcal{M}_X(1)_{n-1}$, hence x must be an element of $\mathcal{M}_X(1)_n$, which contradicts the fact that x is a homogeneous element of V_{n+1}.

LEMMA 4.3. *Let* $J_k^{-2,-2}$ *be the bidegree* $(-2,-2)$ *part of* J_k. *Then we have* i) $J_k^{-2,-2}$ *is independent of* k *if* $k\geqq 2$. ii) J_k *has a decomposition* $J_k=J_k^{-2,-2}\oplus(\oplus_{p\geqq k+1}J_k^{-p,-p})$ *for* $k\geqq 2$.

Proof. Let $j_n\colon\mathcal{M}_X(1)_n\to\mathcal{M}_X(1)_{n+1}$ be the inclusion. By the construction of the 1-minimal model, we have the following exact sequences:

$$(4.4.1)\qquad 0\longrightarrow V_{n+1}\longrightarrow H^2(\mathcal{M}_X(1)_n)\xrightarrow{j_n^*}H^2(\mathcal{M}_X(1)_{n+1})$$

$$(4.4.2)\qquad 0\longrightarrow V_{n+1}\longrightarrow H^2(\mathcal{M}_X(1)_n)\xrightarrow{\rho^*}H^2(X;C)\longrightarrow 0$$

where ρ^* is the homomorphism induced from $\rho\colon\mathcal{M}_{X(1)}\to A_X$. By using the isomorphism $H^2(\mathcal{M}_X(1)_n)\cong\operatorname{Hom}(J_n,C)$, we have the exact sequence

$$(4.4.1)'\qquad 0\longrightarrow V_{n+1}\longrightarrow\operatorname{Hom}(J_n,C)\longrightarrow\operatorname{Hom}(J_{n+1},C)\,.$$

By (4.2) the vector space V_n has a decomposition $\oplus_{p\geqq n}V_n^{p,p}$. On the other hand any element of $H^2(X; C)$ has bidegree (2.2). By using the exact sequence (4.4.1) we have a decomposition

$$J_k = J_k^{-2,-2} \oplus (\bigoplus_{p \geqq k} J_k^{-p,-p})$$

for $k \geqq 2$, where $J_k^{-2,-2}$ is isomorphic to the homology group $H_2(X; C)$ and $\oplus_{p \geqq k+1} J_k^{-p,-p}$ is isomorphic to $\mathrm{Hom}(V_k, C)$, which completes the proof.

(4.5) NOTATIONS. Since $J_k^{-2,-2}$ is independent of k if $k \geqq 2$, we shall denote it by J. Let $\mathscr{J}$ be the homogeneous ideal of $\mathrm{Lib}(A)$ such that $\mathscr{J}/[\mathscr{J}, \mathrm{Lib}(A)] = J$. Let us consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{J}_{n+1} & \longrightarrow & \mathscr{J}_n & \xrightarrow{\tilde{\phi}} & \mathscr{J}_n/\mathscr{J}_{n+1} & \longrightarrow 0 \\
& p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & \\
& J_{n+1} & \longrightarrow & J_n & \xrightarrow{\phi} & \mathrm{Hom}(V_{n+1}, C) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

where we denote by p_j $(j = 1, 2, 3)$ the canonical surjections.

LEMMA 4.6. *Let us assume* $n \geqq 2$. *Then,* i) *the ideal* $\mathscr{J}_n$ *contains the n-th lower central series.* $\Gamma_n \mathrm{Lib}(A)$ *as a subideal,* ii) *the homomorphism* p_2 *restricted on* $\Gamma_n \mathrm{Lib}(A)$ *is surjective and* iii) *we have the equality*

$$p_2(\Gamma_n \mathrm{Lib}(A)) = \bigoplus_{p \geqq n+1} J_n^{-p,-p} .$$

Proof. i) By the theorem of Sullivan (2.10), $\mathscr{L}_n$ is isomorphic to the Lie algebra associated to the Lie group $\pi_1(X)/\Gamma_n \otimes C$. Hence, the n-th lower central series $\Gamma_n \mathrm{Lib}(A)$ contained in the $\mathrm{Ker}(\mathrm{Lib}(A) \to \mathscr{L}_n)$, which proves the assertion i).

ii) The homomorphism $\tilde{\phi} | \Gamma_n \mathrm{Lib}(A)$ is surjective, since $\tilde{\phi}$ factors through the canonical surjection

$$\Gamma_n \mathrm{Lib}(A) \longrightarrow \Gamma_n(\mathrm{Lib}(A)/\mathscr{J}_{n+1}) \cong \mathscr{J}_n/\mathscr{J}_{n+1} .$$

By the commutativity of the diagram in (4.5), we have the assertion ii).

iii) By the decomposition of (4.3) we have

$$J_n = J \oplus (\bigoplus_{p \geqq n+1} J_n^{-p,-p}) .$$

By the proof of (4.3) the homomorphism ϕ gives an isomorphism

$$\mathrm{Hom}\,(V_{n+1}, C) \cong \bigoplus_{p \geqq n+1} J_n^{-p,-p} .$$

By means of the assersion i) the homomorphism $p_2 | \Gamma_n \mathrm{Lib}\,(A) : \Gamma_n \mathrm{Lib}\,(A) \to \bigoplus_{p \geqq n+1} J_n^{-p,-p}$ is surjective, which completes the proof of the assertion iii).

By using Lemma 4.6, we have the following proposition.

PROPOSITION 4.7. *If $n \geqq 2$, we have*

$$J_n = J \oplus (\Gamma_n \mathrm{Lib}\,(A)/(\Gamma_{n+1} \mathrm{Lib}\,(A) + \mathscr{J} \cap \Gamma_n \mathrm{Lib}\,(A))) .$$

Moreover the intersection of $\mathscr{J}_n$, $\bigcap_{n \geqq 1} \mathscr{J}_n$, is isomorphic to the ideal $\mathscr{J}$.

Proof. Let us consider the kernel of the surjective homomorphism

$$p_2 | \Gamma_n \mathrm{Lib}\,(A) : \Gamma_n \mathrm{Lib}\,(A) \longrightarrow \bigoplus_{p \geqq n+1} J_n^{-p,-p} .$$

The kernel is generated by $\Gamma_{n+1} \mathrm{Lib}\,(A)$ and $\mathscr{J} \cap \Gamma_n \mathrm{Lib}\,(A)$, therefore $\bigoplus_{p \geqq n+1} J_n^{-p,-p}$ is isomorphic to

$$\Gamma_n \mathrm{Lib}\,(A)/(\Gamma_{n+1} \mathrm{Lib}\,(A) + \mathscr{J} \cap \Gamma_n \mathrm{Lib}\,(A)) .$$

The second assertion follows from the fact that the intersection of the lower central series of $\mathrm{Lib}\,(A)$ is zero.

We obtain the following corollary which permits us to compute combinatorially the succesive quotients of the lower central series of the fundamental group of $X = W - D$ up to torsion.

COROLLARY 4.8. *Let $\pi_1(X) = \Gamma_0\pi_1(X) \supset \Gamma_1\pi_1(X) \supset \cdots$ be the lower central series of the fundamental group. Then*

$$\begin{aligned}&(\Gamma_n(\pi_1(X))/\Gamma_{n+1}(\pi_1(X)) \otimes C)^* \\ &\quad \cong \Gamma_n (\mathrm{Lib}\,(A))/(\Gamma_{n+1} (\mathrm{Lib}\,(A)) + \mathscr{J} \cap \Gamma_n (\mathrm{Lib}\,(A))) .\end{aligned}$$

To prove our main theorem let us observe the following lemma.

LEMMA 4.9. *Let X be a simplicial complex. Let $\mathscr{L}_X$ be the Lie algebra associated with the nilpotent completion of the fundamental group. Let $\mathscr{J}$ be the subideal of* $\mathrm{Ker}\,(\mathrm{Lib}\,(X_1, \cdots, X_m) \to \mathscr{L}_X)$ *generated by all elements of degree* 2. *Then,* $\mathrm{Lib}\,(X_1, \cdots, X_m)/\mathscr{J}$ *is isomorphic to the holonomy Lie algebra* $\mathfrak{g}$.

Proof. Let $\mathscr{M}_X(1)_1$ be $\Lambda(x_1, \cdots, x_m)$. We choose $\rho: \mathscr{M}_X(1)_1 \to \mathscr{E}(X)$ such that $\rho(x_j) = \omega_j$, $1 \leqq j \leqq m$. Let $\mathscr{M}_X(1)_2$ be $\mathscr{M}_X(1) \otimes \Lambda(y_1, \cdots, y_r)$ with the differential d such that

$$dy_k = \sum_{i,j} \gamma_k^{i,j} x_i \wedge x_j \,.$$

Hence $\sum_{i,j} \gamma_k^{i,j}[\omega_i \wedge \omega_j] = 0$ in $H^2(X; \boldsymbol{C})$. Let us preserve the notations of Section 2. Let $X_1, \cdots, X_m$; $Y_1, \cdots, Y_r$ be the dual basis of $x_1, \cdots, x_m$; $y_1, \cdots, y_r$. From the definition of the Lie algebra $\mathscr{L}$, we have

$$[X_i, X_j] = \sum_k \gamma_k^{i,j} Y_k \qquad \text{in } \mathscr{L} \,.$$

On the other hand, we have $\sum_{i,j} \gamma_k^{i,j} \sum_k c_{i,j}^k [y_k] = 0$, therefore we have the following relation

$$\sum_{i,j} \gamma_k^{i,j} c_{i,j}^k = 0 \,.$$

It follows that $\sum_{i,j} c_{i,j}^k [X_i, X_j] = \sum_{i,j,k} c_{i,j}^k \gamma_k^{i,j} Y_k = 0$ in $\mathrm{Lib}\,(X_1, \cdots, X_m)/\mathscr{J}$, which means the inclusion $\mathscr{N} \subset \mathscr{J}$. We obtain the surjective homomorphism

$$\mathfrak{g} \longrightarrow \mathrm{Lib}\,(X_1, \cdots, X_m)/\mathscr{J} \,.$$

By comparing the dimension of the degree 2-part, we have our lemma.

By means of the Lefschetz type theorem of [LH], if we take a generic plane H, the inclusion $j\colon H \to \boldsymbol{P}^N$ induces an isomorphism

$$j_*\colon \pi_1(H - H \cap S) \longrightarrow \pi_1(\boldsymbol{P}^N - S) \,.$$

Then, we have an isomorphism of the 1-minimal model of $H - H \cap S$ and that of $\boldsymbol{P}^N - S$, which permits us to consider the case $N = 2$. Let $\mu\colon (W, D) \to (\boldsymbol{P}^2, H \cap S)$ be a resolusion of the singularities of $H \cap S$ such that D is a divisor with normal crossings. We can apply the arguments (4.2)–(4.9) to our situation, which completes the proof of our main theorem.

From (4.8) we get the following corollary.

COROLLARY 4.10. *Let $X = \boldsymbol{P}^N - \bigcup_{j=1}^{m+1} S_i$, where S_j denotes a hyperplane. Then, the lower central series of the fundamental group of X is strictly decreasing uncless the hyperplanes are in general position.*

§ 5. Examples and discussion

We give two typical examples of fundamental groups.

EXAMPLE 5.1. Let $S = \bigcup_{j=1}^n L_j$ be a family of n lines in $\boldsymbol{C}^2$ such that $\bigcap_{j=1}^n L_j = \{0\}$. Then, the corresponding holonomy Lie algebra $\mathfrak{g}_S$ is isomorphic to

$$\mathrm{Lib}\,(X_1, \cdots, X_n)/\mathcal{N}$$

where $\mathcal{N}$ denotes the homogeneous ideal generated by

$$\left[X_i, \sum_{j=1}^{n} X_j\right] \qquad (1 \leqq i \leqq n-1)\,.$$

Let $\mathfrak{g}_S = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be the lower central series. Then,

$$\dim \Gamma_0/\Gamma_1 = n \quad \text{and} \quad \dim \Gamma_1/\Gamma_2 = \frac{(n-1)(n-2)}{2}\,.$$

The following remark is suggested to the author by the referee.

Remark 5.1.1. In the situation of the above example we have the following results. Let $\mathscr{E}(\mathfrak{g}_S)$ be the enveloping algebra of $\mathfrak{g}_S$. We put $\deg X_j = 1$ for $1 \leqq j \leqq n$, and we make $\mathscr{E}(\mathfrak{g}_S)$ into the graded algebra in such a way that the products are compatible with the degrees. Let ${}_p\mathscr{E}(\mathfrak{g}_S)$ denote the degree p part of $\mathscr{E}(\mathfrak{g}_S)$ and let $\chi(p)$ be $\dim {}_p\mathscr{E}(\mathfrak{g}_S)$. Our enveloping algebra $\mathscr{E}(\mathfrak{g}_S)$ has the following free resolution

$$0 \longrightarrow \mathscr{E}(\mathfrak{g}_S)^{n-1} \xrightarrow{\rho^2} \mathscr{E}(\mathfrak{g}_S)^n \xrightarrow{\rho^1} \mathscr{E}(\mathfrak{g}_S) \xrightarrow{\mathscr{E}} C$$

where $\mathscr{E}$ is the augmentation and ρ^i $(i = 1, 2)$ are defined in the following way

$$\rho^1(u_1, \cdots, u_n) = \sum_{j=1}^{n} u_j X_j \qquad \text{for } (u_1, \cdots, u_n) \in \mathscr{E}(\mathfrak{g}_S)^n$$

and

$$\rho^2(u'_1, \cdots, u'_n) = \left(\sum_{j=1}^{n-1} u'_j \frac{\partial R_j}{\partial X_1}, \cdots, \sum_{j=1}^{n-1} u'_j \frac{\partial R_j}{\partial X_n}\right)$$

where R_i denotes $[X_{i+1}, \sum_{j=1}^{n} X_j]$, $1 \leqq i \leqq n-1$ and $\partial/\partial X_j$ is the right derivation. Since $b_1(C^2 - S) = n$ and $b_2(C^2 - S) = n-1$, we have the following reccursive equation

$$\chi(p) = n\chi(p-1) - (n-1)\chi(p-2)\,.$$

It follows immediately that

$$\chi(p) = \frac{(n-1)^{p+1} - 1}{n-2}\,.$$

By means of the theorem of Witt (see [MKS]), we have

$$\prod_{p=1}^{\infty}(1-t^p)^{-\varphi(p-1)}=\sum_{p=0}^{\infty}\chi(p)t^p$$

where $\varphi(p)=\operatorname{rank}\Gamma_p/\Gamma_{p+1}$. Therefore we can compute $\varphi(0),\varphi(1),\varphi(2),\cdots$ inductively. In our case

$$\sum_{p=0}^{\infty}\chi(p)t^p=\frac{1}{P_{C-S}(-t)}$$

where we denote by $P_{C-S}(t)$ the Poincaré polynomial of C^2-S.

Let us introduce certain Poincaré-Koszul series for discussing our main result from this point of view. Let

$$C=\mathscr{M}_X(1)_0\subset\mathscr{M}_X(1)_1\subset\cdots\subset\mathscr{M}_X(1)_j\subset\cdots$$

be the canonical series of $\mathscr{M}_X(1)$. We put $\mathscr{M}_X(1)_1=\Lambda(V_1)$ and $\mathscr{M}_X(1)_{j+1}=\mathscr{M}_X(1)_j\otimes\Lambda(V_{j+1})$ for $j\geqq 1$. $\mathscr{M}_X(1)$ has the natural structure of a graded algebra such that $\deg x=1$ for $x\in V_j$ $(j=1,2,\cdots)$. Let us introduce another degree such that $\deg x=j-1$ for $x\in V_j$ $(j=1,2,\cdots)$. We extend this degree to $\mathscr{M}_X(1)$ such that the bidegrees are compatible with the product structure. We denote by $\mathscr{M}_X(1)_{(k)}$ the degree k part with respect to this degree. The gradation $\mathscr{M}_X(1)=\oplus_{k\geqq 0,n\geqq 0}\mathscr{M}_X(1)^n_{(k)}$ induces the gradation

$$H^{\cdot}(\hat{\pi};C)=\bigoplus_{\substack{k\geqq 0\\ n\geqq 0}}H^n_k(\hat{\pi}:C)$$

by means of Theorem 2.10, where we denote by $\hat{\pi}$ the nilpotent completion of $\pi_1(x)$ over C.

THEOREM 5.1.2. *We have the following equalities*

$$\sum_{p=0}^{\infty}\chi(p)t^p=\prod_{j=1}^{\infty}(1-t^j)^{-\varphi(j-1)}=\frac{1}{U_X(-t)}$$

where

$$U_X(t)=\sum_{t=0}^{\infty}\left(\sum_{k=0}^{p}(-1)^k\dim H^{p-k}_k(\hat{\pi};C)\right)t^p\ ,$$

$$\chi(p)=\dim{}_p\mathscr{E}(\mathfrak{g}_S)$$

and

$$\varphi(j)=\operatorname{rk}\Gamma_j(\pi_1(X))/\Gamma_{j+1}(\pi_1(X))\ .$$

Proof. Let $C=\sum_{p\geqq 0,n\geqq 0}C^p_n$ be a bigraded vector space. Let $U_C(t)$ be the Poincaré-Koszul series defined by

$$\sum_{p=0}^{\infty}\left(\sum_{n=0}^{p}(-1)^n \dim C_n^{p-n}\right)t^p .$$

With respect to the bigradation $\mathcal{M}_X(1) = \oplus_{k\geq 0, n\geq 0}\mathcal{M}_X(1)_k^n$, we have

$$U_{A(V_j)}(t) = (1-(-t)^j)^{\varphi(j-1)} .$$

Hence,

$$U_X(t) = U_{\mathcal{M}_X(1)}(t) = \prod_{j=1}^{\infty}(1-(-t)^j)^{\varphi(j-1)} .$$

On the other hand, we have

$$\sum_{p=0}^{\infty}\chi(p)t^p = \prod_{j=1}^{\infty}(1-t^j)^{-\varphi(j-1)}$$

from the theorem of Witt, which completes the proof.

EXAMPLE 5.2. Let X be $\{(x, y, z) \in \boldsymbol{C}^3;\ z^6 = x^2 - y^3,\ z \neq 0\}$. Then, we have $\dim H^1(X; \boldsymbol{C}) = 3$ and this cohomology is generated by the following differential forms

$$\theta = d\log(x^2 - y^3) ,$$
$$\omega_1 = \frac{-\frac{1}{3}ydx + \frac{1}{2}xdy}{(x^2-y^3)^{1/6}} ,$$
$$\omega_2 = \frac{-\frac{1}{3}y^2dx + \frac{1}{2}xydy}{(x^2-y^2)^{5/6}} .$$

The corresponding Lie algebra $\mathfrak{g}$ is $\mathrm{Lib}(\theta^*, \omega_1^*, \omega_2^*)/\mathcal{A}$ where $\mathcal{A}$ is generated by $[\theta^*, \omega_j^*]$ $(j = 1, 2)$. The author studies the relation between the Alexander polynomial of algebraic curves and the rational differential forms in [Ko2].

(5.3) Final remarks. The following construction gives a necessary condition for X to be $K(\pi, 1)$. Let $C_j(\pi_1(X); \boldsymbol{C})$ be the standard j-chains of the group $\pi_1(X)$. We have a homomorphism

$$\varphi_j\colon C_j(\pi_1(X); \boldsymbol{C}) \longrightarrow \mathrm{Hom}(\mathcal{M}_X(1)^j, \boldsymbol{C}) ,$$

which induces a homomorphism

$$\varphi_{j*}\colon H_j(\pi_1(X); \boldsymbol{C}) \longrightarrow H_j(\varprojlim_k \pi_1(X)/\Gamma_k \otimes \boldsymbol{C})$$

(see [Ko2]). Suppose that X has a homotopy type of a n-dimensional CW complex. If we assume that X is $K(\pi, 1)$, we have the following necessary condition (C).

(C) Let c be a cycle in $C_j(\pi_1(X); \boldsymbol{C})$. Then, $\varphi_j(c)$ must be exact in the complex

$$\longrightarrow \mathrm{Hom}\,(\mathcal{M}_X(1)^j; \boldsymbol{C}) \stackrel{\partial}{\longrightarrow} \mathrm{Hom}\,(\mathcal{M}_X(1)^{j-1}; \boldsymbol{C}) \longrightarrow .$$

The condition (C) is satisfied if $\mathcal{M}_X(1)$ is isomorphic to the minimal model $\mathcal{M}_X$.

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