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LEMMA ON LOGARITHMIC DERIVATIVES AND HOLOMORPHIC CURVES IN ALGEBRAIC VARIETIES[1)]

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Nevanlinna's lemma on logarithmic derivatives played an essential role in the proof of the second main theorem for meromorphic functions on the complex plane C (cf., e.g., [17]). In [19, Lemma 2.3] it was generalized for entire holomorphic curves f: $C \to M$ in a compact complex manifold M (Lemma 2.3 in [19] is still valid for non-Kähler M). Here we call, in general, a holomorphic mapping from a domain of C or a Riemann surface into M a holomorphic curve in M, and sometimes use it in the sense of its image if no confusion occurs. Applying the above generalized lemma on logarithmic derivatives to holomorphic curves f: $C \to V$ in a complex projective algebraic smooth variety V and making use of Ochiai [22, Theorem A], we had an inequality of the second main theorem type for f and divisors on V (see [19, Main Theorem] and [20]). Other generalizations of Nevanlinna's lemma on logarithmic derivatives were obtained by Nevanlinna [16], Griffiths-King [10, § 9] and Vitter [23].

In this paper we first deal with holomorphic curves f: $\Delta^* \to M$ from the punctured disc $\Delta^* = \{|z| \geqq 1\}$ with center at the infinity ∞ of the Riemann sphere into a compact Kähler manifold M. Our first aim is to prove the following lemma on logarithmic derivatives which is a generalization of Nevanlinna [16, III, p. 370] and will play a crucial role in §§ 3 and 4 (see § 1 as to the notation):

MAIN LEMMA (2.2). *Let f: $\Delta^* \to M$ be a holomorphic curve in M, $\omega \in H^0(M, \mathfrak{A}^1_M)$ a d-closed meromorphic* 1-*form with logarithmic poles and put $f^*\omega = \zeta(z)dz$. Then we have*

$$m(r, \zeta) \leqq O(\log^+ T_f(r)) + O(\log r)$$

as $r \to \infty$ except for $r \in E$, where E is a subset of $[1, \infty)$ *with finite linear*

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measure.

The difficulty of the present case comes from the fact that the domain Δ^* is not simply connected. In the proof we shall apply the negative curvature method introduced by Griffiths-King [10, Propositions (6.9) and (9.3)] as in Vitter [23].

In § 3 we shall be concerned with the value distribution of holomorphic curves $f: \Delta^* \to V$ in a complex projective algebraic smooth variety V. Let D be an effective reduced divisor on V. Combining Main Lemma (2.2) with Ochiai [22, Theorem A] as in [19, § 3] and [20], we shall obtain an inequality of the second main theorem type

$$KT_f(r) \leqq N(r, \mathrm{Supp}\,(f^*D)) + S(r)\,, \tag{3.2}$$

where K is a positive constant independent of f and $S(r)$ is a small term such as

$$S(r) \leqq O(\log^+ T_f(r)) + O(\log r)$$

as $r \to \infty$ outside a set of r with finite linear measure (see Theorem (3.1)). As a corollary, we shall see that an inequality similar to (3.2) holds for a holomorphic curve from a compact Riemann surface minus a finite number of points into V (Corollary (3.3)).

In § 4 we shall study the extension problem of big Picard type for holomorphic curves $f: \Delta^* \to X$ in an algebraic subvariety X of general type in a quasi-Abelian variety A (cf. § 4). Let W be the union of subvarieties of X which are translations of non-trivial closed algebraic subgroups of A. Then W is a proper algebraic subvariety of X such that each irreducible component of W is foliated by translations of a non-trivial closed algebraic subgroup of A (see Lemma (4.1) whose proof is essentially due to Kawamata [13]). Using Lemma (4.4) due to M. Green by which he completed Ochiai's work [22] on Bloch's conjecture [2], and applying Main Lemma (2.2), we shall prove the following extension theorem of big Picard type:

THEOREM (4.5). *Any holomorphic curve f: $\Delta^* \to X$ has a holomorphic extension $\tilde{f}$: $\Delta = \Delta^* \cup \{\infty\} \to \bar{X}$ unless $f(\Delta^*) \subset W$, where $\bar{X}$ is a completion of X.*

As a corollary of Theorem (4.5) we will see that any holomorphic mapping f: $N - S \to X$ from a complex manifold N minus a thin analytic set S into X extends meromorphically over N unless $f(N-S) \subset W$

(Corollary (4.7)). Fujimoto ([3], [5]) and Green ([8]) obtained extension theorems of big Picard type for holomorphic mappings into projective space omitting hyperplanes in general position or intersecting them with positive defects (cf. also [4] and [7]). We will discuss the relationship between our results and those of Fujimoto and Green.

§ 1. Preliminaries

We set

$$\varDelta^* = \{z \in \boldsymbol{C}; |z| \geqq 1\}, \qquad \varDelta^*(r) = \{1 \leqq |z| < r\},$$
$$\Gamma(r) = \{|z| = r\}, \quad d = \partial + \bar{\partial}, \quad d^c = \frac{i}{4\pi}(\bar{\partial} - \partial).$$

In this paper we assume that functions on $\varDelta^*$ and mappings from $\varDelta^*$ are defined in neighborhoods of $\varDelta^*$ in $\boldsymbol{C}$. Let ξ be a function on $\varDelta^*$ satisfying

(i) ξ is differentiable outside a discrete set of points,

(ii) ξ is locally written as a difference of two subharmonic functions.

Then we have

$$(1.1) \qquad \int_1^r \frac{dt}{t} \int_{\varDelta^*(t)} dd^c\xi = \frac{1}{4\pi}\int_{\Gamma(r)} \xi(re^{i\theta})d\theta - \frac{1}{4\pi}\int_{\Gamma(1)} \xi(e^{i\theta})d\theta - (\log r)\int_{\Gamma(1)} d^c\xi,$$

where $dd^c\xi$ is taken in the sense of currents (cf., e.g., [10]). Let F be a multiplicative meromorphic function on $\varDelta^*$, i.e., F is a many-valued meromorphic function such that the modulus $|F|$ is one-valued. We set

$$m(r, F) = \frac{1}{2\pi}\int_{\Gamma(r)} \log^+|F(re^{i\theta})|\,d\theta,$$

where $\log^+|F| = \max\{0, \log|F|\}$. Let $D = \sum_{i=1}^{\infty} \nu_i a_i$ be a divisor with integral coefficients $\nu_i \in \mathbf{Z}$ on $\varDelta^*$ and set

$$n(t, D) = \sum_{1 \leqq |a_i| < t} \nu_i,$$
$$N(r, D) = \int_1^r \frac{n(t, D)}{t} dt.$$

Since $|F|$ is one-valued, the divisor (F) determined by F is defined on $\varDelta^*$ and so is the divisor $(F)_0$ (resp. $(F)_\infty$) of zeros (resp. poles) of F. We put

$$(1.2) \qquad T(r, F) = N(r, (F)_\infty) + m(r, F).$$

Applying (1.1) to $\xi = \log|F|^2$, we get

$$(1.3)\qquad T\left(r, \frac{1}{F}\right) = T(r, F) - \frac{1}{2\pi}\int_{\Gamma(1)} \log|F|\,d\theta - (\log r)\int_{\Gamma(1)} d^c\log|F|^2$$

(cf. [16, I, p. 369]).

Let M be a compact Kähler manifold and Ω a (1, 1)-form on M. We set

$$T_f(r, \Omega) = \int_1^r \frac{dt}{t}\int_{\Delta^*(t)} f^*\Omega$$

for a holomorphic curve f: $\Delta^* \to M$. Let D be an effective divisor on M and f: $\Delta^* \to M$ a holomorphic curve such that $f(\Delta^*)$ is not contained in the support Supp (D) of D. We take a metric $\|\cdot\|$ in the line bundle $[D]$ determined by D and denote by Ω_0 the curvature form of the metric. Letting $\sigma \in H^0(M, [D])$ be a global holomorphic section of $[D]$ such that the divisor (σ) determined by σ equals D and $\|\sigma\| \leqq 1$, we put

$$m_f(r, D) = \frac{1}{2\pi}\int_{\Gamma(r)} \log\frac{1}{\|\sigma\circ f\|}\,d\theta\,.$$

Applying (1.1) to $\xi = f^*\log\|\sigma\|^2$, we obtain

$$(1.4)\qquad \begin{aligned} T_f(r, \Omega_0) &= N(r, f^*D) + m_f(r, D) - m_f(1, D) \\ &\quad + (\log r)\int_{\Gamma(1)} d^c\log\|\sigma\circ f\|^2\,, \end{aligned}$$

where f^*D denotes the pull-backed divisor of D by f (cf. [10]). Let $\mathfrak{M}_M^*$ be the sheaf of germs of meromorphic functions which do not identically vanish, and define a sheaf $\mathfrak{A}_M^1$ by

$$(1.5)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & C^* & \longrightarrow & \mathfrak{M}_M^* & \xrightarrow{d\log} & \mathfrak{A}_M^1 & \longrightarrow & 0\,, \\ &&&& \cup && \cup && \\ &&&& \gamma & \mapsto & d\log\gamma && \end{array}$$

where C^* denotes the multiplicative group of non-zero complex numbers (cf. [19, § 1(b)]). Let $\omega \in H^0(M, \mathfrak{A}_M^1)$. Then we have the residue Res (ω) which is a divisor homologous to zero such that the line bundle [Res (ω)] equals $\delta\omega$, where δ: $H^0(M, \mathfrak{A}_M^1) \to H^1(M, C^*)$ is the coboundary operator associated with (1.5) (cf. [19, § 1(b)]). By Weil [24, p. 101] there is a multiplicative meromorphic funnction Θ on M such that the divisor (Θ) equals Res (ω). Since $d\log\Theta \in H^0(M, \mathfrak{A}_M^1)$ and $\omega - d\log\Theta$ is holomorphic every-where on M, we have the decomposition

$$\omega = d \log \Theta + \omega_1 , \tag{1.6}$$

where ω_1 is a holomorphic 1-form on M.

§ 2. Lemma on logarithmic derivatives

Let $f\colon \Delta^* \to M$ be a holomorphic curve in a compact Kähler manifold M with Kähler metric h and the associated form Ω, and set

$$T_f(r) = T_f(r, \Omega) .$$

Let $\omega \in H^0(M, \mathfrak{A}^1_M)$ and $\omega = d \log \Theta + \omega_1$ be the decomposition as (1.6). We set

$$\mathrm{Res}^+(\omega) = (\Theta)_0 , \qquad \mathrm{Res}^-(\omega) = (\Theta)_\infty .$$

Then by [24, p. 101] there is respectively a metric $\|\cdot\|$ in each of $[\mathrm{Res}^+(\omega)]$ and $[\mathrm{Res}^-(\omega)]$ such that both metrics have the same curvature form Ω_0; furthermore there are sections $\sigma_1 \in H^0(M, [\mathrm{Res}^-(\omega)])$ and $\sigma_2 \in H^0(M, [\mathrm{Res}^+(\omega)])$ such that $(\sigma_1) = \mathrm{Res}^-(\omega)$, $(\sigma_2) = \mathrm{Res}^+(\omega)$, $\|\sigma_i\| \leqq 1$ and

$$|\Theta| = \frac{\|\sigma_2\|}{\|\sigma_1\|} . \tag{2.1}$$

We put $f^*\omega = \zeta(z)dz$.

MAIN LEMMA (2.2). *Let the notation be as above. Assume that* Supp $(\mathrm{Res}\,(\omega)) \not\supset f(\Delta^*)$. *Then*

$$m(r, \zeta) \leqq 18 \log^+ T_f(r) + O(\log r) \tag{2.3}$$

for $r \geqq 1$ *outside a set of* r *with finite linear measure.*

Proof. Set $f^* d \log \Theta = \zeta_0 dz$ and $f^*\omega_1 = \zeta_1 dz$. Then we have

$$m(r, \zeta) \leqq m(r, \zeta_0) + m(r, \zeta_1) + \log 2 . \tag{2.4}$$

We first estimate the term $m(r, \zeta_1)$. Take a positive constant C_1 so that

$$|\omega_1(v)|^2 \leqq C_1 h(v, v)$$

for every holomorphic tangent vector $v \in T(M)$. Setting $f^*\Omega = s(z)(i/2) dz \wedge d\bar{z}$, we get

$$|\zeta_1(z)|^2 \leqq C_1 s(z) , \tag{2.5}$$

so that

$$\text{(2.6)}\qquad \begin{aligned} m(r,\zeta_1) &\leqq \frac{1}{4\pi}\int_{\Gamma(r)}\log(1+|\zeta_1|^2)d\theta \leqq \frac{1}{2}\log\left(1+\frac{C_1}{2\pi}\int_{\Gamma(r)} s d\theta\right) \\ &\leqq \frac{1}{2}\log\left(1+\frac{C_1}{2\pi r}\frac{d}{dr}\int_{\Delta^*(r)} f^*\Omega\right). \end{aligned}$$

Since $\int_{\Delta^*(r)} f^*\Omega$ is a monotone increasing function in $r \geqq 1$, the inequality

$$\frac{d}{dr}\int_{\Delta^*(r)} f^*\Omega \leqq \left(\int_{\Delta^*(r)} f^*\Omega\right)^2$$

holds for $r \geqq 1$ outside a set E_1 of r with finite linear measure. Combining this with (2.6), we have

$$\text{(2.7)}\qquad m(r,\zeta_1) \leqq \frac{1}{2}\log\left(1+\frac{C_1}{2\pi r}\left(\int_{\Delta^*(r)} f^*\Omega\right)^2\right)$$

for $r \notin E_1$; moreover we have

$$\text{(2.8)}\qquad \int_{\Delta^*(r)} f^*\Omega = r\frac{d}{dr}\int_1^r \frac{dt}{t}\int_{\Delta^*(t)} f^*\Omega = r\frac{d}{dr}T_f(r) \leqq r(T_f(r))^2$$

for $r \notin E_2$, where E_2 is a set similar to E_1. It follows from (2.7) and (2.8) that

$$\text{(2.9)}\qquad m(r,\zeta_1) \leqq 2\log^+ T_f(r) + \frac{1}{2}\log r + \frac{1}{2}\log^+\frac{C_1}{2\pi} + \frac{1}{2}\log 2$$

for $r \notin E_1 \cup E_2$.

Now we estimate the term $m(r,\zeta_0)$ in (2.4). Set $F = f^*\Theta$. Then F is a multiplicative meromorphic function on Δ^* and by (2.1), $|F| = \|\sigma_2\circ f\|/\|\sigma_1\circ f\|$, so that

$$m(r,F) \leqq \frac{1}{2\pi}\int_{\Gamma(r)}\log\frac{1}{\|\sigma_1\circ f\|}d\theta = m_f(r,\mathrm{Res}^-(\omega)) .$$

On the other hand, $N(r,(F)_\infty) \leqq N(r, f^*\mathrm{Res}^-(\omega))$. Thus we see, taking into account (1.4), that

$$\text{(2.10)}\qquad T(r,F) \leqq T_f(r,\Omega_0) + C_2\log r + C_3 ,$$

where C_2 and C_3 are some non-negative constants. Letting C_4 be a positive constant such that $\Omega_0 \leqq C_4\Omega$, we have

$$T_f(r, \Omega_0) \leqq C_4 T_f(r) . \tag{2.11}$$

We complete the proof by combining (2.9) with (2.10), (2.11) and the following one variable lemma.

LEMMA (2.12). *Let G be a multiplicative meromorphic function on Δ^*. Then the inequality*

$$m(r, G'/G) \leqq 16 \log^+ T(r, G) + O(\log r)$$

holds for $r \geqq 1$ outside a set E of r with finite linear measure.

Proof. Let w be an inhomogeneous coordinate of the 1-dimensional complex projective space $\boldsymbol{P}^1$. Then the standard Kähler form ψ_0 on $\boldsymbol{P}^1$ is written as

$$\psi_0 = \frac{1}{(1 + |w|^2)^2} \frac{i}{2\pi} dw \wedge d\bar{w} .$$

By Griffiths-King [10, Proposition (6.9)] we see that the singular form

$$\Psi = \frac{a_0(|w| + |w|^{-1})^{2+2\varepsilon}}{(\log b_0(1 + |w|^2))^2 (\log b_0(1 + |w|^{-2}))^2} \psi_0$$

satisfies

$$\operatorname{Ric} \Psi \geqq (|w| + |w|^{-1})^{-2\varepsilon} \Psi \tag{2.13}$$

for suitably chosen positive constants a_0, b_0 and ε ($\varepsilon < 1$). Since Ψ is invariant by transformations, $w \to e^{i\theta} w$, with real $\theta \in \boldsymbol{R}$ and G is multiplicative, the pull-backed form $G^* \Psi$ of Ψ by G is well-defined. We set

$$\begin{cases} g = \dfrac{G'}{G} , \\ G^* \Psi = \xi \dfrac{i}{2\pi} dz \wedge d\bar{z} = \dfrac{a_0(|G| + |G|^{-1})^{2\varepsilon}}{(\log b_0(1 + |G|^2))^2 (\log b_0(1 + |G|^{-2}))^2} \\ \qquad \times |g|^2 \dfrac{i}{2\pi} dz \wedge d\bar{z} . \end{cases} \tag{2.14}$$

Then by (2.13) we have

$$G^* \operatorname{Ric} \Psi = dd^c \log \xi \geqq (|G| + |G|^{-1})^{-2\varepsilon} \xi \frac{i}{2\pi} dz \wedge d\bar{z} . \tag{2.15}$$

Furthermore, taking $dd^c \log \xi$ in the sense of currents, we get

$$dd^c \log \xi = G^* \mathrm{Ric}\, \Psi - \varepsilon((G)_0 + (G)_\infty) + (g)_0 - (g)_\infty \,. \tag{2.16}$$

Noting that $(g)_\infty = \mathrm{Supp}\,((G)_0 + (G)_\infty) \leqq (G)_0 + (G)_\infty$, we deduce from (2.15) and (2.16) that

$$(|G| + |G|^{-1})^{-2\varepsilon} \xi \frac{i}{2\pi} dz \wedge d\bar{z} \leqq (1+\varepsilon)((G)_0 + (G)_\infty) + dd^c \log \xi \,. \tag{2.17}$$

We infer from (1.1) and (2.17) that

$$\begin{aligned} \int_1^r \frac{dt}{t} \int_{\Delta^*(t)} \frac{\xi}{(|G| + |G|^{-1})^{2\varepsilon}} \frac{i}{2\pi} dz \wedge d\bar{z} &\leqq (1+\varepsilon)(N(r, (G)_0) + N(r, (G)_\infty)) \\ &+ \frac{1}{4\pi} \int_{\Gamma(r)} \log \xi d\theta - (\log r) \int_{\Gamma(1)} d^c \log \xi - \frac{1}{4\pi} \int_{\Gamma(1)} \log \xi d\theta \,. \end{aligned} \tag{2.18}$$

We have by the definition of ξ in (2.14)

$$\begin{aligned} \frac{1}{4\pi} \int_{\Gamma(r)} \log \xi d\theta &\leqq m(r, g) + \varepsilon\Big(m(r, G) + m\Big(r, \frac{1}{G}\Big)\Big) \\ &+ \log^+ a_0 + \log^+ (\log b_0)^{-2} + \varepsilon \log 2 \,. \end{aligned} \tag{2.19}$$

We put

$$\begin{cases} A(t) = \displaystyle\int_{\Delta^*(t)} \frac{\xi}{(|G| + |G|^{-1})^{2\varepsilon}} \frac{i}{2\pi} dz \wedge d\bar{z} \,, \\ B(r) = \displaystyle\int_1^r \frac{A(t)}{t} dt \,. \end{cases} \tag{2.20}$$

Then inequalities (2.18), (2.19), (1.3) and $\varepsilon < 1$ yield

$$B(r) \leqq m(r, g) + 4T(r, G) + O(\log r) + O(1) \,. \tag{2.21}$$

Let us compute $m(r, g)$:

$$\begin{aligned} m(r, g) &= \frac{1}{4\pi} \int_{\Gamma(r)} \log^+ (\xi(|G| + |G|^{-1})^{-2\varepsilon} \frac{1}{a_0} \\ &\quad \times (\log b_0(1 + |G|^2))^2 (\log b_0(1 + |G|^{-2}))^2) d\theta \\ &\leqq \frac{1}{4\pi} \int_{\Gamma(r)} \log (1 + \xi(|G| + |G|^{-1})^{-2\varepsilon}) d\theta \\ &\quad + \frac{1}{2\pi} \int_{\Gamma(r)} \log(1 + \log^+ b_0 + 2 \log^+ |G|) d\theta \\ &\quad + \frac{1}{2\pi} \int_{\Gamma(r)} \log\Big(1 + \log^+ b_0 + 2 \log^+ \frac{1}{|G|}\Big) d\theta + \log^+ \frac{1}{a_0} \end{aligned} \tag{2.22}$$

$$\leqq \frac{1}{2}\log\left(1+\frac{1}{2\pi}\int_{\Gamma(r)}\xi(|G|+|G|^{-1})^{-2\varepsilon}d\theta\right)$$
$$+\log(1+\log^+ b_0+2m(r,G))$$
$$+\log\left(1+\log^+ b_0+2m\left(r,\frac{1}{G}\right)\right)+\log^+\frac{1}{a_0}$$

(by the concavity of "log")

$$\leqq \frac{1}{2}\log\left(1+\frac{1}{2r}\frac{d}{dr}A(r)\right)+2\log^+ T(r,G)+O(\log r)+O(1)\,.$$

Since $A(r)$ and $B(r)$ are monotone increasing, we see that the inequalities

$$\begin{cases}\dfrac{d}{dr}A(r)\leqq (A(r))^2\,,\\[2mm] \dfrac{d}{dr}B(r)\leqq (B(r))^2\end{cases} \tag{2.23}$$

hold for $r \geqq 1$ outside a set E of r with finite linear measure. Using the identity, $dB(r)/dr = A(r)/r$, and combining (2.22) with (2.21) and (2.23), we have

$$m(r,g)\leqq \frac{1}{2}\log\left(1+\frac{1}{2}r(B(r))^4\right)+2\log^+ T(r,G)+O(\log r)+O(1)$$
$$\leqq 2\log^+ m(r,g)+4\log^+ T(r,G)+O(\log r)+O(1)$$

for $r \notin E$. Note that $2\log^+ m(r,g) \leqq 2m(r,g)/e$ and $1-2/e > 1/4$. Hence we infer that

$$m(r,g)\leqq 16\log^+ T(r,G)+O(\log r)+O(1) \tag{2.24}$$

for $r \notin E$. This completes the proof.

Remark 1. In the above proof we used the metric form (cf. (2.14)) due to Griffiths-King [10, Proposition (6.9)] as in Vitter [23], whose curvature behaves nicely. If we use the following metric form due to Grauert-Reckziegel [6] which is simpler than (2.14)

$$\Phi = (1+|G|^{2\varepsilon})|G|^{2\varepsilon}|g|^2\frac{i}{2\pi}dz\wedge d\bar{z}$$

with any $\varepsilon > 0$, we have

$$\mathrm{Ric}\,\Phi = \varepsilon^2(|G|^{\varepsilon}+|G|^{-\varepsilon})^{-2}|g|^2\frac{i}{2\pi}dz\wedge d\bar{z}$$

and obtain the following estimate:

$$m(r, g) \leqq 8\varepsilon T(r, G) + 4 \log^+ \frac{1}{\varepsilon} + 8 \log^+ T(r, G) + (\varepsilon C_1 + 2) \log r + \varepsilon C_2 + C_3 \tag{2.25}$$

for $r \geqq 1$ outside a set E of r with finite linear measure, where C_i, $i = 1, 2, 3$, are non-negative constants independent of r and ε, and E is independent of ε. Because of the presence of the term $8\varepsilon T(r, G)$ in (2.25), inequality (2.24) is better than (2.25), but inequality (2.25) is also sufficient for the later use in §§ 3 and 4.

Remark 2. It is hoped that Main Lemma (2.2) can be applied to the study of holomorphic curves in compact Kähler manifolds.

EXAMPLE. We give an example of $f: \Delta^* \to M$ and Θ such that $f^*\Theta$ is really infinitely many-valued. Let $M = \boldsymbol{C}/(\boldsymbol{Z} + \tau\boldsymbol{Z})$ be an elliptic curve with $\operatorname{Im}\tau > 0$ and $\pi: \boldsymbol{C} \to M$ the universal covering. Take any two points a, b of M so that $n(a - b) \neq 0$ for all $n \in \boldsymbol{Z}$. Then there is a multiplicative meromorphic function Θ on M such that $(\Theta)_0 = a$ and $(\Theta)_\infty = b$. Since $n(a - b) \neq 0$ for all $n \in \boldsymbol{Z}$, Θ is infinitely many-valued. Let γ_1 (resp. γ_2) be the cycle in M defined by γ_1: $[0, 1] \ni t \to \pi(t) \in M$ (resp. γ_2: $[0, 1] \ni t \to \pi(t\tau) \in M$). Then $\{\gamma_1, \gamma_2\}$ is a basis of the first homology group $H_1(M, \boldsymbol{Z})$. One of the periods $\frac{1}{2\pi i}\int_{\gamma_j} d\log\Theta$, $j = 1, 2$, is irrational. Suppose that $\frac{1}{2\pi i}\int_{\gamma_1} d\log\Theta$ is irrational. The covering $\boldsymbol{C} \xrightarrow{\pi} M$ is decomposed as

$$\boldsymbol{C} \xrightarrow{\pi_0} \boldsymbol{C}/\boldsymbol{Z} \xrightarrow{\pi_1} \boldsymbol{C}/(\boldsymbol{Z} + \tau\boldsymbol{Z}) = M .$$

Set $\gamma: [0, 1] \ni t \to \pi_0(t) \in \boldsymbol{C}/\boldsymbol{Z} = \boldsymbol{C}^*$, which is a cycle around ∞ (or 0). Then $\pi_{1*}\gamma = \gamma_1$, so that the period $\frac{1}{2\pi i}\int_\gamma d\log\Theta \circ \pi_1$ is irrational. Let i: $\Delta^* \to \boldsymbol{C}^*$ be the natural inclusion mapping and put $f = \pi_1 \circ i: \Delta^* \to M$. Then $f^*\Theta$ is infinitely many-valued.

Let $\zeta^{(k)}$ denote the k-th derivative of ζ. Using Main Lemma (2.2) inductively, one easily see the following:

COROLLARY (2.26). *Let the notation be as above. Then the inequality*

$$T(r, \zeta^{(k)}) \leqq (k + 1)N(r, \operatorname{Supp}(f^*\operatorname{Res}(\omega))) + O(\log^+ T_f(r)) + O(\log r)$$

holds for $r \geqq 1$ *outside a set* E *with finite linear measure.*

§ 3. Inequality of the second main theorem type

Let V be a complex projective algebraic smooth variety of dimension n, D an effective reduced divisor on V and $\Omega^1_V(\log D)$ the sheaf of logarithmic 1-forms along D (cf., e.g., [12], [19]). Then $\{\omega \in H^0(V, \mathfrak{A}^1_V);\ \mathrm{Supp}(\mathrm{Res}(\omega)) \subset D\}$ spans $H^0(V, \Omega^1_V(\log D))$ over C (see [19, Proposition 1.2]). Assume that there is a system $\{\omega_i\}_{i=1}^{n+1}$ in $H^0(V, \Omega^1_V(\log D))$ such that $\phi_i = \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_{n+1}$, $1 \leqq i \leqq n+1$, are linearly independent over C. Let $f: \Delta^* \to V$ be a holomorphic curve such that $f(\Delta^*) \not\subset D$. Assume that f is non-degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$, i.e., $f(\Delta^*) \not\subset \{\sum c_i \phi_i = 0\}$ for any $(c_i) \in C^{n+1} - \{O\}$. Let Ω be a Kähler form on V and set $T_f(r) = T_f(r, \Omega)$. Making use of Corollary (2.26) and Ochiai [22, Theorem A] as in [19, § 3] and [20], we have the following theorem.

THEOREM (3.1). *Let $\{\omega_i\}_{i=1}^{n+1} \subset H^0(V, \Omega^1_V(\log D))$ and $f: \Delta^* \to V$ be as above. Then there is a positive constant K depending only on Ω and $\{\omega_i\}_{i=1}^{n+1}$, such that*

$$KT_f(r) < N(r, \mathrm{Supp}(f^*D)) + S(r)\,, \tag{3.2}$$

where $S(r) = O(\log^+ T_f(r)) + O(\log r)$ as $r \to \infty$ outside a set of r with finite linear measure.

Let $\bar{R}$ be a compact Riemann surface, $R = \bar{R} - \{a_i\}_{i=1}^q$ with distinct $a_i \in \bar{R}$ and $q < \infty$, and $a_0 \in R$ any point. Then there is a multiplicative meromorphic function α such that $(\alpha) = qa_0 - \sum a_i$. The modulus $|\alpha|$ turns out to be an exhaustion function of R. Set

$$R(t) = \{|\alpha| < t\}\,.$$

Let $f: R \to V$ be a holomorphic curve. Put

$$T_f(r) = \int_1^r \frac{dt}{t} \int_{R(t)} f^*\Omega$$

for f and

$$n\Big(t, \sum_{i=1}^{\infty} \nu_i b_i\Big) = \sum_{|\alpha(b_i)|<t} \nu_i\,, \qquad N\Big(r, \sum_{i=1}^{\infty} \nu_i b_i\Big) = \int_1^r \frac{n(t, \sum \nu_i b_i)}{t}\, dt$$

for a divisor $\sum_{i=1}^{\infty} \nu_i b_i$ on R (cf. § 1 and [10, § 2]). For r_0 large enough, $R - R(r_0)$ is a union of Δ_i^*, $i = 1, \cdots, q$, where $\Delta_i^* \cap \Delta_j^* = \emptyset$ for $i \neq j$ and $\Delta_i = \Delta_i^* \cup \{a_i\}$ are a neighborhood of a_i in $\bar{R}$. Moreover the restriction

$1/z_i = 1/(\alpha|_{\Delta_i})$ of $1/\alpha$ on every Δ_i gives rise to a local coordinate in Δ_i and Δ_1^* is written as $\Delta_i^* = \{r_0 \leqq |z_i| < \infty\}$. Therefore we have the following corollary of Theorem (3.1):

COROLLARY (3.3). *Let* $\{\omega_i\}_{i=1}^{n+1} \subset H^0(V, \Omega_V^1(\log D))$ *be as in Theorem* (3.1). *Let* $f\colon R \to V$ *be a holomorphic curve which is non-degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$. *Then there is a positive constant* K *depending only on* Ω *and* $\{\omega_i\}$ *such that*

$$KT_f(r) \leqq N(r, \operatorname{Supp}(f^*D)) + S(r)\,,$$

where $S(r)$ *is a small quantity as in* (3.2).

Remark. Assume that $\dim V = 1$, and let us calculate sharp K in (3.2) in the way of the proof. The higher dimensional case will be discussed in § 4. Set $T_f(r) = T_f(r, \Omega)$ for Ω such that $\int_V \Omega = 1$.

(1) Let $V = \boldsymbol{P}^1$. If the assumption of Theorem (3.1) for D is satisfied, D must consist of at least three points. Let $D = \sum_{i=1}^q w_i$ be an effective reduced divisor on $\boldsymbol{P}^1$ with inhomogeneous coordinate w such that $w_1 = 0$, $w_2 = \infty$ and $q \geqq 3$. Let $w_0 \in \boldsymbol{P}^1 - D$ and set

$$\omega_1 = d \log w \in H^0(\boldsymbol{P}^1, \Omega_{\boldsymbol{P}^1}^1(\log D))$$
$$\omega_2 = d \log \frac{\prod_{i=3}^q (w - w_i)}{(w - w_0)^{q-2}} \in H^0(\boldsymbol{P}^1, \Omega_{\boldsymbol{P}^1}^1(\log (D + w_0)))\,.$$

Then $\phi = \omega_2/\omega_1$ is a rational function such that the degree $\deg(\phi)_\infty$ of the divisor $(\phi)_\infty$ is $q - 1$. We have by [18, Theorem 1]

$$T(r, f^*\phi) = (q - 1)T_f(r) + O(1)\,. \tag{3.4}$$

Setting $f^*\omega_i = \zeta_i dz$ for $i = 1, 2$, we obtain

$$\begin{aligned} T(r, f^*\phi) &= T\left(r, \frac{\zeta_1}{\zeta_2}\right) \leqq T(r, \zeta_1) + T(r, \zeta_2) + O(\log r) + O(1) \\ &= N(r, f^{-1}(w_0)) + \sum_{i=1}^q N(r, f^{-1}(w_i)) + S(r)\,. \end{aligned} \tag{3.5}$$

Hence we have by (3.4), (3.5) and the first main theorem (1.4)

$$(q - 2)T_f(r) \leqq \sum_{i=1}^q N(r, f^{-1}(w_i)) + S(r)\,,$$

which is the famous second main theorem for meromorphic functions on $\boldsymbol{C}$.

(2) Let V be an elliptic curve. Then inequality (3.2) holds if D

consists of one point $a_0 \in V$. On the other hand, $H^0(V, \Omega^1_V(\log a_0)) = H^0(V, \Omega^1_V)$ is of dimension 1, where Ω^1_V denotes the sheaf of germs of holomorphic 1-forms over V, so that the assumption of Theorem (3.1) is not fulfilled, but we can derive (3.2) for $D = a_0$ by the method of the proof of Theorem (3.1) as follows. Take any point $a_1 \in V - \{a_0\}$. Then there is a multiplicative meromorphic function Θ such that $(\Theta) = a_0 - a_1$. Set $\omega_1 = d \log \Theta \in H^0(V, \Omega^1_V(\log(a_0 + a_1)))$ and let $\omega_2 \in H^0(V, \Omega^1_V)$ and $\omega_2 \neq 0$. We put $\phi = \omega_1/\omega_2$. Then ϕ is a rational function on V such that $\deg (\phi)_\infty = \deg (a_0 + a_1) = 2$, so that by [18, Theorem 1] we have

$$T(r, f^*\phi) = 2T_f(r) + O(1) . \tag{3.6}$$

Letting $f^*\omega_i = \zeta_i dz$, $i = 1, 2$, we see that

$$\begin{aligned} T(r, f^*\phi) &= T\left(r, \frac{\zeta_1}{\zeta_2}\right) \leqq T(r, \zeta_1) + T(r, \zeta_2) + O(\log r) + O(1) \\ &= N(r, f^{-1}(a_0)) + N(r, f^{-1}(a_1)) + S(r) . \end{aligned} \tag{3.7}$$

Therefore it follows from (3.6) and (3.7) that

$$T_f(r) \leqq N(r, f^{-1}(a_0)) + S(r) .$$

(3) Let V be a compact Riemann surface of genus $\geqq 2$. Then $\dim H^0(V, \Omega^1_V) \geqq 2$, so that the condition of Theorem (3.1) is satisfied with $D = 0$. This implies the well-known fact that the isolated singularity of a holomorphic curve in V of genus $\geqq 2$ is removable.

§ 4. Extension theorem of big Picard type

Let A be a quasi-Abelian variety (see [11] and [12]), i.e., A is an algebraic group which is commutative and admits the exact sequence

$$0 \longrightarrow (C^*)^l \longrightarrow A \xrightarrow{\rho} A_0 \longrightarrow 0 ,$$

where A_0 is an Abelian variety. Taking the natural embedding $(C^*)^l \subset (P^1)^l$, we have a smooth completion $\bar{A} = (P^1)^l \times_{(C^*)^l} A$ of A with boundary divisor D which has only normal crossings, and the canonical projection $\bar{\rho}: \bar{A} \to A_0$. One may regard $\bar{\rho}: \bar{A} \to A_0$ as a fibre bundle over A_0 with fibre $(P^1)^l$ and structure group $(C^*)^l$. Let X be an algebraic subvariety of A which is of general type or equally of hyperbolic type (cf. [11]). In the present case, X is of general type if and only if the group $\{a \in A; X + a = X\}$ of translations which preserve X is finite (see [11] and [12]). Let

W be the union of subvarieties of X which are translations of non-trivial closed algebraic subgroups of A.

LEMMA (4.1). *Let X and W be as above. Then W is a proper algebraic subvariety of X, of which each irreducible component is foliated by translations of a non-trivial closed algebraic subgroup of A.*

Remark. This lemma was proved in [21] when $\dim X = 2$. In [13], Kawamata proved it in the case when A is an Abelian variety. To prove it in the present form, we need further consideration. The idea of the following proof is due to Kawamata.

Proof. Let $\pi\colon \boldsymbol{C}^m \to A$ be the universal covering with $m = \dim A$, $A = \boldsymbol{C}^m/\Lambda$ with a discrete subgroup Λ (cf. [12]), and $\lambda\colon \boldsymbol{C}^m - \{O\} \to \boldsymbol{P}^{m-1}$ the natural mapping into the projective space $\boldsymbol{P}^{m-1}$ of lines in $\boldsymbol{C}^m$ through the origin O. Let U be a small open set in $\boldsymbol{P}^{m-1}$ and set

$$s(\bar{X}) = \bigcup_{x \in U} (\bar{X} + \pi(s(x)), x) \subset \bar{A} \times U$$

for a holomorphic section $s \in \Gamma(U, \boldsymbol{C}^m - \{O\})$, where $\bar{X}$ is the Zariski closure of X in $\bar{A}$ and "$+\,\pi(s(x))$" stands for the natural action of A on $\bar{A}$. Hence $s(\bar{X})$ is an analytic subset of $\bar{A} \times U$. We set

$$Y_U = \bigcap_{s \in \Gamma(U, \boldsymbol{C}^m - \{O\})} s(\bar{X}) \subset \bar{A} \times U\,.$$

Then Y_U is again an analytic subset of $\bar{A} \times U$ and we see that a point $(a, x) \in \bar{A} \times U$ belongs to Y_U if and only if $a + \phi(t) \in \bar{X}$ for every $t \in \boldsymbol{C}$, where $\phi(t)$ is the analytic 1-parameter subgroup of A such that $d\phi/dt(0) = x$. Let B_x denote the Zariski closure in A of the analytic 1-parameter subgroup of A associated with the vector x. Then we have that

$$(a, x) \in Y_U \Longleftrightarrow a + B_x \subset \bar{X}\,. \tag{4.2}$$

Let U' be another small open set in $\boldsymbol{P}^{m-1}$. Then it follows from (4.2) that Y_U coincides with $Y_{U'}$ in $\bar{A} \times (U \cap U')$, so that $Y = \bigcup_U Y_U$ is a well-defined analytic subset of $\bar{A} \times \boldsymbol{P}^{m-1}$ and so algebraic in $\bar{A} \times \boldsymbol{P}^{m-1}$. Let $Y_0 = Y \cap (A \times \boldsymbol{P}^{m-1})$ and $p\colon A \times \boldsymbol{P}^{m-1} \to A$ be the projection. Then by (4.2) and the definition of W, $p(Y_0) = W$. Since p is proper and rational, W is a closed algebraic subvariety of X. Now we must show that $W \neq X$ and each irreducible component of W is foliated by translations of a non-trivial closed algebraic subgroup of A. Since there are only countably many non-trivial closed algebraic subgroups in A as in the case of an Abelian variety (cf. [12]), we denote them by $\{B_i\}_{i=1}^{\infty}$. We see by (4.2) that

$$a \in W \Longleftrightarrow a + B_i \subset W \quad \text{for some } B_i . \tag{4.3}$$

Let $h_i: X \to A/B_i$ be the restriction of the natural morphism from A onto the quotient A/B_i on X and put

$$W_i = \{x \in X; \dim_x h_i^{-1}(h_i(x)) = \dim B_i\} .$$

Then W_i is a proper algebraic subvariety of X because X is of general type, and $W = \bigcup_i W_i$ by (4.3). Let $W_i = \bigcup_j W_{ij}$ be the irreducible decomposition of W_i. We get a countable covering $W = \bigcup_{ij} W_{ij}$. It is clear that every $W_{ij} \neq X$. By virtue of Baire's theorem we see that $W \neq X$ and that an irreducible component of W must be one W_{ij} which is foliated by translations of B_i.

Let Z be an algebraic subvariety of A and Z_{reg} the set of regular points of Z with the inclusion mapping $i: Z_{\text{reg}} \to A$. Let $J_\nu(Z_{\text{reg}})$ (resp. $J_\nu(A)$) be the ν-th holomorphic jet bundle over Z_{reg} (resp. A) (see [22]). Then the mapping i naturally induces a bundle homomorphism $i_*: J_\nu(Z_{\text{reg}}) \to J_\nu(A)$. Since A is a quasi-Abelian variety, there is a regular isomorphism $J_\nu(A) \cong A \times C^{\nu m}$. Let $q: A \times C^{\nu m} \to C^{\nu m}$ be the projection and set

$$I_\nu = q \circ i_*: J_\nu(Z_{\text{reg}}) \to C^{\nu m} \quad \text{(cf. [22])} .$$

We denote by $j_\nu g$ the ν-th jet of a holomorphic curve $g: (C, 0) \to Z_{\text{reg}}$ from a neighborhood of the origin 0 of C into Z_{reg}.

LEMMA (4.4). *Let X and W be as in Lemma* (4.1). *Let g: $(C, 0) \to X$ be a holomorphic curve such that $g(0) \notin W$ and $g(0) \in Z_{\text{reg}}$, where Z is the Zariski closure of the image of g in X. Then the differential*

$$dI_\nu: T(J_\nu(Z_{\text{reg}})) \to T(C^{\nu m})$$

is injective at $j_\nu g$ for all large ν, where $T(\cdot)$ denotes the holomorphic tangent bundle.

This lemma is a refined version of a lemma due to M. Green by which he completed the work of Ochiai [22] on Bloch's conjecture [2][2]. M. Green showed it in case A is complete, i.e., A is an Abelian variety, but his proof works in the non-complete case.

2) M. Green gave the proof of the lemma at "Conference on Geometric Function Theory" held at Katata, Sept. 1–6, 1978.

Let $\overline{X}$ be the Zariski closure of X in $\overline{A}$.

THEOREM (4.5) (big Picard theorem). *Let X and W be as above. Then any holomorphic curve $f: \Delta^* \to X$ has a holomorphic extension $\tilde{f}: \Delta = \Delta^* \cup \{\infty\} \to \overline{X}$ unless $f(\Delta^*) \subset W$.*

Proof. We fix a Kähler form Ω on $\overline{A}$ and set $T_f(r) = T_f(r, \Omega)$. By (2.10), (2.11) and [16, I, p. 369], it suffices to prove that $T_f(r)/\log r$ is bounded as $r \to \infty$. Let Z be the Zariski closure of $f(\Delta^*)$ in X. Then $f(z) \notin W$ and $f(z) \in Z_{\mathrm{reg}}$ for $z \in \Delta^*$ except for some discrete set of points. Making use of Lemma (4.4) and Main Lemma (2.2) (more precisely, Corollary (2.26)) as in [19], we have

$$T_f(r) \leqq K_1 \log^+ T_f(r) + K_2 \log r \tag{4.6}$$

for $r \geqq 1$ outside a set E of r with finite linear measure, where K_1 and K_2 are non-negative constants independent of r. We may assume that f is not a constant curve. Then we see that $T_f(r) \uparrow \infty$ as $r \uparrow \infty$. Since $T_f(r)$ is a convex increasing function in $\log r$, $T_f(r)/\log r$ is monotone increasing. Therefore we have by (4.6)

$$\lim_{r\to\infty} \frac{T_f(r)}{\log r} \leqq K_2\ ,$$

which completes the proof.

COROLLARY (4.7). *Let $f: N - S \to X$ be a holomorphic mapping from a complex manifold N minus a thin analytic set S into X. If $f(N - S) \not\subset W$, then f extends to a meromorphic mapping $\tilde{f}: N \to \overline{X}$.*

Proof. We take an embedding $\overline{X} \subset \boldsymbol{P}^N$ into some projective space $\boldsymbol{P}^N$ with a homogeneous coordinate system $(w_0, \cdots, w_N)$ such that $f(N - S) \not\subset \{w_0 = 0\}$. Let $f_i = f^*(w_i/w_0)$. It is enough to prove that every f_i extends to a meromorphic function on N. By virtue of Hartogs' theorem, we may assume that $N = \Delta \times \Delta^{k-1}$ and $S = \{\infty\} \times \Delta^{k-1}$ ($k = \dim N$). Put $S' = \{z' \in \Delta^{k-1}; \Delta^* \times \{z'\} \subset f^{-1}(W)\}$, which is a thin analytic set of Δ^{k-1}. By Hartogs' theorem, it suffices to show that f_i extends meromorphically over $\Delta \times (\Delta^{k-1} - S')$. For each $z_0' \in \Delta^{k-1} - S'$, the holomorphic curve $f(\cdot, z_0')$: $\Delta^* \ni z_1 \mapsto f(z_1, z_0') \in X$ does not lie in W. By Theorem (4.5), f is extendable over Δ, so that $f_i(\cdot, z_0')$ is meromorphic in Δ. We put $f_i(z_1, z_0') = z_1^{\mu(z_0')} \cdot g_i(z_1, z_0')$, where $\mu(z_0') \in Z$ and $g_i(\infty, z_0') \neq 0, \infty$. Take a small neighborhood U of z_0'. Then we see that $\mu(z')$ is bounded in $z' \in U$. Therefore $f_i(z_1, z')$

is meromorphic in $\Delta \times U$, and so is in $\Delta \times (\Delta^{k-1} - S')$.

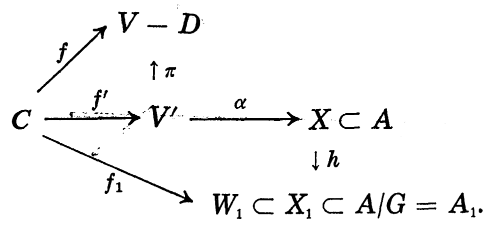
Remark. Fujimoto ([3], [5]) and Green ([8]) proved extension theorems of big Picard type for holomorphic mappings into P^n omitting more than $n + 1$ hyperplanes in general position. Their results will be discussed in Example 1 below. Here, let us give a simple and new observation to another theorem of Green [8, Parts 4 and 5] from the viewpoint of this paper. He proved the following interesting theorem:

Let f: $C \to V \subset P^n$ be a holomorphic curve into a subvariety V of P^N omitting $\dim V + 2$ *non-redundant hyperplane sections of V. Then f is algebraically degenerate,* i.e., *$f(C)$ is contained in a proper subvariety of V.*

Here "*non-redundant*" means that no one of the hyperplane sections is contained in the union of the others. Let D be the sum of the $\dim V + 2$ hyperplane sections of V. Let π: $V' \to V - D$ be a desingularization of $V - D$ and $\bar{V}'$ a smooth completion of V' with boundary divisor D' of normal crossing type. Setting $\bar{q}(V') = \dim H^0(\bar{V}', \Omega^1_{\bar{V}'}(\log D'))$ which is called the logarithmic irregularity of V' ([12]), we have by the assumption for D

$$\bar{q}(V') < \dim V' . \tag{4.8}$$

We may assume that f can be lifted to a holomorphic curve f': $C \to V'$ such that $\pi \circ f' = f$. Let α: $V' \to A$ be the quasi-Albanese mapping (see [12]), $X = \overline{\alpha(V')}$ the Zariski closure of $\alpha(V')$ in A, G the identity component of the group $\{a \in A; X + a = X\}$, h: $A \to A/G = A_1$ the canonical mapping onto the quotient $A/G = A_1$ and $X_1 = \overline{h(X)}$. Then (4.8) implies that X_1 is of positive dimension and of general type. Let W_1 be the union of subvarieties of X_1 which are translations of non-trivial closed algebraic subgroups of A_1. By Lemma (4.1), W_1 is a proper algebraic subvariety of X_1. Put $f_1 = h \circ \alpha \circ f'$:



Then we have $f_1(C) \subset W_1$ by Theorem (4.5) if f_1 is not a constant curve, so that f is algebraically degenerate. Thus inequality (4.8) implies the algebraic degeneracy of f'; this is just a non-complete version of Bloch's conjecture (see [2], [22]).

EXAMPLE 1. Let D_i, $0 \leqq i \leqq n+k$, be $n+k+1$ distinct hyperplanes of P^n and set $V = P^n - \sum_0^{n+k} D_i$. Then we have

$$\bar{q}(V) = \dim H^0(P^n, \Omega^1_{P^n}(\log \textstyle\sum_0^{n+k} D_i)) = n+k\,.$$

Assume that $k \geqq 1$. Then $\bar{q}(V) > \dim V$. Let $\alpha\colon V \to A = (C^*)^{n+k}$ be the quasi-Albanese mapping and $f\colon C \to V$ a holomorphic curve. As in Remark above, we see that $\alpha\circ f(C)$ lies in a translation of a closed algebraic subgroup of A, so that $f(C)$ lies in a proper linear subspace of P^n. This fact was proved in Green [7, Theorem 2].

Suppose that $k=1$ and the D_i's are in general position. We take a system $(w_0, w_1, \cdots, w_n)$ of homogeneous coordinates of P^n so that $D_i = \{w_i = 0\}$ for $i = 0, 1, \cdots, n$ and $D_{n+1} = \{w_0 + \cdots + w_n = 0\}$. Put $x_i = w_i/w_0$ for $i = 1, \cdots, n$. Then the quasi-Albanese mapping $\alpha\colon V \to (C^*)^{n+1}$ is written as

$$\alpha\colon V \ni (x_1, \cdots, x_n) \mapsto \left(x_1, \cdots, x_n, \frac{1 + x_1 + \cdots + x_n}{n}\right) \in (C^*)^{n+1}\,.$$

Set $X = \{(y_1, \cdots, y_{n+1}) \in (C^*)^{n+1};\ ny_{n+1} = 1 + y_1 + \cdots + y_n\}$. Then $\alpha\colon V \to X$ is biregular and so X is of general type. Let Π denotes the union of diagonal hyperplanes of $\sum_1^{n+1} D_i$ (see [15, Example 16, p. 395] and [4, p. 243]). Let W be the proper algebraic subvariety of X as in Lemma (4.1). Then $W = \alpha(\Pi)$. In this case, Fujimoto [4, Theorem 5.5] and Green [8, Part 3] showed Theorem (4.5) (cf. also [1], [5] and [7]). In case $n=2$, the figure of W in X is as follows:

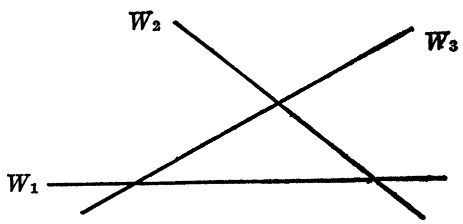


Fig. 1

Here each $W_i \cong C^*$ and $W = W_1 \cup W_2 \cup W_3$.

EXAMPLE 2 ([14, Example 1, p. 92]). Let $Q = \sum_{i=0}^4 L_i$ be a complete quadrilateral in P^2 as in Kobayashi [14, Example 1, p. 92], and set $V = P^2 - Q$. Take a homogeneous coordinate system (w_0, w_1, w_2) of P^2 such that

$$L_0 = \{w_0 = 0\}\,, \quad L_1 = \{w_1 = 0\}\,, \quad L_2 = \{w_0 - w_1 = 0\}\,,$$
$$L_3 = \{w_2 = 0\}\,, \quad L_4 = \{w_0 - w_2 = 0\}\,.$$

Then we have the quasi-Albanese mapping

$$\alpha\colon V \ni (x_1, x_2) \mapsto \left(\frac{1}{2}x_1, x_1 - 1, \frac{1}{2}x_2, x_2 - 1\right) \in (C^*)^4\,,$$

where $x_i = w_i/w_0$, $i = 1, 2$. Thus $\alpha(V) = X = \{(y_1, \cdots, y_4) \in (C^*)^4;\ y_2 = 2y_1 - 1,\ y_4 = 2y_3 - 1\}$ and $\alpha\colon V \to X$ is biregular. Since there is no C^* in X, $W = \emptyset$. Therefore any holomorphic curve $f\colon \Delta^* \to V$ is extendable to a holomorphic curve $\tilde{f}\colon \Delta \to P^2$. Kobayashi [14, p. 92] proved this fact by showing that V is hyperbolically embedded in P^2.

EXAMPLE 3 ([19, § 4(b)]). Let $X = \{(x_1, \cdots, x_{n+2}) \in (C^*)^{n+2};\ x_{n+1} = 1 + x_1 + \cdots + x_{n-1},\ x_{n+2} = x_1 + \cdots + x_n\}$ and $n \geqq 3$. Then X is of general type. For the simplicity, let $n = 3$. Let W be the proper algebraic subvariety of X as in Lemma (4.1). Then we see that

$$W = W_1 \cup W_2 \cup \cdots \cup W_5\,,$$

where $W_1 \cong (C^*)^2$ and $W_i \cong C^*$ for $i = 2, 3, 4, 5$. The figure of W in X is illustrated as follows:

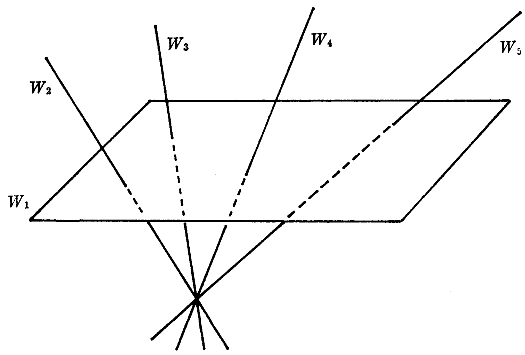


Fig. 2

EXAMPLE 4 ([22, § 5]). Let $A = E_1 \times \cdots \times E_4$ be a product of four elliptic curves E_i belonging to distinct isogeny classes. Let X be the hypersurface of A as defined in Ochiai [22, § 5]. Then the algebraic subvariety W of X as in Lemma (4.1) consists of several elliptic curves which are mutually disjoint.

Lastly we pose a problem and a conjecture related to Theorems (4.5) and (3.1).

PROBLEM. *What can we say of the Kobayashi hyperbolicity of X or $X-W$ in Theorem* (4.5)?

Remark. Green [9] gave a nice criterion of the Kobayashi hyperbolicity, but in the present case his criterion does not work since an irreducible component W' of W may admit a non-constant holomorphic curve $f\colon C \to W'$ omitting the other components of W (see Examples 3 and 4).

The case (2) of Remark to Theorem (3.1) suggests that the following conjecture may be true:

CONJECTURE. Let A be an Abelian variety and D an effective reduced divisor on A. Let $\Omega \in c_1([D])$ be a semi-positive definite (1, 1)-form in the first Chern class $c_1([D]) \in H^{1,1}(A, C)$ of $[D]$. Then we have

$$T_f(r, \Omega) \leqq N(r, f^*D) + S(r)$$

for algebraically non-degenerate holomorphic curves $f\colon \Delta^*$ (or C) $\to A$, where $S(r) = O(\log^+ T_f(r, \Omega)) + O(\log r)$ as $r \to \infty$ outside a set of r with finite linear measure.

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