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ON THE FUNDAMENTAL INEQUALITY FOR DEGENERATE SYSTEMS OF ENTIRE FUNCTIONS

Dedicated to Professor H. Ohtsuka on the occasion of his sixtieth birthday

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§1. Introduction

Let $f = (f_0, f_1, \cdots, f_n)$ $(n \geqq 1)$ be a transcendental system in $|z| < \infty$. That is, $f_0, f_1, \cdots, f_n$ are entire functions without common zeros and the characteristic function of f defined by H. Cartan ([1]):

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} U(re^{i\theta})d\theta - U(0)\,,$$

where

$$U(z) = \max_{0 \leqq j \leqq n} \log |f_j(z)|\,,$$

satisfies the condition

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty\,.$$

Let X be a set of linear combinations ($\not\equiv 0$) of $f_0, f_1, \cdots, f_n$ with coefficients in C in general position; that is, for any $n+1$ elements

$$a_{0j}f_0 + a_{1j}f_1 + \cdots + a_{nj}f_n \qquad (j = 1, \cdots, n+1)$$

in X, $n+1$ vectors $(a_{0j}, a_{1j}, \cdots, a_{nj})$ are linearly independent, and

$$\lambda = \dim \{(c_0, c_1, \cdots, c_n) \in C^{n+1};\ c_0f_0 + c_1f_1 + \cdots + c_nf_n = 0\}\,.$$

It is clear that $0 \leqq \lambda \leqq n-1$. We note that, for any $n+1$ elements $F_0, F_1, \cdots, F_n$ in X,

$$\dim \{(c_0, c_1, \cdots, c_n) \in C^{n+1};\ c_0F_0 + c_1F_1 + \cdots + c_nF_n = 0\}$$

is also equal to λ. We say that the system f is degenerate when $\lambda > 0$.

About fifty years ago, H. Cartan ([1]) proved

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THEOREM A. *When* $\lambda=0$, *for any* q *combinations* $F_1, \cdots, F_q$ *in* X,

$$(q-n-1)T(r,f) \leqq \sum_{j=1}^{q} N_n(r,0,F_j) + S(r),$$

where $N_n(r,0,F_j)=N_n(r,F_j)$ *in* [1] *and*

$$S(r)=O(\log r)+O(\log T(r,f))$$

as $r\to\infty$ *except for a set of finite linear measure.*

He also gave the following conjecture for $\lambda \geqq 1$ (originally in the case of algebroid functions).

CONJECTURE OF CARTAN. *For any* q *combinations* $F_1, \cdots, F_q$ *in* X,

$$(q-n-\lambda-1)T(r,f) \leqq \sum_{j=1}^{q} N_{n-\lambda}(r,0,F_j) + S(r).$$

It is uncertain that this conjecture is true or not in general, except when $\lambda=n-1$ ([1], p. 18). However, it is known that this holds in some special cases. For example,

THEOREM B. *For any* $n+\lambda+2$ *combinations* $F_1, \cdots, F_{n+\lambda+2}$ *in* X,

$$T(r,f) \leqq \sum_{j=1}^{n+\lambda+2} N_{n-\lambda}(r,0,F_j) + S(r)$$

([5]).

This theorem shows that Cartan's conjecture holds when $q=n+\lambda+2$.

The purpose of this paper is to prove that the conjecture is true when $\lambda=1$. Besides, we shall give an improvement of a result of B. Shiffman ([3]).

We use the standard notation of the Nevanlinna theory (See [2]).

§ 2. Lemmas

Let f, X and λ be as in Section 1. In this section, we shall give some lemmas which will be used in Section 3.

LEMMA 1. *For* $H_1, \cdots, H_k$ *in* X $(2 \leqq k \leqq n+1-\lambda)$,

$$m(r, \|H_1, \cdots, H_k\|/H_1 \cdots H_k) = S(r),$$

where $\|H_1, \cdots, H_k\|$ *means the Wronskian of* $H_1, \cdots, H_k$ (See [1]).

LEMMA 2. *For* $F_1, \cdots, F_q$ *in* X $(q \geqq n+1)$, *let*

$$v(z) = \max_{(\beta_1, \cdots, \beta_{q-n})} \log |F_{\beta_1}(z) \cdots F_{\beta_{q-n}}(z)| ,$$

where $\beta_1, \cdots, \beta_{q-n}$ *are mutually disjoint* $q - n$ *numbers from* $\{1, 2, \cdots, q\}$. *Then,*

$$(q - n)T(r, f) \leqq \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta}) d\theta + O(1)$$

(*See* [4], Lemma 3).

LEMMA 3. *For any* $G_1, \cdots, G_q$ *in* X $(q \geqq n + 1)$, *put*

$$u(z) = \min_{j_1 < \cdots < j_{n+1-\lambda}} \log |G_{j_1}(z) \cdots G_{j_{n+1-\lambda}}(z)| ,$$

where $G_{j_1}, \cdots, G_{j_{n+1-\lambda}}$ *are linearly independent and in* $\{G_j\}$. *Then,*

$$-S(r) \leqq \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta .$$

Proof. We suppose without loss of generality that $f_0, f_1, \cdots, f_{n-\lambda}$ are linearly independent. For an arbitrarily fixed $z = re^{i\theta}$, we may suppose that

$$|G_1(z)| \leqq |G_2(z)| \leqq \cdots \leqq |G_q(z)|$$

for brevity. Then, there are $G_{j_1}, \cdots, G_{j_{n+1-\lambda}}$ $(1 \leqq j_1 < \cdots < j_{n+1-\lambda} \leqq n + 1)$ which are linearly independent and satisfy

$$u(z) = \log |G_{j_1}(z) \cdots G_{j_{n+1-\lambda}}(z)| .$$

As

$$\|G_{j_1}, \cdots, G_{j_{n+1-\lambda}}\| = c \|f_0, \cdots, f_{n-\lambda}\| \qquad (c \neq 0, \text{ constant}) ,$$

we have

$$\frac{G_{j_1} \cdots G_{j_{n+1-\lambda}}}{\|G_{j_1}, \cdots, G_{j_{n+1-\lambda}}\|} = \frac{G_{j_1} \cdots G_{j_{n+1-\lambda}}}{c \|f_0, \cdots, f_{n-\lambda}\|} ,$$

so that

$$\log | \|f_0, \cdots, f_{n-\lambda}\| | \leqq u(z) + \sum_{j_1, \cdots, j_{n+1-\lambda}=1}^{q} \log^+ \left| \frac{\|G_{j_1}, \cdots, G_{j_{n+1-\lambda}}\|}{G_{j_1} \cdots G_{j_{n+1-\lambda}}} \right| + O(1) ,$$

where $O(1)$ is a constant dependent only on $G_1, \cdots, G_q$. This inequality holds for any z. Integrating with respect to θ from 0 to 2π and dividing by 2π, we obtain

$$N(r, 0, \|f_0, \cdots, f_{n-\lambda}\|) \leqq \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta + S(r) ,$$

which includes the desired inequality.

According to B. Shiffman ([3]), we let $\mathscr{E}_\rho$ denote the ring of entire functions of the form

$$g(z) = \sum_{k=1}^{p} \phi_k(z) \exp P_k(z)$$

where the P_k are polynomials of degree at most ρ and the ϕ_k are meromorphic functions in $|z| < \infty$ such that

$$T(r, \phi_k) = o(r^\rho) \qquad (r \to \infty) .$$

Moreover, according to Definition 1 ([3]), we say that a system $f = (f_0, \cdots, f_n)$ is of special exponential type of order ρ $(0 < \rho < \infty)$ if

$$c_1 r^\rho < T(r, f) < c_2 r^\rho \qquad \text{as } r \to \infty,$$

where c_1 and c_2 are positive constants, and if $f_0, \cdots, f_n$ belong to $\mathscr{E}_\rho$.

LEMMA 4. *Let $h = (h_1, \cdots, h_N)$ be of special exponential type of order ρ such that $h_j \not\equiv 0$ for $1 \leqq j \leqq N$. Then,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log \sum_{j=1}^{N} (1/|h_j(re^{i\theta})|)d\theta \leqq \frac{1}{2\pi}\int_0^{2\pi} \log \sum_{j=1}^{N} |h_j(re^{i\theta})|\, d\theta + o(r^\rho)$$

as $r \to \infty$ ([3], Lemma 2).

§ 3. Theorems

Let f, X and λ be as in Section 1.

THEOREM 1. *When $\lambda = 1$, for any q $(q \geqq n + 2)$ combinations $F_1, \cdots, F_q$ in X,*

$$(q - n - 2)T(r, f) \leqq \sum_{j=1}^{q} N_{n-1}(r, 0, F_j) + S(r) .$$

Proof. We may suppose that $f_1, \cdots, f_n$ are linearly independent without loss of generality since $\lambda = 1$. Now, there exists an integer k such that any k elements in $X_0 = \{F_j\}_{j=1}^{q}$ are linearly independent, but some $k + 1$ elements in X_0 are linearly dependent. It is clear that $1 \leqq k \leqq n$. For an arbitrarily fixed $z = re^{i\theta}$ $(r > 0)$, let $K_1, \cdots, K_{n+1}$ be $n + 1$ elements of X_0 such that $|K_1(z)|, \cdots, |K_{n+1}(z)|$ are the smallest $n + 1$ elements of

$\{|F_1(z)|, \cdots, |F_q(z)|\}$. As $\lambda = 1$, we suppose without loss of generality that $K_1, \cdots, K_n$ are linearly independent and

$$K_{n+1} = \alpha_1 K_1 + \cdots + \alpha_m K_m \qquad (m \geqq k,\ \alpha_1 \cdots \alpha_m \neq 0)\ .$$

Put

$$W_0 = \|K_1, \cdots, K_n\| \prod_{j=1}^{k} \|K_1, \cdots, K_{j-1}, K_{n+1}, K_{j+1}, \cdots, K_n\|\ ,$$

then, $W_0 \not\equiv 0$ and in $W_0, K_1, \cdots, K_k$ and K_{n+1} appear k times and $K_{k+1}, \cdots, K_n$ appear $k+1$ times. Since

$$\|K_1, \cdots, K_n\| = c_1 \|f_1, \cdots, f_n\|\ ,$$

where c_1 is a constant ($\neq 0$), we have the equality

$$(1) \qquad \frac{(F_1 \cdots F_q)^k}{W_0} = \frac{(F_1 \cdots F_q)^k}{c\|f_1, \cdots, f_n\|^{k+1}} \qquad (c = c_1^{k+1}\alpha_1 \cdots \alpha_k)$$

so that we obtain the following inequality as usual (cf. [1], [4]):

$$k(q-n-1)U(z) \leqq k\sum_{j=1}^{q} \log|F_j(z)| + \sum_{j_1,\cdots,j_n=1}^{q} \log^+ \left| \frac{\|F_{j_1}, \cdots, F_{j_n}\|}{F_{j_1} \cdots F_{j_n}} \right|$$
$$+ (n-k)U(z) - (k+1)\log|\|f_1, \cdots, f_n\|| + O(1)\ ,$$

where $O(1)$ is a constant depending only on X_0. This inequality holds for every z, so that, integrating with respect to θ from 0 to 2π and dividing by 2π, we obtain

$$k(q-n-1)T(r,f) \leqq k\sum_{j=1}^{q} N(r,0,F_j) + (n-k)T(r,f)$$
$$- (k+1)N(r,0,\|f_1, \cdots, f_n\|) + S(r)$$

by Lemma 1; that is,

$$(2) \qquad \begin{aligned} &(q-n-1-(n-k)/k)T(r,f) \\ &\leqq \sum_{j=1}^{q} N(r,0,F_j) - (1+1/k)N(r,0,\|f_1, \cdots, f_n\|) \\ &+ S(r) \leqq \sum_{j=1}^{q} N_{n-1}(r,0,F_j) + S(r)\ . \end{aligned}$$

We have the last inequality by calculating the multiplicity of zero at z of the righthand side of (1) as in the case of the fundamental theorem of Cartan ([1], p. 14).

I. Therefore, when $(n-k)/k \leqq 1$, that is, $n/2 \leqq k$, we have the theorem.

II. Next, we prove this theorem when $1 \leqq k < n/2$. To begin with, we note that there exists an element G in X_0 such that any $n-k$ elements in $X_0 - \{G\}$ are linearly independent. Indeed, let $G, H_1, \cdots, H_k$ be $k+1$ elements in X_0 which are linearly dependent, then G may be represented by $H_1, \cdots, H_k$:

$$G = d_1 H_1 + \cdots + d_k H_k \qquad (d_1 \cdots d_k \neq 0)$$

because of the definition of the number k. If there exist $I_1, \cdots, I_{n-k}$ in $X_0 - \{G\}$ which are linearly dependent, there are at least two distinct linear relations among $G, H_1, \cdots, H_k, I_1, \cdots, I_{n-k}$. This is a contradiction to the hypothesis of $\lambda = 1$. Let

$$X_0 - \{G\} = \{G_1, G_2, \cdots, G_{q-1}\} .$$

For a fixed $z = re^{i\theta}$ ($\neq 0$), we may suppose for brevity that

$$|G_j(z)| \leqq |G_{n+1}(z)| \leqq \cdots \leqq |G_{q-1}(z)| \qquad (j = 1, \cdots, n) .$$

We consider the following two cases.

(i) The case when $G_1, \cdots, G_n$ are linearly dependent.

Let, for example, without loss of generality

$$G_n = \beta_1 G_1 + \cdots + \beta_\nu G_\nu \qquad (\beta_1 \cdots \beta_\nu \neq 0)$$

then $\nu \geqq n - k$ and $G_1, \cdots, G_{n-1}, G$ are linearly independent. Consider the following product

$$W_1 = \|G_1, \cdots, G_{n-1}, G\| \prod_{j=1}^{n-k} \|G_1, \cdots, G_{j-1}, G_n, G_{j+1}, \cdots, G_{n-1}, G\| .$$

Then, $W_1 \not\equiv 0$ and in W_1, $G_1, \cdots, G_{n-k}$ appear $n-k$ times and $G_{n+1-k}, \cdots, G_{n-1}, G$ appear $n+1-k$ times. As in (1), we obtain

$$(3) \qquad \frac{(G_1 \cdots G_{q-1})^{n-k}}{W_1} = \frac{(G_1 \cdots G_{q-1})^{n-k}}{c_1 \|f_1, \cdots, f_n\|^{n+1-k}} \qquad (c_1 \neq 0, \text{ constant})$$

so that we have the following inequality:

$$\begin{aligned}(n-k)v_1(z) \leqq{}& (n-k)\sum_{j=1}^{q-1} \log |G_j(z)| + (n-k)\log|G(z)| + kU(z) \\ &+ \sum_{j_1, \cdots, j_n = 1}^{q} \log^+ \left| \frac{\|F_{j_1}, \cdots, F_{j_n}\|}{F_{j_1} \cdots F_{j_n}} \right| \\ &- (n+1-k)\log \big| \|f_1, \cdots, f_n\| \big| + O(1) ,\end{aligned}$$

where $v_1(z)$ is equal to $v(z)$ given in Lemma 2 for $G_1, \cdots, G_{q-1}$ and $O(1)$ is dependent only on X_0.

As

$$\log|G(z)| \leqq U(z) + O(1)$$

and $n-k>k$, we have

$$(n-k)\log|G(z)| + kU(z) \leqq k\log|G(z)| + (n-k)U(z) + O(1)\,.$$

Therefore,

$$\begin{aligned}(n-k)v_1(z) \leqq (n-k)\sum_{j=1}^{q-1}\log|G_j(z)| &+ k\log|G(z)| + (n-k)U(z)\\ &-(n+1-k)\log|\|f_1, \cdots, f_n\||\\ &+\sum_{j_1,\cdots,j_n=1}^{q}\log^+\left|\frac{\|F_{j_1}, \cdots, F_{j_n}\|}{F_{j_1}\cdots F_{j_n}}\right| + O(1)\,.\end{aligned} \tag{4}$$

(ii) The case when $G_1, \cdots, G_n$ are linearly independent.

In this case G can be represented by $G_1, \cdots, G_n$; that is, without loss of generality we may write

$$G = \gamma_1 G_1 + \cdots + \gamma_\mu G_\mu \qquad (\mu \geqq k,\ \gamma_1\cdots\gamma_\mu \neq 0)\,.$$

Consider the following product

$$W_2 = \|G_1, \cdots, G_n\|^{n+1-2k}\prod_{j=1}^{k}\|G_1, \cdots, G_{j-1}, G, G_{j+1}, \cdots, G_n\|\,.$$

Then, $W_2 \not\equiv 0$ and in W_2, $G_1, \cdots, G_k$ appear $n-k$ times, $G_{k+1}, \cdots, G_n$ appear $n+1-k$ times and G appears k times. As in (3), it holds the following equality:

$$\frac{(G_1\cdots G_{q-1})^{n-k}}{W_2} = \frac{(G_1\cdots G_{q-1})^{n-k}}{c_2\|f_1, \cdots, f_n\|^{n+1-k}} \qquad (c_2 \neq 0,\ \text{constant}) \tag{5}$$

from which we obtain the following inequality:

$$\begin{aligned}(n-k)v_1(z) \leqq (n-k)\sum_{j=1}^{q-1}\log|G_j(z)| &+ k\log|G(z)| + (n-k)U(z)\\ &-(n+1-k)\log|\|f_1, \cdots, f_n\||\\ &+\sum_{j_1,\cdots,j_n=1}^{q}\log^+\left|\frac{\|F_{j_1}, \cdots, F_{j_n}\|}{F_{j_1}\cdots F_{j_n}}\right| + O(1)\,.\end{aligned} \tag{6}$$

In both cases (i) and (ii), we obtain the same inequality (4) or (6) which holds for any z ($\neq 0$). Integrating the inequality with respect to θ from

0 to 2π, dividing by 2π and applying Lemmas 1 and 2, we have

$$\begin{aligned}(n-k)(q-n-1)T(r,f) \leqq (n-k)\sum_{j=1}^{q-1} N(r,0,G_j) + kN(r,0,G) \\ + (n-k)T(r,f) \\ - (n+1-k)N(r,0,\|f_1,\cdots,f_n\|) + S(r)\,,\end{aligned}$$

that is,

$$(7)\qquad \begin{aligned}(q-n-2)T(r,f) \leqq \sum_{j=1}^{q-1} N(r,0,G_j) + kN(r,0,G)/(n-k) \\ - (1+1/(n-k))N(r,0,\|f_1,\cdots,f_n\|) \\ + S(r) \leqq \sum_{j=1}^{q} N_{n-1}(r,0,F_j) + S(r)\,.\end{aligned}$$

We can easily prove the last inequality using the following inequality (8). Let m_j be the multiplicity of zero of G_j at z $(j=1,\cdots,q-1)$ and m that of G at z, then we obtain

$$(8)\qquad \begin{aligned}\text{the multiplicity of zero of } \frac{(G_1\cdots G_{q-1})^{n-k}G^k}{\|f_1,\cdots,f_n\|^{n+1-k}} \\ \leqq (n-k)\sum_{j=1}^{q-1}\min(m_j, n-1) + k\min(m, n-1)\end{aligned}$$

applying the method used in the proof of the fundamental theorem of Cartan ([1]) to

$$\frac{(G_1\cdots G_{q-1})^{n-k}G^k}{W_j} = \frac{(G_1\cdots G_{q-1})^{n-k}G^k}{c_j\|f_1,\cdots,f_n\|^{n+1-k}} \qquad (j=1 \text{ or } 2)\,.$$

Thus the proof of our theorem is complete.

COROLLARY 1. *Under the same assumption as in Theorem* 1,

$$(9)\qquad \sum_{F\in X}\delta(F) \leqq n+2\,.$$

If the equality holds in (9) *and if n is odd, there are at least two F in X for which $\delta(F)=1$. Here, $\delta(F) = 1 - \limsup_{r\to\infty} N(r,0,F)/T(r,f)$.*

Proof. We can prove easily (9) as usual. Now, suppose that n is odd and

$$\sum_{F\in X}\delta(F) = n+2\,.$$

In the sequel in this proof, we use the same notation as in the proof of

Theorem 1. Let ε be any positive number smaller than $1/n$. Then, there are $F_1, \cdots, F_q$ in X for which $\delta(F_j) > 0$ $(j = 1, \cdots, q)$ and such that

$$n + 2 - \varepsilon < \sum_{j=1}^{q} \delta(F_j) . \tag{10}$$

Then for $X_0 = \{F_1, \cdots, F_q\}$, $k < n/2$. Because, if $k \geqq n/2$, then $k \geqq (n + 1)/2$ since n is odd and from (2) we have

$$\sum_{j=1}^{q} \delta(F_j) \leqq n + 1 + (n - k)/k \leqq n + 2 - 2/(n + 2),$$

which contradicts (10).

There are $G, H_1, \cdots, H_k$ in X_0 such that

$$G = d_1 H_1 + \cdots + d_k H_k \qquad (d_1 \cdots d_k \neq 0)$$

as in II. Suppose

$$\delta = \min \{\delta(G), \delta(H_1), \cdots, \delta(H_k)\} < 1$$

and let ε' be any positive number smaller than $(1 - \delta)/n$. Let X_1 be a finite subset of X which contains X_0 such that

$$n + 2 - \varepsilon' < \sum_{F \in X_1} \delta(F) . \tag{11}$$

Then any $k + 1$ elements in X_1 which are not in coincidence with $\{G, H_1, \cdots, H_k\}$ are linearly independent as $k \leqq (n - 1)/2$ and $\lambda = 1$. Indeed, if there are $k + 1$ elements $I_1, \cdots, I_{k+1}$ in X_1 which are linearly dependent and don't coincide with $\{G, H_1, \cdots, H_k\}$, then $2(k + 1) \leqq n + 1$ and there are at least two linearly independent linear relations among $G, H_1, \cdots, H_k$, $I_1, \cdots, I_{k+1}$. That is, $\lambda \geqq 2$, which is a contradiction

Now, as is easily seen, we can use any one of $\{H_j\}_{j=1}^{k}$ instead of G in II so that from the first inequality in (7), we have

$$\sum_{F \in X_1} \delta(F) \leqq n + 1 + (n - 2k)(1 - \delta)/(n - k) ,$$

which contradicts (11). This shows that δ must be equal to 1 and so

$$\delta(G) = \delta(H_1) = \cdots = \delta(H_k) = 1 .$$

This completes the proof.

THEOREM 2. *Suppose that f is of special exponential type of order ρ. Then for any $F_1, \cdots, F_q$ in X,*

$$(q-n-\lambda-1)T(r,f) \leqq \sum_{j=1}^{q} N(r,0,F_j) + o(T(r,f)) + S(r) .$$

Proof. We have only to prove this theorem when $q \geqq n+\lambda+2$. Let $h_1 = F_1F_2 \cdots F_\lambda$, $h_2 = F_1 \cdots F_{\lambda-1}F_{\lambda+1}, \cdots, h_N = F_{q+1-\lambda} \cdots F_q$ $(N = \binom{q}{\lambda})$. Then, $h_j \not\equiv 0$ and $h = (h_1, \cdots, h_N)$ is a system of special exponential type of order ρ. Now, for an arbitrarily fixed $z = re^{i\theta}$ $(\neq 0)$, we suppose without loss of generality that

$$|F_1(z)| \leqq |F_2(z)| \leqq \cdots \leqq |F_q(z)| .$$

Then

$$u(z) + (q-n-1)U(z) \leqq \sum_{j=1}^{q} \log|F_j(z)| + \log \sum_{j=1}^{N} |1/h_j(z)| + O(1) ,$$

where $O(1)$ is dependent only on $\{F_j\}_{j=1}^{q}$, so that we have by Lemmas 3 and 4

$$(q-n-1)T(r,f) \leqq \sum_{j=1}^{q} N(r,0,F_j) + \frac{1}{2\pi}\int_0^{2\pi} \log \sum_{j=1}^{N} |h_j(re^{i\theta})|\, d\theta + o(r^\rho) + S(r) .$$

Here we use the following inequalities

$$|h_j(z)| \leqq a_j \exp \lambda U(z) \qquad (j = 1, \cdots, N) ,$$

where a_j are constants. These are true because

$$|F_\nu(z)| \leqq b_\nu \max_{0 \leqq j \leqq n} |f_j(z)| \qquad (\nu = 1, \cdots, q)$$

and

$$|h_j(z)| \leqq a_j (\max_{0 \leqq j \leqq n} |f_j(z)|)^\lambda = a_j \exp \lambda U(z) \qquad (j = 1, \cdots, N) .$$

That is, we obtain

$$(q-n-\lambda-1)T(r,f) \leqq \sum_{j=1}^{q} N(r,0,F_j) + o(r^\rho) + S(r) ,$$

which is the desired inequality.

COROLLARY 2. *Under the same assumption of Theorem 2,*

$$\sum_{F \in X} \delta(F) \leqq n + \lambda + 1 .$$

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