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SOME LIE ALGEBRAS OF VECTOR FIELDS AND THEIR DERIVATIONS CASE OF PARTIALLY CLASSICAL TYPE

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Introduction

Let $(M, \mathscr{F})$ be a smooth foliated manifold, and $\mathscr{T}(M, \mathscr{F})$ the Lie algebra of all leaf-tangent vector fields on M.

Assume that $(M, \mathscr{F})$ admits a partially classical structure τ, ω or θ (see § 4.1). Then we have natural Lie subalgebras $\mathscr{T}_\tau(M, \mathscr{F})$, $\mathscr{T}_{c\tau}(M, \mathscr{F})$, $\mathscr{T}_\omega(M, \mathscr{F})$, $\mathscr{T}_{c\omega}(M, \mathscr{F})$, $\mathscr{T}_\theta(M, \mathscr{F})$ of the Lie algebra $\mathscr{T}(M, \mathscr{F}) = \mathscr{T}_0(M, \mathscr{F})$ (see § 4.2). These Lie algebras including $\mathscr{T}(M, \mathscr{F})$ itself are called of partially classical type. Here we study the structures of those Lie algebras and their derivation algebras.

The derivation algebra of $\mathscr{T}(M, \mathscr{F})$ is naturally isomorphic to the Lie algebra $\mathscr{L}(M, \mathscr{F})$ of all locally foliation-preserving vector fields on M (see [4]). We get similarly natural Lie subalgebras $\mathscr{L}_\tau(M, \mathscr{F})$, $\mathscr{L}_{c\tau}(M, \mathscr{F})$, $\mathscr{L}_\omega(M, \mathscr{F})$, $\mathscr{L}_{c\omega}(M, \mathscr{F})$, $\mathscr{L}_\theta(M, \mathscr{F})$ of $\mathscr{L}(M, \mathscr{F}) = \mathscr{L}_0(M, \mathscr{F})$ (see § 4.2).

Our main results (announced in [11]) are

MAIN THEOREM. *Let M be a smooth $(p + q)$-dimensional manifold, and $\mathscr{F}$ a codimension q foliation on M. Assume that $(M, \mathscr{F})$ is equipped with a partially classical structure τ, ω or θ.*

(a) *Let $\sigma = 0$, $c\tau (p \neq 1)$, $c\omega$ or θ. Then*

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) = 0 ,$$
$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_\sigma(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) .$$

(b) *Let $\sigma = \tau (p \neq 1)$ or ω. Then*

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{L}_\sigma(M, \mathscr{F}) ,$$
$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) .$$

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The Lie algebras of partially classical type correspond in the formal case to some parts of É. Cartan's classification of infinite intransitive Lie algebras (see T. Morimoto [5]), and N. Nakanishi [6] discusses about derivations of such Lie algebras.

This work is in a series of F. Takens' work [9] and the author's [2], [3], [4] which we use in this paper for general references. However this work is also an attempt to define natural and typical Lie algebras of vector fields in the intransitive case.

The content of this paper is arranged as follows. In § 1, we introduce Lie algebras $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$ for standard foliations on Euclidean spaces, and study their structures. In § 2, we introduce the grading of subalgebras of $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$, consisting of vector fields with polynomial coefficients, and the finite dimensional Lie subalgebras $\mathfrak{B}_\sigma$ of $\mathscr{T}_\sigma$, on which any derivations of $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$ are determined. We prove Main Theorem (Theorem 3.6) for $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$ (flat case) in § 3. In § 4, we define partially classical structures on $(M, \mathscr{F})$ and Lie algebras $\mathscr{T}_\sigma(M, \mathscr{F})$ and $\mathscr{L}_\sigma(M, \mathscr{F})$, and prove Main Theorem (Theorem 4.10). Here it is essential that derivations of $\mathscr{T}_\sigma(M, \mathscr{F})$ $(\sigma = \tau, \omega, \theta)$ are localizable (Proposition 4.8). In § 5, we give a further discussion on $H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F}))$ and $H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F}))$ for $\sigma = \tau$ and ω. In § 6, we treat the pathological case ($p = 1$ and $\sigma = \tau$ or $c\tau$), prove our theorem for $\mathscr{T}_{c\tau}$ and $\mathscr{L}_{c\tau}$, and remark that there are derivations of $\mathscr{T}_\tau$ and $\mathscr{L}_\tau$ which cannot be realized by vector fields (properly outer derivations).

All manifolds, foliations, vector fields, etc. are assumed to be of C^∞-class, throughout this paper. However, for flat case, our method here is applicable without any change to the case of analytic or complex category. Similarly the results in [2], [3], [4] for flat case are valid in those categories.

§ 1. Lie algebras $\mathscr{T}, \mathscr{L}$ and their subalgebras

1.1. *Notations and definitions.* Fix a coordinate system $v_1, \cdots, v_p$ in a p-dimensional Euclidean space $U = \boldsymbol{R}^p$, and $w_1, \cdots, w_q$ in a q-dimensional $W = \boldsymbol{R}^q$. We consider vector fields on the $(p+q)$-dimensional space $V = U \oplus W = \boldsymbol{R}^{p+q}$, and the Lie algebra $\mathfrak{A}(V)$ of all vector fields on V. Denote $\partial/\partial v_i$ by ∂_i $(i = 1, \cdots, p)$, and $\partial/\partial w_\alpha$ by ∂_α $(\alpha = 1, \cdots, q)$. Use Latin indices $i, j, k, \cdots$ for variables in U, and Greek indices $\alpha, \beta, \cdots$ for variables in W, otherwise stated.

Consider the standard codimension q foliation $\mathscr{F}$ on V, defined by parallel p-planes: π_W^{-1} (a point), where π_W is a canonical projection of V onto W. Let $\mathscr{T}$ be the Lie algebra of all leaf-tangent vector fields on V, then by [4], the derivation algebra $\mathscr{D}er(\mathscr{T})$ of $\mathscr{T}$ is naturally isomorphic to the Lie algebra $\mathscr{L}$ of foliation-preserving vector fields, and $\mathscr{L}$ is decomposed as

$$\mathscr{L} = \mathscr{T} + \mathscr{L}' ,$$

where $\mathscr{L}'$ is naturally isomorphic to the Lie algebra $\mathfrak{A}(W)$.

Let $\Omega(V)$ be the exterior algebra of all differential forms on V, and $\mathscr{I}(\mathscr{F})$ be the ideal of $\Omega(V)$, generated by $dw_1, \cdots, dw_q$, that is,

$$\mathscr{I}(\mathscr{F}) = \{\alpha \in \Omega(V); \alpha(X_1, X_2, \cdots) = 0 \text{ for } X_i \in \mathscr{T}\} .$$

Denote by $\Omega(\mathscr{F})$ the complement of $\mathscr{I}(\mathscr{F})$ in $\Omega(V)$, that is,

$$\Omega(V) = \Omega(\mathscr{F}) + \mathscr{I}(\mathscr{F}) ,$$

and $\Omega(\mathscr{F})$ is the exterior algebra over $C^\infty(V)$, generated by $dv_1, \cdots, dv_p$.

LEMMA 1.1. (i) *The ideal $\mathscr{I}(\mathscr{F})$ is L_X-stable for $X \in \mathscr{L}$, and i_X-stable for $X \in \mathscr{T}$, where L_X means the Lie derivative with respect to X, and $i_X\alpha$ means the interior product of X and α.*

(ii) *The ideal $\mathscr{I}(\mathscr{F})$ is a differential ideal, that is, $d\mathscr{I}(\mathscr{F}) \subset \mathscr{I}(\mathscr{F})$.*

Proof. It is enough to use the fact that $\mathscr{T}$ is an ideal of $\mathscr{L}$. Q.E.D.

1.2. Put $p = n$, $x_i = v_i$ $(i = 1, \cdots, n)$, and $\tau = dx_1 \wedge \cdots \wedge dx_n$. A leaf-tangent vector field X is called *partially conformally unimodular*, if $L_X\tau$ is congruent to $\phi(w)\tau$ modulo $\mathscr{I}(\mathscr{F})$ for some function $\phi(w) \in C^\infty(W)$. Moreover, if the function $\phi(w)$ is zero, X is called *partially unimodular.* Then by Lemma 1.1, we get two Lie subalgebras of $\mathscr{T}$:

$$\mathscr{T}_\tau = \{X \in \mathscr{T}; L_X\tau \equiv 0 \ (\text{mod}\ \mathscr{I}(\mathscr{F}))\} ,$$
$$\mathscr{T}_{c\tau} = \{X \in \mathscr{T}; L_X\tau \equiv \phi(w)\tau \ (\text{mod}\ \mathscr{I}(\mathscr{F})) \quad \text{for some } \phi(w) \in C^\infty(W)\} .$$

LEMMA 1.2. *Write $X \in \mathscr{T}$ as $X = \sum_{i=1}^n f_i(x, w)\partial_i$.*

(i) *X is partially unimodular, if and only if $\sum_{i=1}^n \partial_i f_i = 0$.*

(ii) *X is partially conformally unimodular, if and only if $\sum_{i=1}^n \partial_i f_i \in C^\infty(W)$.*

(iii) *$\mathscr{T}_\tau$ is an ideal of $\mathscr{T}_{c\tau}$, and $[\mathscr{T}_{c\tau}, \mathscr{T}_{c\tau}] \subset \mathscr{T}_\tau$.*

(iv) *Put* $I_\tau = \sum_{i=1}^n x_i\partial_i \in \mathscr{T}_{c\tau}$. *Then, any* $X \in \mathscr{T}_{c\tau}$ *is decomposed as* $X = X_1 + X_2$, *where* $X_1 \in \mathscr{T}_\tau$, $X_2 = n^{-1}\phi(w)I_\tau$ *and* $L_X\tau \equiv \phi(w)\tau \pmod{\mathscr{I}(\mathscr{F})}$. *Namely,*

$$\mathscr{T}_{c\tau} = \mathscr{T}_\tau + n^{-1}C^\infty(W)I_\tau .$$

(v) *If* X *is partially unimodular,* $i_X\tau \in \Omega(\mathscr{F})$ *is a partially closed* $(n-1)$*-form, that is,* $di_X\tau \in \mathscr{I}(\mathscr{F})$. *The mapping which assigns* $i_X\tau$ *to* $X \in \mathscr{T}_\tau$ *is the linear isomorphism of* $\mathscr{T}_\tau$ *onto the subspace of* $\Omega^{n-1}(\mathscr{F})$, *consisting of partially closed* $(n-1)$*-forms, that is,*

$$\mathscr{T}_\tau \cong \{\alpha \in \Omega^{n-1}(\mathscr{F});\ d\alpha \in \mathscr{I}(\mathscr{F})\} .$$

(vi) *Let* $\alpha \in \Omega^{n-2}(\mathscr{F})$. *Denote by* $X[\alpha]$ *the partially unimodular vector field corresponding to the* $\Omega(\mathscr{F})$*-part of* $d\alpha$. *The assignment of* $X[\alpha]$ *to* α *defines a mapping of* $\Omega^{n-2}(\mathscr{F})$ *onto* $\mathscr{T}_\tau$. *Put*

$$\alpha_{ij} = dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n ,$$

then for any functions f *and* g *in* $C^\infty(V)$,

$$X[f\alpha_{ij}] = (-1)^{i+j-1}\{(\partial_j f)\partial_i - (\partial_i f)\partial_j\} \qquad (1 \leqq i < j \leqq n) ,$$
$$[X[f\alpha_{ij}], X[g\alpha_{ij}]] = (-1)^{i+j}X[\{f, g\}_{ij}\alpha_{ij}] ,$$

where $\{\ ,\ \}_{ij}$ *is the Poisson bracket in* x_i *and* x_j, *that is,*

$$\{f, g\}_{ij} = (\partial_i f)(\partial_j g) - (\partial_j f)(\partial_i g) .$$

1.3. Put $p = 2n$, $x_i = v_i$, $y_i = v_{n+i}$ $(1 \leqq i \leqq n)$, and $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. A leaf-tangent vector field X is called *partially conformally symplectic*, if $L_X\omega$ is congruent to a form $\phi(w)\omega$ modulo $\mathscr{I}(\mathscr{F})$ for some function $\phi(w) \in C^\infty(W)$. Moreover, if the function $\phi(w)$ is zero, X is called *partially symplectic*. Then by Lemma 1.1, we get two Lie subalgebras of $\mathscr{T}$:

$$\mathscr{T}_\omega = \{X \in \mathscr{T};\ L_X\omega \equiv 0 \pmod{\mathscr{I}(\mathscr{F})}\} .$$
$$\mathscr{T}_{c\omega} = \{X \in \mathscr{T};\ L_X\omega \equiv \phi(w)\omega \pmod{\mathscr{I}(\mathscr{F})} \text{ for some } \phi(w) \in C^\infty(W)\} .$$

LEMMA 1.3. *Write* $X \in \mathscr{T}$ *as* $X = \sum_{i=1}^{2n} f_i(x, y, w)\partial_i$.

(i) *The following three conditions are equivalent:*

(a) X *is partially symplectic;*

(b) $\partial_i f_j = -\partial_{j+n} f_{i+n}$, $\partial_{i+n} f_j = \partial_{j+n} f_i$, $\partial_i f_{j+n} = \partial_j f_{i+n}$ $(1 \leqq i, j \leqq n)$;

(c) *there is a unique function* $H \in C^\infty(V)$ *up to functions in* $C^\infty(W)$ *such that* $f_i = \partial_{i+n}H$ *and* $f_{i+n} = -\partial_i H$ *for any* i $(1 \leqq i \leqq n)$.

(ii) *X is partially conformally symplectic, if and only if for any* $1 \leqq i, j \leqq n$, $\partial_{i+n} f_j = \partial_{j+n} f_i$, $\partial_i f_{j+n} = \partial_j f_{i+n}$, $\partial_i f_j + \partial_{j+n} f_{i+n} = \delta_{ij} \phi(w)$ *for some function* $\phi(w) \in C^\infty(W)$, *where* δ_{ij} *is Kronecker's delta.*

(iii) $\mathscr{T}_\omega$ *is an ideal of* $\mathscr{T}_{c\omega}$, *and* $[\mathscr{T}_{c\omega}, \mathscr{T}_{c\omega}] \subset \mathscr{T}_\omega$.

(iv) *Put* $I_\omega = \sum_{i=1}^{2n} v_i \partial_i \in \mathscr{T}_{c\omega}$. *Then, any* $X \in \mathscr{T}_{c\omega}$ *is decomposed as* $X = X_1 + X_2$ *where* $X_1 \in \mathscr{T}_\omega$, $X_2 = 2^{-1}\phi(w) I_\omega$ *and* $L_X \omega \equiv \phi(w)\omega \pmod{\mathscr{I}(\mathscr{F})}$. *Namely,*

$$\mathscr{T}_{c\omega} = \mathscr{T}_\omega + 2^{-1} C^\infty(W) I_\omega .$$

(v) *Denote by* X_H *the partially symplectic vector field corresponding to a function* $H \in C^\infty(V)$ *as in* (i). *Then for any functions* H *and* K *in* $C^\infty(V)$,

$$[X_H, X_K] = -X_{\{H,K\}} ,$$

where $\{\ ,\ \}$ *is the Poisson bracket in the variables* $x_1, \cdots, x_n, y_1, \cdots, y_n$, *that is,*

$$\{H, K\} = \sum_{i=1}^{n} (H_{x_i} K_{y_i} - H_{y_i} K_{x_i}) .$$

1.4. Put $p = 2n + 1$, $x_i = v_i$, $y_i = v_{i+n}$ $(1 \leqq i \leqq n)$, $z = v_{2n+1}$, and $\theta = dz - \sum_{i=1}^{2n} y_i dx_i$. A leaf-tangent vector field X is called *partially contact*, if $L_X\theta$ is congruent to a form $\phi(x, y, z, w)\theta$ modulo $\mathscr{I}(\mathscr{F})$ for some function $\phi(x, y, z, w) \in C^\infty(V)$. We denote by $\mathscr{T}_\theta$ the Lie subalgebra of $\mathscr{T}$, consisting of all partially contact vector fields.

LEMMA 1.4. *Write* $X \in \mathscr{T}$ *as* $X = \sum_{i=1}^{2n+1} f_i(x, y, z, w)\partial_i$.

(i) *X is partially contact, if and only if there is a unique function* $k(x, y, z, w) \in C^\infty(V)$ *such that for any* i $(1 \leqq i \leqq n)$,

$$f_i = -\partial_{i+n} k , \quad f_{i+n} = (\partial_i k) + y_i(\partial_{2n+1} k) , \quad \text{and} \quad f_{2n+1} = k - \sum_{i=1}^{n} y_i(\partial_{i+n} k) .$$

Here, k *is obtained as* $k = i_X\theta = f_{2n+1} - \sum_{i=1}^{n} y_i f_i$.

(ii) *Let* $\sharp$ *be a mapping from* $\mathscr{T}_\theta$ *to* $C^\infty(V)$, *which assigns* $X^\sharp = i_X\theta$ *to* $X \in \mathscr{T}_\theta$. *Then the linear mapping* $\sharp$ *is bijective.*

(iii) *If X is partially contact, it satisfies the following equalities:*

$$(\sharp)_1 \quad \partial_{i+n} f_{2n+1} = \sum_{i=1}^{n} y_j(\partial_{i+n} f_j) \qquad (1 \leqq i \leqq n) ,$$

$$(\sharp)_2 \quad y_i(\partial_{2n+1} f_{2n+1} - \sum_{j=1}^{n} y_j(\partial_{2n+1} f_j)) = f_{i+n} - \partial_i f_{2n+1} + \sum_{j=1}^{n} y_j(\partial_i f_i) .$$

(iv) *Denote by* $\flat$ *the inverse mapping of* $\sharp$. *We can introduce the generalized Poisson bracket* $((\ ,\ ))$ *in* $C^\infty(V)$ *as follows*:

$$\begin{aligned}((f, g)) &= [f^\flat, g^\flat]^\sharp \qquad (f, g \in C^\infty(V)) \\ &= \{f, g\}_{xy} - f_z\Big(g - \sum_{j=1}^{n} y_j g_{y_j}\Big) + g_z\Big(f - \sum_{j=1}^{n} y_j f_{y_j}\Big)\,,\end{aligned}$$

where $\{\ ,\ \}_{xy}$ *is the Poisson bracket in the variables* $x_1, \cdots, x_n, y_1, \cdots, y_n$.

Using Poincaré's lemma with parameters, we can get these Lemmata 1.2 ~ 1.4 similarly as in the transitive case (see [2] and [3]).

1.5.

Remark. Our Lie algebras $\mathscr{T}, \mathscr{T}_\tau, \mathscr{T}_{c\tau}, \mathscr{T}_\omega, \mathscr{T}_{c\omega}$ and $\mathscr{T}_\theta$ correspond in the formal case to "intransitive Lie algebras whose transitive parts are infinite and primitive" (see E. Cartan [1] and T. Morimoto [5]) with the exception of $\mathscr{T}_\tau$ for $p = 1$. In fact, $\mathscr{T}_\sigma$ is isomorphic to the topological completion of the Lie algebra $C^\infty(W) \otimes \mathfrak{A}_\sigma(U)$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ and θ, where $\mathfrak{A}_\sigma(U)$ is the Lie algebra of classical type, consisting of vector fields on the Euclidean space U (see [3]). In our notation, $\mathscr{T}_0 = \mathscr{T}$ and $\mathfrak{A}_0(U) = \mathfrak{A}(U)$.

1.6. *Lie algebras* $\mathscr{L}_\sigma$. We can similarly define Lie subalgebras of $\mathscr{L}$ as follows: for $\sigma = \tau$ or ω,

$$\begin{aligned}\mathscr{L}_\sigma &= \{X \in \mathscr{L}\,;\, L_X\sigma \in \mathscr{I}(\mathscr{F})\}\,, \\ \mathscr{L}_{c\sigma} &= \{X \in \mathscr{L}\,;\, L_X\sigma \equiv \phi(w)\sigma \pmod{\mathscr{I}(\mathscr{F})}\}\,,\end{aligned}$$

and

$$\mathscr{L}_\theta = \{X \in \mathscr{L}\,;\, L_X\theta \equiv \phi(x, y, z, w)\theta \pmod{\mathscr{I}(\mathscr{F})}\}\,.$$

Then we get easily

LEMMA 1.5. (i) *Let* $\sigma = \tau, \omega$ *or* θ. *Then* $L_X\sigma = 0$ *for any* $X \in \mathscr{L}'$.

(ii) *Let* $\sigma = \tau, c\tau, \omega, c\omega$ *or* θ. *Then* $\mathscr{T}_\sigma$ *is an ideal of* $\mathscr{L}_\sigma$, $\mathscr{L}'$ *is a subalgebra of* $\mathscr{L}_\sigma$, *and* $\mathscr{L}_\sigma$ *is a direct sum of* $\mathscr{T}_\sigma$ *and* $\mathscr{L}'$ *as vector spaces*:

$$\mathscr{L}_\sigma = \mathscr{T}_\sigma + \mathscr{L}'\,.$$

Proof. (i) By the definition of σ, σ and $d\sigma$ belong to $\Omega(\mathscr{F})$, and $i_X = 0$ on $\Omega(\mathscr{F})$ for $X \in \mathscr{L}'$. Hence, $L_X\sigma = 0$.

(ii) Since $\mathscr{T}$ is an ideal of $\mathscr{L}$, the assertion (ii) follows from (i).

Q.E.D.

By Lemmata 1.1 and 1.5, we get easily

PROPOSITION 1.6. *Let σ_1 be congruent to σ modulo $\mathscr{I}(\mathscr{F})$ for $\sigma = \tau, \omega$ or θ. Then Lie algebras $\mathscr{T}_{\sigma_1}$ and $\mathscr{L}_{\sigma_1}$ given by σ_1 are the same as by σ:*

$$\mathscr{T}_{\sigma_1} = \mathscr{T}_{\sigma}\,, \qquad \mathscr{L}_{\sigma_1} = \mathscr{L}_{\sigma} \quad (\sigma = \tau, \omega, \theta)\,,$$
$$\mathscr{T}_{c\sigma_1} = \mathscr{T}_{c\sigma}\,, \quad \mathscr{L}_{c\sigma_1} = \mathscr{L}_{c\sigma} \quad (\sigma = \tau, \omega)\,,$$

and $L_X\sigma_1 \in \mathscr{I}(\mathscr{F})$ for $X \in \mathscr{L}'$, hence $\mathscr{L}'$ is a subalgebra of $\mathscr{L}_{\sigma_1}$ for any σ.

1.7.

Remarks. Let $\sigma_a = a(w)\sigma$ $(\sigma = \tau, \omega, \theta)$ for some non-vanishing function $a(w) \in C^\infty(W)$. Put $X = \sum_{i=1}^{p} f_i(v, w)\partial_i \in \mathscr{T}$ and $Y = \sum_{\alpha=1}^{q} g_\alpha(w)\partial_\alpha \in \mathscr{L}'$, then

$$L_X\sigma_a = a(w)L_X\sigma\ ,$$
$$L_Y\sigma_a \equiv \frac{1}{a(w)}\sum_{\alpha=1}^{q}(\partial_\alpha a)(w)g_\alpha(w)\sigma_a \qquad (\mathrm{mod}\ \mathscr{I}(\mathscr{F}))\ .$$

Hence, if we consider analogously Lie algebras $\mathscr{T}_{\sigma_a}$ for σ_a, these are exactly the same as those for σ. However, if the function $a(w)$ is not constant, $L_Y\sigma_a \not\equiv 0$ $(\mathrm{mod}\ \mathscr{I}(\mathscr{F}))$ for $Y \in \mathscr{L}'$, and so $\mathscr{L}'$ is not a subalgebra of $\mathscr{L}_{\tau_a}$ or $\mathscr{L}_{\omega_a}$. To avoid this difficulty, we have two ways. One way is to replace $\mathscr{L}'$ by the Lie subalgebra $\mathscr{L}'_a$ of $\mathscr{L}_{\sigma_a}$ $(\sigma = \tau, \omega, \theta)$ which is isomorphic to $\mathscr{L}'$ under the Lie algebra homomorphism Ψ. Here Ψ is defined for $Y = \sum_{\alpha=1}^{q} g_\alpha(w)\partial_\alpha \in \mathscr{L}'$ as $\Psi(Y) = Y - (1/n_\sigma a(w))\,(\sum_\alpha (\partial_\alpha a)g_\alpha)I_\sigma \in \mathscr{L}_{\sigma_a}$, where $n_\tau = n$ and $n_\omega = n_\theta = 2$; I_σ is defined in Lemmata 1.2 and 1.3 $(\sigma = \tau, \omega)$; $I_\theta = 2z\partial_{2n+1} + \sum_{i=1}^{2n} v_i\partial_i$. Then we get that $L_{\Psi(Y)}\sigma_a \in \mathscr{I}(\mathscr{F})$, and that $\mathscr{L}_{\sigma_a}$ is a direct sum of $\mathscr{T}_\sigma = \mathscr{T}_{\sigma_a}$ and $\mathscr{L}'_a$.

The second way which we take in the following is to use coordinate transformations ψ_σ:

$$\psi_\tau\colon \begin{cases} \bar{x}_1 = a(w)x_1\ , & \\ \bar{x}_i = x_i & (2 \leqq i \leqq n)\ , \\ \bar{w}_\alpha = w_\alpha\ , & \end{cases}$$

$$\psi_\omega\colon \begin{cases} \bar{x}_i = a(w)x_i\ , \\ \bar{y}_i = y_i\ , \\ \bar{w}_\alpha = w_\alpha\ , \end{cases} \qquad \psi_\theta\colon \begin{cases} \bar{x}_i = x_i\ , & \\ \bar{y}_i = a(w)y_i & (1 \leqq i \leqq n)\ , \\ \bar{z} = a(w)z\ , & \\ \bar{w}_\alpha = w_\alpha & (1 \leqq \alpha \leqq q)\ . \end{cases}$$

Then we get

$$\begin{cases} \psi_\tau^* \tau_a \equiv d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_n\,, & \psi_\omega^* \omega_a \equiv \sum_{i=1}^n d\bar{x}_i \wedge d\bar{y}_i\,, \\ \psi_\theta^* \theta_a \equiv d\bar{z} - \sum_{i=1}^n \bar{y}_i d\bar{x}_i & (\operatorname{mod} \mathscr{I}(\mathscr{F}))\,. \end{cases}$$

Moreover we get that $L_Y(\psi_\sigma^* \sigma_a) \in \mathscr{I}(\mathscr{F})$ for

$$Y = \sum_\alpha g_\alpha(w)\partial_\alpha = \sum_\alpha g_\alpha(\bar{w})\bar{\partial}_\alpha \in \mathscr{L}' \qquad (\bar{\partial}_\alpha = \partial/\partial\bar{w}_\alpha)\,.$$

In fact, for $\sigma = \tau$

$$dx_1 = d\left(\frac{\bar{x}_1}{a}\right) = \frac{1}{a(\bar{w})} d\bar{x}_1 + \bar{x}_1 \sum_{\alpha=1}^q \frac{(\bar{\partial}_\alpha a)(\bar{w})}{a^2(\bar{w})} d\bar{w}_\alpha\,,$$

then

$$\psi_\tau^* \tau_a = d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_n + \bar{x}_1 \sum_\alpha \frac{(\bar{\partial}_\alpha a)(\bar{w})}{a(\bar{w})} d\bar{w}_\alpha \wedge d\bar{x}_2 \wedge \cdots \wedge d\bar{x}_n\,,$$

$$\begin{aligned} d(\psi_\tau^* \tau_a) = {} & d\bar{x}_1 \wedge \sum_\alpha \frac{(\bar{\partial}_\alpha a)}{a} d\bar{w}_\alpha \wedge d\bar{x}_2 \wedge \cdots \wedge d\bar{x}_n \\ & + \bar{x}_1 \sum_{\alpha,\beta} \frac{(\bar{\partial}_\beta \bar{\partial}_\alpha a)a - (\bar{\partial}_\alpha a)(\bar{\partial}_\beta a)}{a^2} d\bar{w}_\beta \wedge d\bar{w}_\alpha \wedge d\bar{x}_2 \wedge \cdots \wedge d\bar{x}_n\,. \end{aligned}$$

Hence

$$\begin{aligned} L_Y(\psi_\tau^* \tau_a) &= i_Y d(\psi_\tau^* \tau_a) + d i_Y(\psi_\tau^* \tau_a) \\ &\equiv -\sum_\alpha \frac{(\bar{\partial}_\alpha a)}{a} g_\alpha d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_n + d\left(\bar{x}_1 \sum_\alpha \frac{(\bar{\partial}_\alpha a)}{a} g_\alpha d\bar{x}_2 \wedge \cdots \wedge d\bar{x}_n\right) \\ &\equiv 0 \qquad (\operatorname{mod} \mathscr{I}(\mathscr{F}))\,. \end{aligned}$$

We get similarly the assertion for $\sigma = \omega$ and θ.

1.8. *Property* (A). Denote by $\mathfrak{A}(M)$ the Lie algebra of all vector fields on a smooth manifold M. Let $\mathfrak{B}(M)$ be a Lie subalgebra of $\mathfrak{A}(M)$ defined by local conditions such that we can define a Lie subalgebra $\mathfrak{B}(U)$ of $\mathfrak{A}(U)$ for any open subset U of M. We say that $\mathfrak{B}(M)$ has *the property* (A) for an open set U, if $r_U(\mathfrak{B}(M)) = r_U(\mathfrak{B}(U'))$ for any open set U' such that $\bar{U} \subset U'$. Here r_U is the restriction mapping onto U.

Then we get the following similarly as for the transitive case.

PROPOSITION 1.7. *The Lie algebras $\mathscr{T}_\sigma$ for $\sigma = 0, \tau, \omega$ and θ have the property* (A) *as subalgebras of $\mathfrak{A}(V)$ for sufficiently many open sets (such as in Proof).*

Proof. For any open set U, the Lie algebra $\mathscr{T}(U)$ is a module over $C^\infty(U)$. And the Lie algebra $\mathscr{T}_\theta(U)$ is isomorphic to $C^\infty(U)$ because of Lemma 1.4 (i) and (ii).

For any simply connected open set U, we get, by Lemma 1.3 (i), a function H on U for any $X \in \mathscr{T}_\omega(U)$ such that $X = X_H$ on U.

Let U be a simply connected open set whose closure is a compact manifold. Let X be partially unimodular, then we get, by Lemma 1.2 (vi), an $(n-2)$-form $\alpha \in \Omega(F)$ on U such that $X = X[\alpha]$ on U. Q.E.D.

1.9. *Commutators in $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$.*

PROPOSITION 1.8. (i) *Let $p \geqq 2$. The Lie algebras $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$ are perfect for $\sigma = 0, \tau, \omega$ and θ. Moreover there hold equalities:*

$$\mathscr{T}_\sigma = [\mathscr{T}_\sigma, \mathscr{T}_\sigma] = [\mathscr{L}_\sigma, \mathscr{T}_\sigma] \qquad (\sigma = 0, \theta)\,,$$
$$\mathscr{T}_\sigma = [\mathscr{T}_\sigma, \mathscr{T}_\sigma] = [\mathscr{T}_{c\sigma}, \mathscr{T}_{c\sigma}] = [\mathscr{L}_{c\sigma}, \mathscr{T}_{c\sigma}] \qquad (\sigma = \tau, \omega)\,.$$

(ii) *Let $p = 1$. The Lie algebra $\mathscr{T}_\tau$ is abelian, and $\mathscr{L}_\tau$, $\mathscr{T}$ and $\mathscr{L}$ are perfect. Moreover*

$$\mathscr{T}_\tau = [\mathscr{T}_{c\tau}, \mathscr{T}_{c\tau}] = [\mathscr{L}_{c\tau}, \mathscr{T}_{c\tau}]\,.$$

Proof. (i) Using the fact that $[\mathscr{L}', \mathscr{T}_\sigma] \subset \mathscr{T}_\sigma$ and $\mathscr{L}' = [\mathscr{L}', \mathscr{L}']$, it is sufficient to show that $\mathscr{T}_\sigma = [\mathscr{T}_\sigma, \mathscr{T}_\sigma]$ for $\sigma = 0, \tau, \omega$ and θ.

Case of $\mathscr{T}$. Let $X = f(v, w)\partial_i$, then $X = [\partial_i, h(v, w)\partial_i]$, where

$$h(v, w) = \int_0^{v_i} f(v, w)dv_i\,.$$

Case of $\mathscr{T}_\tau$. Any function $f(x, w)$ can be written as $f = \{x_i, g\}_{ij}$, where

$$g(x, w) = \int_0^{x_j} f(x, w)dx_j\,.$$

Hence we get the assertion of the proposition by Lemma 1.2 (vi).

Case of $\mathscr{T}_\omega$. Similarly any function $H(x, y, w)$ can be written as $H = \{x_1, G\}$, where

$$G(x, y, w) = \int_0^{y_1} H(x, y, w)dy_1\,.$$

Hence the assertion follows from Lemma 1.3. (i) and (v).

Case of $\mathscr{T}_\theta$. For a function $k(x, y, w)$ independent of z, the assertion can be reduced to Case of $\mathscr{T}_\omega$ by Lemma 1.4 (ii) and (iv). Hence we may assume that k is written as $k = zh$ for some function $h \in C^\infty(V)$. Put

$$g = \int_0^{y_1} h dy_1 ,$$

then we get

$$k = ((x_1 z, g)) + ((x_1 g, z)) .$$

(ii) By Lemma 1.2 (i) and (ii), we get that $\mathscr{T}_\tau = C^\infty(W)\partial_1$, and $X \in \mathscr{T}_{c\tau}$ can be written as $X = f(w)\partial_1 + x_1 g(w)\partial_1$ for some $f, g \in C^\infty(W)$. Q.E.D.

PROPOSITION 1.9. *Put $m_\sigma = 3, 2, 2$ and 4 for $\sigma = 0, \tau, \omega$ and θ respectively. If a vector field $X \in \mathscr{T}_\sigma$ satisfies $j^{m_\sigma}(X)(0) = 0$, then there exists a finite number of vector fields $X_1, \cdots, X_{2r} \in \mathscr{T}_\sigma$ such that*

$$X = \sum_{i=1}^{r} [X_i, X_{i+r}] \quad \textit{and} \quad j^1(X_i)(0) = 0 \quad (i = 1, \cdots, 2r) .$$

Here, the case where $\sigma = \tau$ and $p = 1$ is excluded.

Proof. For $\sigma = 0$, this is Proposition 1.4 in [4].

For $\sigma = \tau$, by Lemma 1.2 (vi), we get this proposition similarly as Proposition 4.7 in [3].

Let $\sigma = \omega$. By Lemma 1.3 (i) and (v), it is sufficient to show that any function H written as

$$H = \prod_{i=1}^{n} x_i^{\ell_i} y_i^{m_i} \prod_{\alpha=1}^{q} w_\alpha^{\lambda_\alpha} G(x, y, w)$$

with $\sum_i (\ell_i + m_i) + \sum_\alpha \lambda_\alpha \geqq 4$ can be decomposed as

$$H = \sum_{i=1}^{r} \{F_i, F_{i+r}\}$$

with $j^2 F_i(0) = 0$ for $i = 1, \cdots, 2r$.

If $\sum_i \ell_i \geqq 2$ or $\sum_i m_i \geqq 2$, the assertion follows from similarly as Proposition 2 in [2]. So we may assume that $\sum_i \ell_i \leqq 1$ and $\sum_i m_i \leqq 1$. Hence $\sum_\alpha \lambda_\alpha \geqq 2$, and we may write H as $H = w_\alpha w_\beta G$. Put

$$K = \int_0^{y_1} G(x, y, w) dy_1 ,$$

then we get that $j^2 K(0) = 0$ and $H = \{x_1 w_\alpha w_\beta, K\}$.

Let $\sigma = \theta$. By Lemma 1.4 (i), (ii) and (iv), it is sufficient to show that any function $f \in C^\infty(V)$ with $j^4 f(0) = 0$ can be written as

$$f=\sum_{i=1}^{r}((g_i, g_{i+r}))\ ,$$

and $j^1(g_i^\flat)(0)=0$ for $i=1,\cdots,2r$.

If $\partial_\alpha f=0$ for any α, this is reduced to Proposition 2.7 in [3]. If $\partial_{2n+1}f=0$, this is reduced to the case where $\sigma=\omega$. So we may assume that f is written as $f=zw_\alpha g$ for some α. Put

$$h=\int_0^{y_1} g dy_1\ ,$$

then we get

$$f=((x_1zw_\alpha, h))+((x_1h, zw_\alpha))\ . \qquad \text{Q.E.D.}$$

Note. The case where $p=1$ and $\sigma=\tau, c\tau$ is pathological. So we exclude this case in § 2 ~ § 5, and treat the case in § 6.

§ 2. Vector fields with polynomial coefficients

2.1. *Grading of Lie subalgebra* $\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}}$. Denote by $\boldsymbol{R}[V]$ the algebra of all polynomials in the variables $v_1,\cdots,v_p$, $w_1,\cdots,w_q$. The vector field $X=\sum_{i=1}^p f_i(v,w)\partial_i+\sum_{\alpha=1}^q g_\alpha(v,w)\partial_\alpha$ on V is said to be with polynomial coefficients, if $f_i(v,w)$ and $g_\alpha(v,w)$ are in $\boldsymbol{R}[V]$ $(i=1,\cdots,p, \alpha=1,\cdots,q)$. Such vector fields form a Lie subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}(V)$, and we get Lie subalgebras $\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}}$ of $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma\cap\tilde{\mathfrak{A}}$ of $\mathscr{L}_\sigma$ for $\sigma=0,\tau,c\tau,\omega,c\omega$ and θ. Put $I_\sigma=\sum_{i=1}^p v_i\partial_i$ for $\sigma=0,\tau,c\tau,\omega$ and $c\omega$, and $I_\theta=2v_{2n+1}\partial_{2n+1}+\sum_{i=1}^{2n}v_i\partial_i$. Then, we get

$$\begin{aligned} &I_\sigma\in\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}} && (\sigma=0, c\tau, c\omega, \theta)\ ,\\ &I_\sigma\in(\mathscr{T}_{c\sigma}\backslash\mathscr{T}_\sigma)\cap\tilde{\mathfrak{A}} && (\sigma=\tau,\omega)\ . \end{aligned}$$

Put $J=\sum_{\alpha=1}^q w_\alpha\partial_\alpha\in\mathscr{L}'$, then we have a natural grading of $\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}}$ as follows.

LEMMA 2.1. (i) *Let* $\sigma=0,\tau,c\tau,\omega$ *or* $c\omega$. *For any* $n\geqq -1$ *and* $m\geqq -1$, *put*

$$\begin{aligned}\mathscr{T}_\sigma(n,m)&=\{X\in\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}}; [I_\sigma, X]=nX,\ [J,X]=(m+1)X\}\\ &=\{X=\textstyle\sum_{i=1}^p f_i(v,w)\partial_i\in\mathscr{T}_\sigma; f_i \text{ are homogeneous polynomials}\\ &\qquad \text{of degree } n+1 \text{ in } v_1,\cdots,v_p, \text{ and}\\ &\qquad \text{of degree } m+1 \text{ in } w_1,\cdots,w_q\}\ ,\end{aligned}$$

then the Lie algebra $\mathscr{T}_\sigma\cap\tilde{\mathfrak{A}}$ *is decomposed as*

$$\mathscr{T}_\sigma \cap \tilde{\mathfrak{A}} = \sum_{n,m\geqq -1} \mathscr{T}_\sigma(n, m) .$$

(ii) *For any $n \geqq -2$ and $m \geqq -1$, put*

$$\mathscr{T}_\theta(n, m) = \{X \in \mathscr{T}_\sigma \cap \tilde{\mathfrak{A}};\ [I_\theta, X] = nX,\ [J, X] = (m+1)X\} ,$$

then the Lie algebra $\mathscr{T}_\theta \cap \tilde{\mathfrak{A}}$ is decomposed as

$$\mathscr{T}_\theta \cap \tilde{\mathfrak{A}} = \sum_{n\geqq -2, m\geqq -1} \mathscr{T}_\theta(n, m) .$$

Moreover, under the isomorphism $\sharp$, $(\mathscr{T}_\theta \cap \tilde{\mathfrak{A}})^\sharp = \boldsymbol{R}[V]$, and $\mathscr{T}_\theta(n, m)$ is isomorphic to

$$\mathscr{T}_\theta(n, m)^\sharp = \{f \in \boldsymbol{R}[V];\ f \text{ is weighted homogeneous of degree } n+2 \text{ in } x_1, \cdots, x_n, y_1, \cdots, y_n, z, \text{ and of degree } m+1 \text{ in } w_1, \cdots, w_q\} ,$$

where $\deg x_i = \deg y_i = \deg w_\alpha = 1$ and $\deg z = 2$.

(iii) *Let $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ, then*

$$[\mathscr{T}_\sigma(n, m),\ \mathscr{T}_\sigma(n', m')] \subset \mathscr{T}_\sigma(n+n', m+m'+1) ,$$

where $\mathscr{T}_\sigma(n, m) = 0$ for n or $m \leqq -2$ $(\sigma \neq \theta)$, and $\mathscr{T}_\theta(n, m) = 0$ for $n \leqq -3$ or $m \leqq -2$.

2.2. *Subalgebras $\mathfrak{B}_\sigma$.* For $\sigma = 0, \tau, c\tau, \omega$ and $c\omega$, put

$$\mathfrak{B}_\sigma = \sum_{n+m\leqq -1} \mathscr{T}_\sigma(n, m) ,$$

and put

$$\mathfrak{B}_\theta = \sum_{n\leqq 0} \mathscr{T}_\theta(n, -1) + \sum_{n\leqq -1} \mathscr{T}_\theta(n, 0) .$$

Then these $\mathfrak{B}_\sigma$ are finite-dimensional subalgebras of $\mathscr{T}_\sigma \cap \tilde{\mathfrak{A}}$ for all σ. Our aim of this section is to show that any derivation of $\mathscr{T}_\sigma$ or $\mathscr{L}_\sigma$ is determined on the subalgebra $\mathfrak{B}_\sigma$.

LEMMA 2.2. *Let $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. If a derivation D of $\mathscr{T}_\sigma$ is zero on $\mathfrak{B}_\sigma$, then D is zero on $\mathscr{T}_\sigma \cap \tilde{\mathfrak{A}}$.*

LEMMA 2.3. *Let $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. If a derivation D of $\mathscr{L}_\sigma$ is zero on $\mathfrak{B}_\sigma$, then D is zero on $\mathscr{L}_\sigma \cap \tilde{\mathfrak{A}}$.*

The proof of these two lemmata will be given in § 2.4 ~ 2.6. Here we remark the following proposition which easily follows from Propositions 1.3 and 1.4 in [3], Proposition 1.5 in [4], and Proposition 1.9.

PROPOSITION 2.4. *Let $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. If a derivation D of $\mathscr{T}_\sigma$ or $\mathscr{L}_\sigma$ is zero on $\mathfrak{B}_\sigma$, then D is zero on $\mathscr{T}_\sigma$ or $\mathscr{L}_\sigma$ respectively.*

2.3. Here we summarize the facts which will be applied later. Put

$$\mathscr{T}_\sigma(v) = \{X \in \mathscr{T}_\sigma;\ [\partial_\alpha, X] = 0\ (1 \leqq \alpha \leqq q)\}\ ,$$
$$\mathscr{T}_\sigma(w) = \{X \in \mathscr{T}_\sigma;\ [\partial_i, X] = 0\ (1 \leqq i \leqq p)\}\ .$$

LEMMA 2.5. *Let $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ.*

(i) *Let $X \in \mathfrak{A}(V)$. If $[\partial_i, X] = 0$ for $i = 1, \cdots, p$, then X is independent of the variables $v_1, \cdots, v_p$, that is, $X \in \mathscr{T}(w)$.*

(ii) *Let $X \in \mathscr{T}_\sigma$. If $[X, Y] = 0$ for any $Y \in \mathscr{T}_\sigma(n, -1)$ with $n \leqq -1$, then X is in $\mathscr{T}_\sigma(w)$, and moreover X is in $C^\infty(W)^p$ for $\sigma = \theta$.*

(iii) *Let $X \in \mathscr{L}_\sigma$. If $[X, Y] \in \mathscr{L}'$ for any $Y \in \mathscr{T}_\sigma(n, -1)$ with $n \leqq -1$, then X is in $\mathscr{T}_\sigma(w) + \mathscr{L}'$.*

(iv) *Let $X \in \mathscr{T}_\sigma(w)$. If $[X, I_\sigma] = 0$, then $X = 0$.*

(v) *$[\mathscr{T}_\sigma(v), \mathscr{L}'] = 0$.*

(vi) *Let $X \in \mathscr{L}'$. If $[X, Y] = 0$ for any $Y \in \mathscr{T}_\sigma(n, 0)$ with $n \leqq -1$, then $X = 0$.*

(vii) *Let $\sigma = \tau$ or ω. Let $X \in \mathscr{T}_\sigma(w)$ or $\mathscr{T}(w)$. If $[X, \mathscr{T}_\sigma(0, -1)] = 0$, then $X = 0$.*

Proof. For $\sigma \neq \theta$, this lemma follows similarly as Lemma 1.1 in [4]. For $\sigma = \theta$, it is enough to note Lemma 1.3 (iii) or (iv), and Lemma 2.1 (ii).

We can prove (vii) similarly as (iv), by using vector fields $x_i\partial_j$ $(i \neq j)$ or $x_i\partial_{x_i} - y_i\partial_{y_i}$ instead of I_τ or I_ω respectively. Q.E.D.

2.4.

Proof of Lemma 2.2. Our proof is similar to the proof of Proposition 2.1 in [4] (the case of $\sigma = 0$), and consists of following four steps.

Step 1. *D is zero* on $\mathscr{T}_\sigma(v) \cap \tilde{\mathfrak{A}}$.

Step 2. *D is zero on $\mathscr{T}_\sigma(0, 0)$.*

Step 3. *D is zero* on $\sum_{n \leqq -1} \sum_{m \geqq -1} \mathscr{T}_\sigma(n, m)$.

Step 4. *D is zero* on $\mathscr{T}_\sigma \cap \tilde{\mathfrak{A}}$.

We can prove *Step* 1 similarly as the corresponding transitive cases (see [2] and [3]).

Let $\sigma = c\tau, c\omega$ and θ. Then I_σ is in $\mathscr{T}_\sigma$. So, by using the grading $\sum_n (\sum_{m \geqq -1} \mathscr{T}_\sigma(n, m))$ of $\mathscr{T}_\sigma \cap \tilde{\mathfrak{A}}$, we can prove *Step* 4 by the induction on n, similarly as Lemmata 3.4 and 5.9 in [3].

Step 2. Let $X \in \mathcal{T}_\sigma(0, 0)$. By Lemma 2.1, we get

$$[\mathcal{T}_\sigma(n, -1), X] \subset \mathcal{T}_\sigma(n, 0) \quad \text{and} \quad [I_\sigma, X] = 0\,.$$

Applying D to these formulae, we get by the assumption

$$[\mathcal{T}_\sigma(n, -1), D(X)] = 0 \ (n \leqq -1) \quad \text{and} \quad [I_\sigma, D(X)] = 0\,.$$

Hence Lemma 2.5 (ii) and (iv) imply that $D(X) = 0$.

Step 3. The proof is carried out by the induction on m. When m is nonpositive, the assertion holds by the assumption. Assume that D is zero on $\sum_{n \leqq -1} \sum_{m \leqq k-1} \mathcal{T}_\sigma(n, m)$ for some $k > 0$. Clearly it is enough to show that $D(X) = 0$ for the case where X is written as

$$X = w_1^{\lambda_1} \cdots w_q^{\lambda_q} Y\,,$$

where $\sum_\alpha \lambda_\alpha = k + 1$ and $Y \in \mathcal{T}_\sigma(n, -1)$ with $n \leqq -1$. There is an index β with $\lambda_\beta > 0$. Apply D to

$$X = n^{-1}[w_\beta I_\sigma, w_\beta^{-1} X]\,,$$

then $D(X) = 0$ because $w_\beta^{-1} X \in \mathcal{T}_\sigma(n, k-1)$ and $w_\beta I_\sigma \in \mathcal{T}_\sigma(0, 0)$.

2.5. *Case of $\sigma = \tau$ or ω.* Let $\sigma = \tau$ or ω. For *Step* 3 we can prove it similarly as above by noting the following equalities: (for $\lambda_\beta > 0$),

$$f\partial_i = w_1^{\lambda_1} \cdots w_q^{\lambda_q} \partial_i = [w_\beta^{-1} f \partial_j, x_j w_\beta \partial_i] \quad (i \neq j) \qquad (\sigma = \tau)\,,$$

$$\left.\begin{aligned} X &= w_1^{\lambda_1} \cdots w_q^{\lambda_q} \partial_{x_i} = [w_\beta^{-1} X, w_\beta(x_i \partial_i - y_i \partial_i)]\,, \\ Y &= w_1^{\lambda_1} \cdots w_q^{\lambda_q} \partial_{y_i} = [w_\beta^{-1} Y, w_\beta(y_i \partial_i - x_i \partial_i)] \end{aligned}\right\} \qquad (\sigma = \omega)\,.$$

We can prove *Step* 2 and *Step* 4 similarly as above by elementary calculations. However, here we give a simple proof by using some facts about graded Lie algebras (see Singer-Sternberg [8], for definitions).

FACTS. (a) *There are natural isomorphisms*

$$\mathcal{T}_\sigma(-1, -1) \cong U\,, \quad \mathcal{T}_\sigma(0, -1) \cong \mathfrak{g}_\sigma \quad \text{and} \quad \mathcal{T}_\sigma(k, -1) \cong \mathfrak{g}_\sigma^{(k)}\,,$$

where $\mathfrak{g}_\tau = \mathfrak{sl}(U)$, $\mathfrak{g}_\omega = \mathfrak{sp}(U)$ *and* $\mathfrak{g}_\sigma^{(k)}$ *is the k-th prolongation of* $\mathfrak{g}_\sigma$.

(b) $\mathfrak{g}_\sigma = [U, \mathfrak{g}_\sigma^{(1)}]$.

(c) *The adjoint action of* $\mathcal{T}_\sigma(0, -1)$ *on* $\mathcal{T}_\sigma(k, -1)$ *is the natural action of* $\mathfrak{g}_\sigma$ *on* $\mathfrak{g}_\sigma^{(k)}$, *and is irreducible* (*the irreducibility is due to H. Weyl* [10]).

Step 2. It is enough to show that $D(w_\alpha X) = 0$ for any $X \in \mathcal{T}_\sigma(0, -1)$. By *Fact* (b), we get vector fields $Y_i \in \mathcal{T}_\sigma(-1, -1)$ and $Z_i \in \mathcal{T}_\sigma(1, -1)$ $(1 \leqq i \leqq r)$ such that $X = \sum_i [Y_i, Z_i]$. So $w_\alpha X$ is written as

$$w_\alpha X = \sum_{i=1}^{r} [w_\alpha Y_i, Z_i] .$$

Hence $D(w_\alpha X) = 0$, because $w_\alpha Y_i \in \mathscr{T}_\sigma(-1, 0)$ and $Z_i \in \mathscr{T}_\sigma(1, -1)$.

Step 4. Decompose $\mathscr{T}_\sigma \cap \mathfrak{A}$ as $\mathscr{T}_\sigma \cap \mathfrak{A} = \sum_{k \geqq -1} (\sum_{m \geqq -1} \mathscr{T}_\sigma(k, m))$. The proof is carried out by the induction on k. The assertion for $k = -1$ holds by *Step* 3. Assume that D is zero on $\sum_{n \leqq k-1} (\sum_{m \geqq -1} \mathscr{T}_\sigma(n, m))$ for some $k \geqq 0$. Let $X \in \mathscr{T}_\sigma(k, m)$. Applying D to $[\partial_i, X] \in \mathscr{T}_\sigma(k-1, m)$, we get that $D(X) \in \mathscr{T}_\sigma(w)$. Thus we get the mapping $D_k = D|_{\mathscr{T}_\sigma(k,m)}$ of $\mathscr{T}_\sigma(k, m)$ to $\mathscr{T}_\sigma(w)$, and we want to show $D_k = 0$.

There are the natural isomorphisms $\mathscr{T}_\sigma(n, m) \cong \boldsymbol{P}^{m+1}(W) \otimes \mathfrak{g}_\sigma^{(n)}$, where $\boldsymbol{P}^{m+1}(W)$ is the space of homogeneous polynomials of degree $m+1$ in the variables $w_1, \cdots, w_q$. Let $Y \in \mathscr{T}_\sigma(0, -1) \cong \mathfrak{g}_\sigma$, then we get that $D[Y, X] = [Y, D(X)]$, so $D_k \circ \operatorname{ad} Y = \operatorname{ad} Y \circ D_k$, that is, the following diagram commutes:

$$\begin{tikzcd}
\boldsymbol{P}^{m+1}(W) \otimes \mathfrak{g}_\sigma^{(k)} \arrow[r, "1 \otimes \operatorname{ad} Y"'] & \boldsymbol{P}^{m+1}(W) \otimes \mathfrak{g}_\sigma^{(k)} \\
\mathscr{T}_\sigma(k, m) \arrow[u, phantom, "\wr\|"] \arrow[r, "\operatorname{ad} Y"'] \arrow[d, "D_k"] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{T}_\sigma(k, m) \arrow[u, phantom, "\wr\|"] \arrow[d, "D_k"] \\
\mathscr{T}_\sigma(w) \arrow[r, "\operatorname{ad} Y"'] & \mathscr{T}_\sigma(w) .
\end{tikzcd}$$

Since the adjoint action of $\mathfrak{g}_\sigma$ on $\mathfrak{g}_\sigma^{(k)}$ is irreducible by *Fact* (c), it is enough to show that $D_k^{-1}(0) \neq 0$.

Let $\sigma = \tau$, $f(\neq 0) \in \boldsymbol{P}^{m+1}(W)$ and $X = x_2^{k+1}\partial_1$. Then, $fX \in \mathscr{T}_\sigma(k, m)$ is written as

$$fX = [f(w)\partial_2, (k+2)^{-1}x_2^{k+2}\partial_1] ,$$

so $fX \in D_k^{-1}(0)$ by *Steps* 1 and 2.

Let $\sigma = \omega$, $f(\neq 0) \in \boldsymbol{P}^{m+1}(W)$ and $X = x_1^{k+1}\partial_{y_1}$, then $fX \in \mathscr{T}_\omega(k, m)$. We want to show $D(fX) = 0$. Write $D(fX)$ as $D(fX) = \sum (a_k(w)\partial_{x_k} + b_k(w)\partial_{y_k})$. Applying D to $[x_i\partial_{x_i} - y_i\partial_{y_i}, fX] = \delta_{i1}(k+2)fX$ (where δ_{i1} is the Kronecker's delta), we get

$$\begin{aligned}
[x_i\partial_{x_i} - y_i\partial_{y_i}, D(fX)] &= -a_i(w)\partial_{x_i} + b_i(w)\partial_{y_i} \\
&= \delta_{i1}(k+2)D(fX) = \delta_{i1}(k+2)\sum_j (a_j(w)\partial_{x_j} + b_j(w)\partial_{y_j}) .
\end{aligned}$$

For $i \neq 1$, we get that $a_i(w) = b_i(w) = 0$. For $i = 1$, we get

$$-a_1 = (k+2)a_1 \quad \text{and} \quad b_1 = (k+2)b_1 ,$$

so $a_1 = b_1 = 0$ because $k \geqq 0$. Thus, $D(fX) = 0$. Q.E.D.

2.6.

Proof of Lemma 2.3. We need only to prove that $D = 0$ on $\mathscr{L}' \cap \mathfrak{A}$. For $\sigma = c\tau, c\omega$ and θ, we can prove similarly as in the proof of Proposition 2.3 in [4] ($\sigma = 0$), by noting Lemma 2.5 (ii) and (iv) and the fact that $I_\sigma \in \mathscr{T}_\sigma$.

For $\sigma = \tau$ and ω, we can also prove similarly as above, by using Lemma 2.5 (vii). Q.E.D.

§ 3. Derivations of $\mathscr{T}_\sigma$ and $\mathscr{L}_\sigma$

3.1. *Derivation.* Let Z be a vector field on V. We define $\operatorname{ad} Z$ as $\operatorname{ad} Z(X) = [Z, X]$ for $X \in \mathfrak{A}(V)$. Then we have

LEMMA 3.1. *For $\sigma = 0, \tau, c\tau, \omega, c\omega$ and θ, the mapping $Z \to \operatorname{ad} Z|_{\mathscr{T}_\sigma}$ or $Z \to \operatorname{ad} Z|_{\mathscr{L}_\sigma}$ of $\mathscr{L}_{\sigma'}$ into $\mathscr{D}er(\mathscr{T}_\sigma)$ or $\mathscr{D}er(\mathscr{L}_\sigma)$ respectively is an into isomorphism, where $\sigma' = \sigma$ for $\sigma = 0, c\tau, c\omega$ and θ, and $\sigma' = c\sigma$ for $\sigma = \tau, \omega$.*

Proof. It is sufficient to show the injectivity. Let $Z \in \mathscr{L}_{\sigma'}$. Assume that $\operatorname{ad} Z(\mathscr{T}_\sigma) = 0$. By Lemma 2.5 (iii), we can write Z as $Z = X + Y$ with $X \in \mathscr{T}_{\sigma'}(w)$ and $Y \in \mathscr{L}'$. Then we get that $X = 0$ by Lemma 2.5 (v) and (iv), and that $Y = 0$ by Lemma 2.5 (vi). Q.E.D.

LEMMA 3.2. (i) *Let D be a derivation of $\mathscr{L}_\sigma$ for $\sigma = 0, \tau, \omega$ or θ. Then $D(\mathscr{T}_\sigma) \subset \mathscr{T}_\sigma$.*

(ii) *Let D be a derivation of $\mathscr{T}_{c\sigma}$ or $\mathscr{L}_{c\sigma}$ for $\sigma = \tau$ or ω. Then $D(\mathscr{T}_\sigma) \subset \mathscr{T}_\sigma$.*

Proof. This follows immediately from Proposition 1.8 and the following general proposition. Q.E.D.

PROPOSITION 3.3. *Let T be an ideal of a Lie algebra L such that $[T, T] = T$. Let D be a derivation of L. Then $D(T) \subset T$, that is, D induces a derivation $D|_T$ of the Lie algebra T.*

Proof. Let $X \in T$. Write X as $X = \sum_{i=1}^{r} [Y_i, Z_i]$ with $Y_i, Z_i \in T$. So, since T is an ideal, we get

$$D(X) = D\Big(\sum_i [Y_i, Z_i]\Big) = \sum_i \{[D(Y_i), Z_i] + [Y_i, D(Z_i)]\} \in T\,.$$

Q.E.D.

3.2. *Determination of $\mathscr{D}er(\mathscr{T}_\sigma)$.*

PROPOSITION 3.4. *Let D be a derivation of $\mathscr{T}_\sigma$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. Then there exists a unique vector field Z on V such that $D = \mathrm{ad}\, Z$ on $\mathfrak{B}_\sigma$. Moreover, Z is in $\mathscr{L}_{\sigma'}$, where $\sigma' = \sigma$ for $\sigma = 0, c\tau, c\omega$ or θ, and $\sigma' = c\sigma$ for $\sigma = \tau$ or ω.*

The proof of this proposition will be given in § 3.3 ~ 3.4.

PROPOSITION 3.5. *Let D be a derivation of $\mathscr{T}_\sigma$ or $\mathscr{L}_\sigma$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ, then there exists a unique vector field $Z \in \mathfrak{X}(V)$ such that $D = \mathrm{ad}\, Z|_{\mathscr{T}_\sigma}$ or $\mathrm{ad}\, Z|_{\mathscr{L}_\sigma}$ respectively. Moreover, Z is in $\mathscr{L}_{\sigma'}$, where σ' is as in Proposition 3.4.*

Proof. Let D be a derivation of $\mathscr{T}_\sigma$. Then by Proposition 3.4, we get a unique vector field $Z \in \mathscr{L}_{\sigma'}$, such that $D = \mathrm{ad}\, Z$ on $\mathfrak{B}_\sigma$. Since $\mathrm{ad}\, Z(\mathscr{T}_\sigma) \subset \mathscr{T}_\sigma$, $D_1 = D - \mathrm{ad}\, Z$ is also a derivation of $\mathscr{T}_\sigma$ and vanishes on $\mathfrak{B}_\sigma$. Hence by Propositon 2.4, we get that $D_1 = 0$, that is, $D = \mathrm{ad}\, Z$ on $\mathscr{T}_\sigma$.

Let D be a derivation of $\mathscr{L}_\sigma$, then we can restrict D to $\mathscr{T}_{\sigma_0}$ by Lemma 3.2, where $\sigma_0 = \sigma$ for $\sigma \neq c\tau$ or $c\omega$; and $\sigma_0 = \tau$ or ω for $\sigma = c\tau$ or $c\omega$ respectively. Q.E.D.

THEOREM 3.6. (i) *Let $\sigma = 0, c\tau, c\omega$ or θ. All derivations of $\mathscr{L}_\sigma$ are inner, that is, $\mathscr{D}er\,(\mathscr{L}_\sigma) = \mathrm{ad}\, \mathscr{L}_\sigma \cong \mathscr{L}_\sigma$. Hence,*

$$H^1(\mathscr{L}_\sigma; \mathscr{L}_\sigma) = 0 .$$

(ii) *Let $\sigma = \tau$ $(p \neq 1)$ or ω. The derivation algebra of $\mathscr{L}_\sigma$ is naturally isomorphic to $\mathscr{L}_{c\sigma}$, that is, $\mathscr{D}er\,(\mathscr{L}_\sigma) = \{\mathrm{ad}\, Z|_{\mathscr{L}_\sigma}; Z \in \mathscr{L}_{c\sigma}\} \cong \mathscr{L}_{c\sigma}$. Hence,*

$$H^1(\mathscr{L}_\sigma; \mathscr{L}_\sigma) \cong \mathscr{L}_{c\sigma}/\mathscr{L}_\sigma \cong C^\infty(W) .$$

(iii) *Let $\sigma = 0, c\tau, c\omega$ or θ. The derivation algebra of $\mathscr{T}_\sigma$ is naturally isomorphic to $\mathscr{L}_\sigma$, that is, $\mathscr{D}er\,(\mathscr{T}_\sigma) = \{\mathrm{ad}\, Z|_{\mathscr{T}_\sigma}; Z \in \mathscr{L}_\sigma\} \cong \mathscr{L}_\sigma$. Hence,*

$$H^1(\mathscr{T}_\sigma; \mathscr{T}_\sigma) \cong \mathscr{L}_\sigma/\mathscr{T}_\sigma \cong \mathscr{L}' .$$

(iv) *Let $\sigma = \tau$ $(p \neq 1)$ or ω. The derivation algebra of $\mathscr{T}_\sigma$ is naturally isomorphic to $\mathscr{L}_{c\sigma}$, that is, $\mathscr{D}er\,(\mathscr{T}_\sigma) = \{\mathrm{ad}\, Z|_{\mathscr{T}_\sigma}; Z \in \mathscr{L}_{c\sigma}\} \cong \mathscr{L}_{c\sigma}$. Hence,*

$$H^1(\mathscr{T}_\sigma; \mathscr{T}_\sigma) \cong \mathscr{L}_{c\sigma}/\mathscr{T}_\sigma \cong C^\infty(W)I_\sigma + \mathscr{L}' .$$

Proof. This follows from Lemma 3.1 and Proposition 3.5. For the latter half, remember that $H^1(L; L) \cong \mathscr{D}er\,(L)/\mathrm{ad}\, L$ (see e.g. § 1 in [2]).

Q.E.D.

3.3.

Proof of Proposition 3.4. It is sufficient to show the following two lemmata. Here we can determine Z as $Z = Z_1 + Z_2$, where Z_1 and Z_2 are given in the lemmata below.

LEMMA 3.7. *Let $D \in \mathscr{D}er(\mathscr{T}_\sigma)$. Then there exists a vector field $Z_1 \in \mathscr{T}_{\sigma'}$ such that $D = \operatorname{ad} Z_1$ on $\mathscr{T}_\sigma(n, -1)$ with $n \leqq 0$.*

Proof. By using Poincaré lemma with parameters, we can get the assertion similarly as in the corresponding transitive cases (see Lemma 5 in [2] and Propositions 3.2 and 5.5 in [3]). Q.E.D.

LEMMA 3.8. *Let $D \in \mathscr{D}er(\mathscr{T}_\sigma)$. Assume that $D = 0$ on $\mathscr{T}_\sigma(n, -1)$ with $n \leqq 0$. Then there exists a unique vector field Z_2 on V such that $D = \operatorname{ad} Z_2$ on $\mathfrak{B}_\sigma$. Moreover, Z_2 is in $\mathscr{L}'$.*

Proof. (i) Let $\sigma \neq \theta$. Let $X \in \mathscr{T}_\sigma(-1, 0)$. Since $[X, \mathscr{T}_\sigma(-1, -1)] = 0$, we get $[D(X), \mathscr{T}_\sigma(-1, -1)] = 0$, so $D(X) \in \mathscr{T}_\sigma(w)$ by Lemma 2.5 (ii). Hence we get the linear map $D' = D|_{\mathscr{T}_\sigma(-1,0)}$ of $\mathscr{T}_\sigma(-1, 0) = W^* \otimes U$ to $\mathscr{T}_\sigma(w) = C^\infty(W) \otimes U$, which commutes with $\operatorname{ad} X$ for $X \in \mathscr{T}_\sigma(0, -1)$. Thus we get the following commutative diagram: for $X \in \mathfrak{sl}(U)$ or $\mathfrak{sp}(U)$ ($\sigma = \tau$, $c\tau$ or $\omega, c\omega$ respectively)

$$\begin{tikzcd}
\mathscr{T}_\sigma(-1,0) \arrow[r, "\operatorname{ad} X"'] \arrow[d, equal, "\wr"] & \mathscr{T}_\sigma(-1,0) \arrow[d, equal, "\wr"] \\
W^* \otimes U \arrow[r, "1 \otimes \operatorname{ad} X"'] \arrow[d, "D'"] \arrow[dr, phantom, "\circlearrowleft"] & W^* \otimes U \arrow[d, "D'"] \\
C^\infty(W) \otimes U \arrow[r, "1 \otimes \operatorname{ad} X"'] & C^\infty(W) \otimes U\,.
\end{tikzcd}$$

Since $w_\alpha \partial_i$ are basis of $\mathscr{T}_\sigma(-1, 0)$, we get by Schur's lemma

$$D(w_\alpha \partial_i) = h_\alpha(w)\partial_i \qquad (1 \leqq i \leqq p,\ 1 \leqq \alpha \leqq q)$$

for some functions $h_\alpha(w) \in C^\infty(W)$. Here we remark that we can show this fact also by simple calculations. In fact, apply D to

$$w_\alpha \partial_i = [w_\alpha \partial_j, x_j \partial_i] \qquad (\sigma = \tau (i \neq j), \sigma = c\tau)\,,$$

and

$$w_\alpha \partial_{x_i} = [w_\alpha \partial_{x_j}, x_j \partial_{x_i} - y_i \partial_{y_j}]\,, \quad w_\alpha \partial_{y_i} = [w_\alpha \partial_{y_j}, y_j \partial_{y_i} - x_i \partial_{x_j}] \quad (\sigma = \omega, c\omega)\,.$$

Let Z_2 be a vector field on V such that $D = \operatorname{ad} Z_2$ on $\mathfrak{B}_\sigma$. Since

$[Z_2, \mathscr{T}_\sigma(n, -1)] = 0$ for $n \leqq -1$, we get that $Z_2 \in \mathscr{L}'$, by Lemma 2.5 (i), (iv) and (vii). Write Z_2 as $Z_2 = \sum_\alpha g_\alpha(w)\partial_\alpha$, then

$$[Z_2, w_\alpha\partial_i] = g_\alpha(w)\partial_i \qquad (1 \leqq i \leqq p,\ 1 \leqq \alpha \leqq q)\ .$$

Hence, $g_\alpha(w)$ must be equal to $h_\alpha(w)$ for all α.

(ii) Let $\sigma = \theta$. By Lemma 1.4 we get the derivation $\tilde{D}$ of $C^\infty(V)$ such that $\tilde{D} \circ \sharp = \sharp \circ D$. Take bases $\{1\}$ of $\mathscr{T}_\theta(-2, -1)^\sharp$, $\{x_i, y_i\}_{(1 \leqq i \leqq n)}$ of $\mathscr{T}_\theta(-1, -1)^\sharp$, $\{w_\alpha\}_{(1 \leqq \alpha \leqq q)}$ of $\mathscr{T}_\theta(-2, 0)^\sharp$ and $\{w_\alpha x_i, w_\alpha y_i\}_{(1 \leqq i \leqq n, 1 \leqq \alpha \leqq q)}$ of $\mathscr{T}_\theta(-1, 0)^\sharp$. We want to show that $\tilde{D}(w_\alpha) = h_\alpha(w)$, $\tilde{D}(w_\alpha x_i) = h_\alpha(w)x_i$ and $\tilde{D}(w_\alpha y_i) = h_\alpha(w)y_i$ for some functions $h_\alpha(w) \in C^\infty(W)$.

Since $((w_\alpha, 1)) = ((w_\alpha, x_i)) = ((w_\alpha, y_i)) = 0$, we get that $((\tilde{D}(w_\alpha), 1)) = ((\tilde{D}(w_\alpha), x_i)) = ((\tilde{D}(w_\alpha), y_i)) = 0$, so by Lemma 2.5 (ii), $\tilde{D}(w_\alpha) \in C^\infty(W)$. Put $h_\alpha(w) = \tilde{D}(w_\alpha)$.

Put $f_{\alpha i} = \tilde{D}(w_\alpha x_i)$. Apply $\tilde{D}$ to $((w_\alpha x_i, 1)) = ((w_\alpha x_i, x_j)) = 0$ and $((w_\alpha x_i, y_j)) = \delta_{ij} w_\alpha$, then we get that $\partial_z f_{\alpha i} = \partial_{y_j} f_{\alpha i} = 0$ and $\partial_{x_j} f_{\alpha i} = \delta_{ij} h_\alpha(w)$, so that $f_{\alpha i} = h_\alpha(w)x_i + h_{\alpha i}(w)$ for some functions $h_{\alpha i} \in C^\infty(W)$. Apply $\tilde{D}$ to $((I_\theta^\sharp, w_\alpha x_i)) = -w_\alpha x_i$, then by Proposition 2.1 (ii)

$$-f_{\alpha i} = ((I_\theta^\sharp, h_\alpha(w)x_i + h_{\alpha i}(w))) = -h_\alpha(w)x_i - 2h_{\alpha i}(w)\ ,$$

hence we get that $h_{\alpha i} = 0$, that is, $\tilde{D}(w_\alpha x_i) = h_\alpha(w)x_i$.

Similarly we get that $\tilde{D}(w_\alpha y_i) = h_\alpha(w)y_i$.

Let Z_2 be a vector field on V such that $D = \operatorname{ad} Z_2$ on $\mathfrak{V}_\theta$. Since $1^\flat = \partial_z$, $x_i^\flat = \partial_{y_i} + x_i\partial_z$, $y_i^\flat = -\partial_{x_i}$, easily we get that $Z_2 \in \mathscr{T}(w) + \mathscr{L}'$. Similarly we get $Z_2 \in \mathscr{L}'$, since $[I_\theta, Z_2] = 0$. Write Z_2 as $Z_2 = \sum_\beta g_\beta(w)\partial_\beta$, then

$$[Z_2, w_\alpha^\flat] = \left[\sum_\beta g_\beta(w)\partial_\beta,\ w_\alpha\partial_z\right] = g_\alpha(w)\partial_z = g_\alpha(w)^\flat\ ,$$

hence, $g_\alpha(w)$ must be equal to $h_\alpha(w)$ for all α. Q.E.D.

§4. Lie algebras $\mathscr{T}_\sigma(M, \mathscr{F})$ and $\mathscr{L}_\sigma(M, \mathscr{F})$, and their derivations

4.1. *Structures on leaves.* Let M be a $(p+q)$-dimensional manifold and $\mathscr{F}$ a codimension q foliation on M. Around any point M, there is a distinguished coordinate neighbourhood $(U; v_1, \cdots, v_p, w_1, \cdots, w_q)$, for which a plate represented as $w_1 = \text{constant}, \cdots, w_q = \text{constant}$ in U is a connected component of $L \cap U$ for some leaf L of $\mathscr{F}$ (see e.g. [7] for definitions).

Let $\mathscr{T}(M, \mathscr{F})$ be the Lie algebra of all leaf-tangent vector fields on M, then by [4], the derivation algebra of $\mathscr{T}(M, \mathscr{F})$ is isomorphic to the

Lie algebra $\mathscr{L}(M, \mathscr{F})$ of all locally foliation-preserving vector fields on M.

Let $\Omega(M)$ be the exterior algebra of all differential forms on M, and $\mathscr{I}(M, \mathscr{F})$ be its ideal defined by

$$\begin{aligned}\mathscr{I}(M, \mathscr{F}) &= \{\alpha \in \Omega(M); \alpha(X_1, X_2, \cdots) = 0 \text{ for } X_i \in \mathscr{T}(M, \mathscr{F})\} \\ &= \{\alpha \in \Omega(M); \iota_L^* \alpha = 0 \text{ for every leaf } L \text{ of } \mathscr{F}\},\end{aligned}$$

where ι_L is the inclusion mapping of L in M. Then, we get the following lemma similarly as Lemma 1.1.

LEMMA 4.1. (i) *The ideal $\mathscr{I}(M, \mathscr{F})$ is L_X-stable for $X \in \mathscr{L}(M, \mathscr{F})$, and i_X-stable for $X \in \mathscr{T}(M, \mathscr{F})$.*

(ii) *The ideal $\mathscr{I}(M, \mathscr{F})$ is a differential ideal.*

DEFINITION 4.2. A p-form τ on M is called a *partially unimodular structure* on $(M, \mathscr{F})$, if $\iota_L^* \tau \neq 0$ for every leaf L of $\mathscr{F}$, that is, $\iota_L^* \tau$ is a volume form on L. Then we get $d\tau \in \mathscr{I}(M, \mathscr{F})$.

DEFINITION 4.3. Let $p = 2n$. A 2-form ω on M is called a *partially symplectic structure* on $(M, \mathscr{F})$, if $d\omega \in \mathscr{I}(M, \mathscr{F})$ and $\iota_L^* \omega$ is of rank $2n$ for every leaf L of $\mathscr{F}$.

DEFINITION 4.4. Let $p = 2n + 1$. A 1-form θ on M is called a *partially contact structure* on $(M, \mathscr{F})$, if $(\iota_L^* \theta) \wedge (d\iota_L^* \theta)^n \neq 0$ for every leaf L of $\mathscr{F}$.

These τ, ω and θ are called *partially classical* structures, and we get their normal forms as follows.

PROPOSITION 4.5. *Let τ, ω or θ be a partially unimodular, symplectic, or contact structure on $(M, \mathscr{F})$ respectively. Then we can take distinguished coordinates $(U; v_1, \cdots, v_p, w_1, \cdots, w_q)$ around any point of M such that on U*

$$\left.\begin{aligned}\tau &\equiv dv_1 \wedge \cdots \wedge dv_p \\ \omega &\equiv \sum_{i=1}^{n} dv_i \wedge dv_{i+n} \\ \theta &\equiv dv_{2n+1} - \sum_{i=1}^{n} v_{i+n} dv_i\end{aligned}\right\} \quad (\operatorname{mod} \mathscr{I}(M, \mathscr{F})).$$

Proof. At first choose distinguished coordinates $(U; v_1, \cdots, v_p, w_1, \cdots, w_q)$, then by similar arguments to the proof of Darboux's theorem for the variables $v_1, \cdots, v_p$, we get the above normal form up to a multiplicative factor depending on the variables $w_1, \cdots, w_q$. But this factor does not vanish anywhere, so we can take the coordinate transformation ψ_σ as in § 1.7. Q.E.D.

4.2. *Lie algebras $\mathscr{T}_\sigma(M, \mathscr{F})$ and $\mathscr{L}_\sigma(M, \mathscr{F})$.* Let τ be a partially unimodular structure on $(M, \mathscr{F})$. A vector field $X \in \mathscr{T}(M, \mathscr{F})$ is called *partially conformally unimodular*, if $L_X\tau$ is congruent to $\phi\tau$ modulo $\mathscr{I}(M, \mathscr{F})$ for some function $\phi \in C^\infty(M)^{\mathscr{F}}$, where $C^\infty(M)^{\mathscr{F}}$ is the space of functions which are constant on each leaves of $\mathscr{F}$. Moreover, if the function ϕ is zero, X is called *partially unimodular*. Then by Lemma 4.1, we get two Lie subalgebras of $\mathscr{T}(M, \mathscr{F})$:

$$\mathscr{T}_\tau(M, \mathscr{F}) = \{X \in \mathscr{T}(M, \mathscr{F});\ L_X\tau \in \mathscr{I}(M, \mathscr{F})\}\ ,$$
$$\mathscr{T}_{c\tau}(M, \mathscr{F}) = \{X \in \mathscr{T}(M, \mathscr{F});\ L_X\tau \equiv \phi\tau \pmod{\mathscr{I}(M, \mathscr{F})}\ \text{for some } \phi \in C^\infty(M)^{\mathscr{F}}\}\ .$$

Let ω be a partially symplectic structure on $(M, \mathscr{F})$. A vector field $X \in \mathscr{T}(M, \mathscr{F})$ is called *partially conformally symplectic*, if $L_X\omega$ is congruent to $\phi\omega$ modulo $\mathscr{I}(M, \mathscr{F})$ for some function $\phi \in C^\infty(M)^{\mathscr{F}}$. Moreover, if the function ϕ is zero, X is called *partially symplectic*. Then by Lemma 4.1, we get two Lie subalgebras of $\mathscr{T}(M, \mathscr{F})$:

$$\mathscr{T}_\omega(M, \mathscr{F}) = \{X \in \mathscr{T}(M, \mathscr{F});\ L_X\omega \in \mathscr{I}(M, \mathscr{F})\}\ ,$$
$$\mathscr{T}_{c\omega}(M, \mathscr{F}) = \{X \in \mathscr{T}(M, \mathscr{F});\ L_X\omega \equiv \phi\omega \pmod{\mathscr{I}(M, \mathscr{F})}\ \text{for some } \phi \in C^\infty(M)^{\mathscr{F}}\}\ .$$

Let θ be a partially contact structure on $(M, \mathscr{F})$. A vector field $X \in \mathscr{T}(M, \mathscr{F})$ is called *partially contact*, if $L_X\theta$ is congruent to $\phi\theta$ modulo $\mathscr{I}(M, \mathscr{F})$ for some function $\phi \in C^\infty(M)$. Such vector fields form the Lie subalgebra $\mathscr{T}_\theta(M, \mathscr{F})$ of $\mathscr{T}(M, \mathscr{F})$ by Lemma 4.1.

These Lie algebras $\mathscr{T}(M, \mathscr{F})$, $\mathscr{T}_\tau(M, \mathscr{F})$, $\mathscr{T}_{c\tau}(M, \mathscr{F})$, $\mathscr{T}_\omega(M, \mathscr{F})$, $\mathscr{T}_{c\omega}(M, \mathscr{F})$ and $\mathscr{T}_\theta(M, \mathscr{F})$ are called of *partially classical type* (see Remark in § 1.5).

Similarly as above, we can define Lie subalgebras of $\mathscr{L}(M, \mathscr{F})$: for $\sigma = \tau$ or ω,

$$\mathscr{L}_\sigma(M, \mathscr{F}) = \{X \in \mathscr{L}(M, \mathscr{F});\ L_X\sigma \in \mathscr{I}(M, \mathscr{F})\}\ ,$$
$$\mathscr{L}_{c\sigma}(M, \mathscr{F}) = \{X \in \mathscr{L}(M, \mathscr{F});\ L_X\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}\ \text{for some } \phi \in C^\infty(M)^{\mathscr{F}}\}\ ,$$

and

$$\mathscr{L}_\theta(M, \mathscr{F}) = \{X \in \mathscr{L}(M, \mathscr{F});\ L_X\theta \equiv \phi\theta \pmod{\mathscr{I}(M, \mathscr{F})}\ \text{for some } \phi \in C^\infty(M)\}\ .$$

Then by Lemma 4.1, we get that $\mathscr{T}_\sigma(M, \mathscr{F})$ is an ideal of $\mathscr{L}_\sigma(M, \mathscr{F})$ for every σ, and these Lie algebras $\mathscr{T}_\sigma(M, \mathscr{F})$ and $\mathscr{L}_\sigma(M, \mathscr{F})$ are defined by classes of partially classical structures modulo $\mathscr{I}(M, \mathscr{F})$.

If we take distinguished coordinates such as in Proposition 4.5, vector fields in $\mathscr{T}_\sigma(M, \mathscr{F})$ or $\mathscr{L}_\sigma(M, \mathscr{F})$ for $\sigma = \tau, c\tau, \omega, c\omega$ and θ can be described, in local, similarly as Lemmata 1.2 $\sim$ 1.4 (see Remarks in § 1.7).

4.3. *Derivations.*

LEMMA 4.6. *Let U be an open subset of M, and $X \in \mathscr{L}_\sigma(M, F)$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. Assume that $[X, Y] = 0$ on U for any $Y \in \mathscr{T}_\sigma(M, F)$ with its support contained in U. Then, $X = 0$ on U.*

Proof. Let P be a point of U, and fix a distinguished coordinate neighbourhood $(U'; v_1, \cdots, v_p, w_1, \cdots, w_q)$ around P such as in Proposition 4.5. Take a neighbourhood U'' of P such that $\bar{U}'' \subset U' \cap U$. Let Y' be in $\mathscr{T}_\sigma(n, -1)$ $(n \leqq 0)$ with respect to the fixed coordinates. For $\sigma = 0, \tau, \omega$ and θ, we get by Proposition 1.7 a vector field $Y \in \mathscr{T}_\sigma(M, \mathscr{F})$ such that $Y = Y'$ on U'' and the support of Y is contained in U. Then we have that $[X, Y] = 0$ on U, by the assumption. By Lemma 2.5, we have that $X = 0$ on U'', in particular at P. Hence we get that $X = 0$ on U. Q.E.D.

From this lemma, we get the following proposition similarly as Proposition 2.4 in [3].

PROPOSITION 4.7. *Let D be a derivation of $\mathscr{T}_\sigma(M, \mathscr{F})$ or $\mathscr{L}_\sigma(M, \mathscr{F})$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. Then, D is local.*

PROPOSITION 4.8. *Let D be a derivation of $\mathscr{T}_\sigma(M, \mathscr{F})$ for $\sigma = 0, \tau, \omega$ or θ. Then D is localizable (see § 1.2 in [3] for definitions).*

Proof. This follows from Proposition 1.2 in [3] and Propositions 1.7 and 4.7. Q.E.D.

4.4.

PROPOSITION 4.9. *Let D be a derivation of $\mathscr{T}_\sigma(M, \mathscr{F})$ or $\mathscr{L}_\sigma(M, \mathscr{F})$ for $\sigma = 0, \tau, c\tau, \omega, c\omega$ or θ. Then there exists a unique vector field Z on M such that $D = \operatorname{ad} Z$ on $\mathscr{T}_\sigma(M, \mathscr{F})$ or $\mathscr{L}_\sigma(M, \mathscr{F})$ respectively. Moreover, Z is in $\mathscr{L}_{\sigma'}(M, \mathscr{F})$, where σ' is as in Proposition 3.4.*

The proof of this proposition will be given in § 4.5. Here we get Main Theorem similarly as Theorem 3.6:

THEOREM 4.10. (i) *Let* $\sigma = 0, c\tau, c\omega$ *or* θ. *All derivations of* $\mathscr{L}_\sigma(M, \mathscr{F})$ *are inner, that is,* $\mathscr{D}er(\mathscr{L}_\sigma(M, \mathscr{F})) = \mathrm{ad}\, \mathscr{L}_\sigma(M, \mathscr{F}) \cong \mathscr{L}_\sigma(M, \mathscr{F})$. *Hence*

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) = 0 .$$

(ii) *Let* $\sigma = \tau$ $(p \neq 1)$ *or* ω. *The derivation algebra of* $\mathscr{L}_\sigma(M, \mathscr{F})$ *is naturally isomorphic to* $\mathscr{L}_{c\sigma}(M, \mathscr{F})$, *that is,*

$$\mathscr{D}er(\mathscr{L}_\sigma(M, \mathscr{F})) = \{\mathrm{ad}\, Z|_{\mathscr{L}_\sigma(M,\mathscr{F})};\ Z \in \mathscr{L}_\sigma(M, \mathscr{F})\} \cong \mathscr{L}_{c\sigma}(M, \mathscr{F}) .$$

Hence,

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{L}_\sigma(M, \mathscr{F}) .$$

(iii) *Let* $\sigma = 0, c\tau, c\omega$ *or* θ. *The derivation algebra of* $\mathscr{T}_\sigma(M, \mathscr{F})$ *is naturally isomorphic to* $\mathscr{L}_\sigma(M, \mathscr{F})$, *that is,*

$$\mathscr{D}er(\mathscr{T}_\sigma(M, \mathscr{F})) = \{\mathrm{ad}\, Z|_{\mathscr{T}_\sigma(M,\mathscr{F})};\ Z \in \mathscr{L}_\sigma(M, \mathscr{F})\} \cong \mathscr{L}_\sigma(M, \mathscr{F}) .$$

Hence,

$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_\sigma(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) .$$

(iv) *Let* $\sigma = \tau$ $(p \neq 1)$ *or* ω. *The derivation algebra of* $\mathscr{T}_\sigma(M, \mathscr{F})$ *is naturally isomorphic to* $\mathscr{L}_{c\sigma}(M, \mathscr{F})$, *that is,*

$$\mathscr{D}er(\mathscr{T}_\sigma(M, \mathscr{F})) = \{\mathrm{ad}\, Z|_{\mathscr{T}_\sigma(M,\mathscr{F})};\ Z \in \mathscr{L}_{c\sigma}(M, \mathscr{F})\} \cong \mathscr{L}_{c\sigma}(M, \mathscr{F}) .$$

Hence,

$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) .$$

4.5.

Proof of Proposition 4.9. (a) *Case of* $\mathscr{T}_\sigma(M, \mathscr{F})$ $(\sigma = 0, \tau, \omega, \theta)$. Take a distinguished coordinate neighborhood system $\{(U^\lambda; v_1^\lambda, \cdots, v_p^\lambda, w_1^\lambda, \cdots, w_q^\lambda)\}_{\lambda \in \Lambda}$ on $(M, \mathscr{F})$ such as in Proposition 4.5.

Let D be a derivation of $\mathscr{T}_\sigma(M, \mathscr{F})$. Since D is localizable (Proposition 4.8), the derivation $D^\lambda \in \mathscr{D}er(\mathscr{T}_\sigma(U^\lambda))$ can be defined for all $\lambda \in \Lambda$ in such a way that $D(X)|_{U^\lambda} = D^\lambda(X|_{U^\lambda})$ for all $X \in \mathscr{T}_\sigma(M, \mathscr{F})$. Then by Proposition 3.5, there exists a unique vector field Z^λ on U^λ for any $\lambda \in \Lambda$ such that $D^\lambda = \mathrm{ad}\, Z^\lambda|_{\mathscr{T}_\sigma(U^\lambda)}$. On the other hand, since D is local, we get that $D^\lambda|_{U^\lambda \cap U^\mu} = D^\mu|_{U^\lambda \cap U^\mu}$, so $Z^\lambda = Z^\mu$ on $U^\lambda \cap U^\mu$. Hence there is a vector field Z on M such that $Z = Z^\lambda$ on U^λ for all $\lambda \in \Lambda$ and $D = \mathrm{ad}\, Z|_{\mathscr{T}_\sigma(M,\mathscr{F})}$. Moreover, we have $Z \in \mathscr{L}_{\sigma'}(M, \mathscr{F})$, because $Z^\lambda \in \mathscr{L}_{\sigma'}(U^\lambda, \mathscr{F}|_{U^\lambda})$ for all $\lambda \in \Lambda$.

(b) *Case of* $\mathscr{T}_{c\sigma}(M, \mathscr{F})$ $(\sigma = \tau, \omega)$. Let D be a derivation of $\mathscr{T}_{c\sigma}(M, \mathscr{F})$. Restrict D to $\mathscr{T}_{\sigma}(M, \mathscr{F})$, then we get the derivation $D' = D|_{\mathscr{T}_\sigma(M,\mathscr{F})}$ of $\mathscr{T}_{\sigma}(M, \mathscr{F})$ with values in $\mathscr{T}_{c\sigma}(M, \mathscr{F})$. Similarly as Proposition 4.8, we can show that D' is localizable. So for any open set U, we can define the derivation D'_U of $\mathscr{T}_{\sigma}(U, \mathscr{F}|_U)$ with values in $\mathscr{T}_{c\sigma}(U, \mathscr{F}|_U)$ in such a way that $r_U \circ D' = D'_U \circ r_U$. If U is sufficiently small, $\mathscr{T}_{\sigma}(U)$ is perfect, and so $D'_U(\mathscr{T}_{\sigma}(U)) \subset \mathscr{T}_{\sigma}(U)$. Then by Proposition 3.5, we get a unique vector field $Z_U \in \mathscr{L}_{c\sigma}(U)$ such that $D'_U = \operatorname{ad} Z_U$ on $\mathscr{T}_{\sigma}(U)$. Similarly as (a), there is a vector field $Z \in \mathscr{L}_{c\sigma}(M, \mathscr{F})$ such that $Z|_U = Z_U$ and $D' = \operatorname{ad} Z$ on $\mathscr{T}_{\sigma}(M, \mathscr{F})$.

For any $X \in \mathscr{T}_{c\sigma}(M, \mathscr{F})$ and all $Y \in \mathscr{T}_{\sigma}(M, \mathscr{F})$, we get

$$\begin{aligned}[D(X), Y] &= D([X, Y]) - [X, D(Y)] \\ &= [Z, [X, Y]] - [X, [Z, Y]] \\ &= [[Z, X], Y] .\end{aligned}$$

Then by Lemma 4.6, we get that $D(X) = [Z, X]$, hence $D = \operatorname{ad} Z$ on $\mathscr{T}_{c\sigma}(M, \mathscr{F})$.

(c) The proof for the case of $\mathscr{L}_{\sigma}(M, \mathscr{F})$ is similarly obtained as in (b). Q.E.D.

§ 5. Partially exactness of differential forms

5.1. *Partially exactness.* In this section, we treat only the case where $\sigma = \tau$ or ω. Recall that τ and ω are partially closed, that is, $d\tau, d\omega \in \mathscr{I}(M, \mathscr{F})$. Let $n_\tau = n(=p)$ and $n_\omega = 2$.

DEFINITION 5.1. Let σ be τ or ω.

(i) Let $\phi \in C^\infty(M)^{\mathscr{F}}$. We call σ *partially semi-exact* with respect to ϕ, if there exists an $(n_\sigma - 1)$-form α on M such that $\phi\sigma$ is congruent to $d\alpha$ modulo $\mathscr{I}(M, \mathscr{F})$.

(ii) σ is called *partially exact*, if σ is partially semi-exact with respect to constant functions on M.

(iii) σ is called *partially non-exact*, if σ is not partially semi-exact with respect to any function $\phi(\neq 0) \in C^\infty(M)^{\mathscr{F}}$.

Easily from the definition, we get the following proposition.

PROPOSITION 5.2. (i) *Assume that* $\iota_L^*\sigma$ *represents a non-zero class of* $H^{n_\sigma}(L; \boldsymbol{R})$ *for each leaf* L *of* $\mathscr{F}$. *Then,* σ *is partially non-exact.*

(ii) *If all leaves of* $\mathscr{F}$ *are compact, then any partially unimodular structure* τ *on* $(M, \mathscr{F})$ *is partially non-exact.*

LEMMA 5.3. (i) *Let $\phi \in C^\infty(M)^{\mathscr{F}}$. There exists a vector field $X \in \mathscr{T}_{c\sigma}(M, \mathscr{F})$ such that $L_X\sigma$ is congruent to $\phi\sigma$ modulo $\mathscr{I}(M, \mathscr{F})$, if and only if σ is partially semi-exact with respect to ϕ.*

(ii) *σ is partially non-exact, if and only if $\mathscr{T}_{c\sigma}(M, \mathscr{F}) = \mathscr{T}_\sigma(M, \mathscr{F})$.*

(iii) *If σ is partially semi-exact with respect to a non-vanishing function $\phi \in C^\infty(M)^{\mathscr{F}}$, then σ is partially exact.*

Proof. (i) *Proof on the "only if" part.* Let X be a vector field such that $L_X\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}$. Put $\alpha = i_X\sigma$, then by Lemma 4.1,

$$d\alpha = di_X\sigma = L_X\sigma - i_Xd\sigma \equiv L_X\sigma \equiv \phi\sigma \qquad (\mathrm{mod}\ \mathscr{I}(M, \mathscr{F}))\ .$$

Proof on the "if" part. Let α be a form on M such that $d\alpha \equiv \phi\sigma$ $(\mathrm{mod}\ \mathscr{I}(M, \mathscr{F}))$. By the partial nondegeneracy of σ, there exists a unique vector field $X \in \mathscr{T}(M, \mathscr{F})$ such that $i_X\sigma \equiv \alpha \pmod{\mathscr{I}(M, \mathscr{F})}$. Then

$$L_X\sigma \equiv di_X\sigma \equiv d\alpha \equiv \phi\sigma \qquad (\mathrm{mod}\ \mathscr{I}(M, \mathscr{F}))\ .$$

(ii) This follows from (i).

(iii) From the assumption, there is an $(n_\sigma - 1)$-form α such that $\phi\sigma \equiv d\alpha \pmod{\mathscr{I}(M, \mathscr{F})}$. Put $\psi = 1/\phi \in C^\infty(M)^{\mathscr{F}}$. Then

$$d(\psi\alpha) = d\psi \wedge \alpha + \psi d\alpha \equiv \psi d\alpha \equiv \psi\phi\sigma = \sigma \qquad (\mathrm{mod}\ \mathscr{I}(M, \mathscr{F}))\ ,$$

because $d\psi \in \mathscr{I}(M, \mathscr{F})$. Q.E.D.

5.2. *The mapping Φ.* Denote by Φ_0 the mapping which assigns to $X \in \mathscr{L}_{c\sigma}(M, \mathscr{F})$ the function $\phi \in C^\infty(M)^{\mathscr{F}}$ such that $L_X\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}$. Then, by factoring $\mathscr{T}_\sigma(M, \mathscr{F})$ and $\mathscr{L}_\sigma(M, \mathscr{F})$, we get the mapping Φ' and Φ as follows:

$$\begin{tikzcd}
 & \mathscr{L}_{c\sigma}(M, \mathscr{F}) \arrow[d, "\text{projection}"] \arrow[ddr, "\Phi_0"] & \\
H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong & \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) \arrow[d, "\text{projection}"] \arrow[dr, "\Phi'"] & \\
H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) \cong & \mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{L}_\sigma(M, \mathscr{F}) \arrow[r, "\Phi"] & C^\infty(M)^{\mathscr{F}}\ .
\end{tikzcd}$$

Since the kernel of Φ_0 is $\mathscr{L}_\sigma(M, \mathscr{F})$, the mapping Φ is injective. Hence we get

LEMMA 5.4. (i) *Φ is a zero map, if and only if $\mathscr{L}_{c\sigma}(M, \mathscr{F}) = \mathscr{L}_\sigma(M, \mathscr{F})$.*

(ii) *If Φ is a zero map, then σ is partially non-exact.*

Proof. (ii) Since $\mathscr{T}_\sigma(M, \mathscr{F}) = \mathscr{T}_{c\sigma}(M, \mathscr{F}) \cap \mathscr{L}_\sigma(M, \mathscr{F})$, then $\mathscr{T}_{c\sigma}(M, \mathscr{F}) = \mathscr{T}_\sigma(M, \mathscr{F})$, so we get the assertion by Lemma 5.3 (ii). Q.E.D.

LEMMA 5.5. *Let σ be partially semi-exact with respect to a function $\phi \in C^\infty(M)^{\mathscr{F}}$. Then the image of Φ includes the ideal $\phi \cdot C^\infty(M)^{\mathscr{F}}$ of $C^\infty(M)^{\mathscr{F}}$.*

Proof. By Lemma 5.3 (i), there is a vector field $X \in \mathscr{T}_{c\sigma}(M, \mathscr{F})$ such that $L_X\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}$. Let $\psi \in C^\infty(M)^{\mathscr{F}}$, then

$$L_{\psi X}\sigma = \psi L_X\sigma + d\psi \wedge i_X\sigma \equiv \psi L_X\sigma \equiv \psi\phi\sigma \qquad (\mathrm{mod}\, \mathscr{I}(M, \mathscr{F}))\,.$$

Hence $\psi X \in \mathscr{T}_{c\sigma}(M, \mathscr{F}) \subset \mathscr{L}_{c\sigma}(M, \mathscr{F})$, and $\Phi(\psi X) = \psi\phi$, that is, $\phi C^\infty(M)^{\mathscr{F}} \subset \mathrm{Im}\,\Phi$. Q.E.D.

LEMMA 5.6. *Let σ be partially exact. Then*

(i) *Φ is surjective, hence Φ is the isomorphism.*

(ii) *The inclusion map η of $\mathscr{T}_{c\sigma}(M, \mathscr{F})$ into $\mathscr{L}_{c\sigma}(M, \mathscr{F})$ induces the isomorphism η_* of $\mathscr{T}_{c\sigma}(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F})$ onto $\mathscr{L}_{c\sigma}(M, \mathscr{F})/\mathscr{L}_\sigma(M, \mathscr{F})$.*

Proof. (i) This follows from Lemma 5.5.

(ii) Since $\mathscr{L}_\sigma(M, \mathscr{F}) \cap \mathscr{T}_{c\sigma}(M, \mathscr{F}) = \mathscr{T}_\sigma(M, \mathscr{F})$, then η_* is injective. Let $X \in \mathscr{L}_{c\sigma}(M, \mathscr{F}) \setminus \mathscr{L}_\sigma(M, \mathscr{F})$. Then there is a function $\phi(\neq 0) \in C^\infty(M)^{\mathscr{F}}$ such that $L_X\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}$. Similarly as in the proof of Lemma 5.5, we get a vector field $Y \in \mathscr{T}_{c\sigma}(M, \mathscr{F})$ such that $L_Y\sigma \equiv \phi\sigma \pmod{\mathscr{I}(M, \mathscr{F})}$. Put $Z = X - Y \in \mathscr{L}_{c\sigma}(M, \mathscr{F})$, then $L_Z\sigma \equiv L_X\sigma - L_Y\sigma \equiv 0 \pmod{\mathscr{I}(M, \mathscr{F})}$, so $Z \in \mathscr{L}_\sigma(M, \mathscr{F})$. Thus, $\eta(Y)$ represents a class of X, that is, η_* is surjective. Q.E.D.

THEOREM 5.7. *Let σ be τ or ω.*

(i) *Assume that $C^\infty(M)^{\mathscr{F}} \cong \boldsymbol{R}$ and σ is not partially exact, then*

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) = 0\,,$$
$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong \mathscr{L}_\sigma(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F})\,.$$

(ii) *Assume that σ is partially exact, then*

$$H^1(\mathscr{L}_\sigma(M, \mathscr{F}); \mathscr{L}_\sigma(M, \mathscr{F})) \cong \mathscr{T}_{c\sigma}(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F}) \cong C^\infty(M)^{\mathscr{F}}\,.$$
$$H^1(\mathscr{T}_\sigma(M, \mathscr{F}); \mathscr{T}_\sigma(M, \mathscr{F})) \cong C^\infty(M)^{\mathscr{F}} + \mathscr{L}_\sigma(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F})\,.$$

Proof. Recall that $\pi^{-1}(0) \cong \mathscr{L}_\sigma(M, \mathscr{F})/\mathscr{T}_\sigma(M, \mathscr{F})$. (i) The assumption implies that Φ is a zero map. (ii) This follows from Lemma 5.6 (i) and (ii). Q.E.D.

5.3.

EXAMPLE 1. Let $M = \boldsymbol{R}^{p+q}$, and the foliation $\mathscr{F}$ on M be given by p-planes parallel to the coordinate p-plane. Then $C^\infty(M)^{\mathscr{F}} \cong C^\infty(\boldsymbol{R}^q)$. Let $\tau = dx_1 \wedge \cdots \wedge dx_n$ $(p = n)$ and $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ $(p = 2n)$, then τ and ω are partially exact. In fact,

$$\tau = d(x_1 dx_2 \wedge \cdots \wedge dx_n) \quad \text{and} \quad \omega = d\left(\sum_{i=1}^n x_i dy_i\right).$$

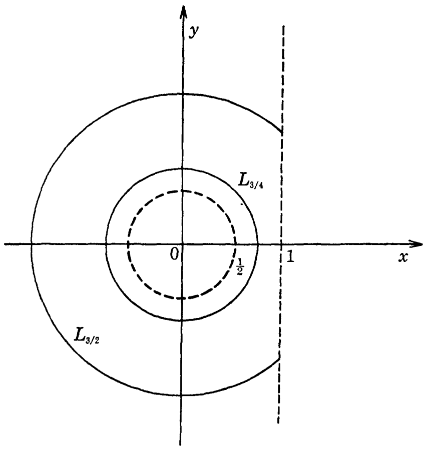
EXAMPLE 2. Let M be the product of manifolds X and Y. Consider the product foliation $\mathscr{F}_{pr}: M = \bigcup_{x \in X} \{x\} \times Y$. Then $C^\infty(M)^{\mathscr{F}_{pr}} \cong C^\infty(X)$. Assume that Y admits a volume form τ or a symplectic form ω. Let π be the projection of M onto Y. Then, $\pi^*\tau$ or $\pi^*\omega$ is a partially unimodular, or symplectic structure on $(M, \mathscr{F}_{pr})$ respectively. Moreover, $\pi^*\sigma (\sigma = \tau, \omega)$ is partially exact, or non-exact, if and only if σ is exact or not respectively.

EXAMPLE 3. Consider the foliation $\mathscr{F}_e$ on $\boldsymbol{R}^3 = \{(x, y, z)\}$ as $\boldsymbol{R}^3 = \{z = 0\} \cup \bigcup_{c \in \boldsymbol{R}, \varepsilon = \pm 1} \{z = \varepsilon e^{x-c}\}$, and a partially unimodular and symplectic structure $\tau = dx \wedge dy$ on $(\boldsymbol{R}^3, \mathscr{F}_e)$. Then $C^\infty(\boldsymbol{R}^3)^{\mathscr{F}_e} \cong \boldsymbol{R}$ and τ is partially exact. In fact, $\tau = d(xdy)$.

EXAMPLE 4. Consider the linear foliation $\mathscr{F}_\lambda$ on the 2-dimensional torus $T^2 = \boldsymbol{R}^2/\boldsymbol{Z}^2$ with a slope λ. If λ is a rational number, $C^\infty(T^2)^{\mathscr{F}_\lambda} \cong C^\infty(S^1)$ and any partially unimodular structure on $(T^2, \mathscr{F}_\lambda)$ is partially non-exact by virtue of Proposition 5.2 (ii). If λ is irrational, $C^\infty(T^2)^{\mathscr{F}_\lambda} \cong \boldsymbol{R}$ and any partially unimodular structure τ is also partially non-exact. In fact, suppose that $\tau \equiv d\phi \pmod{\mathscr{I}(T^2, \mathscr{F}_\lambda)}$ for some function $\phi \in C^\infty(T^2)$. Since $\iota_L^*\tau \neq 0$ for every leaf L of $\mathscr{F}_\lambda$, we can easily show that the function $\iota_L^*\phi$ increases or decreases infinitely along the leaf L. But this contradicts the periodicity of ϕ.

We can modify this example to get linear foliations of codimension 1 of the 3-dimensional torus T^3 such that $C^\infty(T^3)^{\mathscr{F}} \cong \boldsymbol{R}$ and any partially symplectic structure on $(T^3, \mathscr{F})$ is partially non-exact.

EXAMPLE 5. Let $M = \{(x, y) \in \boldsymbol{R}^2; x < 1 \text{ and } x^2 + y^2 > 1/4\}$. Consider the foliation $\mathscr{F}: M = \bigcup_{r > 1/2} L_r$, where $L_r = \{(x, y) \in M; x^2 + y^2 = r^2\}$. Using polar coordinates (r, θ), we get that $C^\infty(M)^{\mathscr{F}} \cong C^\infty((1/2, \infty))$ and the form $\tau = d\theta$ is a partially unimodular structure on $(M, \mathscr{F})$. Let $\mathscr{K}$ be the subspace of $C^\infty((1/2, \infty))$, consisting of functions which vanishes on the open interval $(1/2, 1)$. Since the leaves L_r are compact only for $r < 1$, easily we get



PROPOSITION 5.8. *Let $\phi \in C^\infty((1/2, \infty))$. Then, τ is partially semi-exact with respect to ϕ, if and only if the function ϕ belongs to $\mathscr{K}$.*

On $M \times \boldsymbol{R}$, we get the foliation: $M \times \boldsymbol{R} = \bigcup_{r>1/2} L_r \times \boldsymbol{R}$, and the partially symplectic structure $d\theta \wedge dz$ which is partially semi-exact with respect to $\phi \in \mathscr{K}$.

§ 6. Properly outer derivations

6.1. In the following we treat the case where $\sigma = \tau$ and $p = 1$. By Lemma 1.2, we get the isomorphisms

$$\mathscr{T}_\tau = C^\infty(W)\partial \quad \text{and} \quad \mathscr{T}_{c\tau} = C^\infty(W)\partial + C^\infty(W)I\,,$$

where we omit the index 1 of x_1 and ∂_1; $I = I_\tau = x\partial$. Hence

$$\mathscr{T}_\tau \cap \tilde{\mathfrak{A}} = \boldsymbol{R}[W]\partial = \sum_{m \geqq -1} \mathscr{T}_\tau(-1, m) \cong \sum_{m \geqq 0} \boldsymbol{P}^m(W)\,,$$

$$\mathscr{T}_{c\tau} \cap \tilde{\mathfrak{A}} = \boldsymbol{R}[W]\partial + \boldsymbol{R}[W]I = \sum_{n=-1}^{0} \sum_{m \geqq -1} \mathscr{T}_{c\tau}(n, m)$$
$$\cong \sum_{m \geqq 0} \boldsymbol{P}^m(W) + \sum_{m \geqq 0} \boldsymbol{P}^m(W)I\,.$$

Then we get the following lemma similarly as Lemmata 2.2 and 2.3.

LEMMA 6.1. *Let D be a derivation of $\mathscr{T}_{c\tau}$ or $\mathscr{L}_{c\tau}$. Assume that $D = 0$ on $\sum_{n+m \leqq -1} \mathscr{T}_{c\tau}(n, m)$. Then $D = 0$ on $\mathscr{T}_{c\tau} \cap \tilde{\mathfrak{A}}$ or $\mathscr{L}_{c\tau} \cap \tilde{\mathfrak{A}}$ respectively.*

Here we get for $\mathscr{D}er\,(\mathscr{L}_\tau)$

LEMMA 6.2. *Let D be a derivation of $\mathscr{L}_\tau$. Assume that $D(X) = 0$ for $X = \partial, \partial_\alpha$ and $J\,(1 \leqq \alpha \leqq q)$. Then $D = 0$ on $\mathscr{L}_\tau \cap \tilde{\mathfrak{A}}$.*

Proof. At first we show that $D=0$ on $\mathscr{T}_\tau \cap \tilde{\mathfrak{A}} = \sum_{m \geqq -1} \mathscr{T}_\tau(-1, m)$, by the induction on m. When m is negative, the assertion holds by the assumption. Assume that $D=0$ on $\mathscr{T}_\tau(-1, k)$ ($k \leqq m-1, m \geqq 0$). Let $X \in \mathscr{T}_\tau(-1, m)$, and define the vector fields $Y \in \mathscr{T}_\tau$ and $Z \in \mathscr{L}'$ as $D(X) = Y + Z$.

Apply D to $[\partial_\alpha, X] \in \mathscr{T}_\tau(-1, m-1)$, then we get that Y and Z are with constant coefficients.

Apply D to $(m+1)X = [J, X]$, then we get

$$(m+1)(Y+Z) = [J, Y+Z] = 0 - Z\,,$$

hence $Y = Z = 0$, so $D(X) = 0$. Thus $D = 0$ on $\mathscr{T}_\tau \cap \tilde{\mathfrak{A}}$.

We can show that $D = 0$ on $\mathscr{L}' \cap \tilde{\mathfrak{A}}$, similarly as Last Step of the proof of Proposition 2.3 in [4]. Q.E.D.

6.2.

LEMMA 6.3. *Let D be a derivation of $\mathscr{T}_{c\tau}$ or $\mathscr{L}_{c\tau}$. Then there exists a unique vector field Z on V such that $D = \operatorname{ad} Z$ on $\sum_{n+m \leqq -1} \mathscr{T}_{c\tau}(n, m)$. Moreover, Z is in $\mathscr{L}_{c\tau}$.*

Proof. Let $D \in \mathscr{D}er(\mathscr{T}_{c\tau})$ or $\mathscr{D}er(\mathscr{L}_{c\tau})$. Since $\mathscr{T}_\tau = [\mathscr{T}_{c\tau}, \mathscr{L}_{c\tau}]$ and $f(w)\partial = [f(w)\partial, I]$, then we get that $D(\mathscr{T}_\tau) \subset \mathscr{T}_\tau$, similarly as Proposition 2.3. Define functions $g, g_\alpha \in C^\infty(W)$ as $D(\partial) = g(w)\partial$ and $D(w_\alpha\partial) = (g_\alpha(w) + w_\alpha g(w))\partial$ for $1 \leqq \alpha \leqq q$. Put $Z_1 = -g(w)I + \sum_\alpha g_\alpha(w)\partial_\alpha \in \mathscr{T}_{c\tau} + \mathscr{L}'$, then we get

$$\begin{cases} [Z_1, \partial] = g(w)\partial = D(\partial)\,, \\ [Z_1, w_\alpha\partial] = g_\alpha(w)\partial + w_\alpha g(w)\partial = D(w_\alpha\partial)\,. \end{cases}$$

Let $D_1 = D - \operatorname{ad} Z_1$, then D_1 is a derivation of $\mathscr{T}_{c\tau}$, and $D_1(\partial) = D_1(w_\alpha\partial) = 0$. Apply D_1 to $\partial = [\partial, I]$ and $w_\alpha\partial = [w_\alpha\partial, I]$, then we get easily that $D_1(I) \in \mathscr{T}_\tau$. Put $Z_2 = D_1(I)$, then $[Z_2, I] = Z_2 = D_1(I)$ and $[Z_2, \partial] = [Z_2, w_\alpha\partial] = 0$.

Thus we get a vector field $Z = Z_1 + Z_2 \in \mathscr{L}_{c\tau}$ such that $D = \operatorname{ad} Z$ on $\sum_{n+m \leqq -1} \mathscr{T}_{c\tau}(n, m)$.

To prove the uniqueness of Z, it is sufficient to show that a vector field Z on V is zero, if $[Z, \partial] = [Z, I] = [Z, w_\alpha\partial] = 0$. Q.E.D.

6.3.

PROPOSITION 6.4. *Let D be a derivation of $\mathscr{T}_{c\tau}$ or $\mathscr{L}_{c\tau}$. Then there* *exists a unique vector field Z on V such that $D = \mathrm{ad}\, Z$ on $\mathscr{T}_{c\tau}$ or $\mathscr{L}_{c\tau}$ respectively. Moreover, Z is in $\mathscr{L}_{c\tau}$.*

Proof. We can prove this for $\mathscr{L}_{c\tau}$ similarly as Proposition 3.5, by the following lemma.

LEMMA 6.5. *If a vector field $X \in \mathscr{T}_{c\tau}$ satisfies $j^2(X)(0) = 0$, then there exists a finite number of vector fields $Y_i \in \mathscr{L}_{c\tau}$ and $Z_i \in \mathscr{T}_{c\tau}$ $(1 \leqq i \leqq r)$ such that $X = \sum_{i=1}^{r} [Y_i, Z_i]$ and $j^1(Y_i)(0) = j^1(Z_i)(0) = 0$.*

Moreover, for $X \in \mathscr{T}_{\tau}$, we can take vector fields such that $Y_i \in \mathscr{T}_{\tau}$ and $Z_i \in \mathscr{T}_{c\tau}$ or $\mathscr{L}'$.

Proof. It is enough to remark the formulae

$$w_\alpha f(w)\partial = [f(w)\partial, w_\alpha x\partial] = [f(w)\partial_\alpha, 2^{-1}w_\alpha^2\partial]\,, \qquad (1 \leqq \alpha \leqq q)$$

and

$$g(w)I = [g(w)\partial_\alpha, w_\alpha x\partial]\,. \qquad \text{Q.E.D.}$$

We return to the proof of Proposition 6.4 for $\mathscr{T}_{c\tau}$. We get that $D = 0$ on $\mathscr{T}_{\tau}$ similarly as for $\mathscr{L}_{c\tau}$ by the lemma above. So it is enough to show that $D(X) = [Z, X]$ for $X = f(w)I$, where Z is obtained in Lemma 6.3.

Apply $D_1 = D - \mathrm{ad}\, Z$ to $[\partial, X] = f(w)\partial \in \mathscr{T}_{\tau}$ and $[I, X] = 0$, then easily we get $D_1(X) = 0$, that is, $D(X) = [Z, X]$. Q.E.D.

From Proposition 6.6, we get

THEOREM 6.7. (i) *The derivation algebra of $\mathscr{T}_{c\tau}$ is naturally isomorphic to $\mathscr{L}_{c\tau}$, that is, $\mathscr{D}er\,(\mathscr{T}_{c\tau}) = \mathrm{ad}\,\mathscr{L}_{c\tau} \cong \mathscr{L}_{c\tau}$. Hence,*

$$H^1(\mathscr{T}_{c\tau}; \mathscr{T}_{c\tau}) \cong \mathscr{L}_{c\tau}/\mathscr{T}_{c\tau} \cong \mathfrak{A}(W)\,.$$

(ii) *All derivations of $\mathscr{L}_{c\tau}$ are inner. Hence,*

$$H^1(\mathscr{L}_{c\tau}; \mathscr{L}_{c\tau}) = 0\,.$$

6.4.

LEMMA 6.8. *Let D be a derivation of $\mathscr{L}_{\tau}$. Then there exists a vector field Z on V such that $D(X) = [Z, X]$ for $X = \partial$ and ∂_α, and $D(J) \equiv [Z, X]$ (mod $\mathfrak{z}$), where $\mathfrak{z} = \boldsymbol{R}\cdot\partial$ is the center of $\mathscr{L}_{\tau}$. Moreover, Z is in $\mathscr{L}_{\tau} + \boldsymbol{R}\cdot I$ and is unique modulo $\mathfrak{z}$.*

Proof. Let D be a derivation of $\mathscr{L}_{\tau}$. Define the functions $f_\alpha(w)$ and $g_\alpha^\beta(w)$ in $C^\infty(W)$ as $D(\partial_\alpha) = f_\alpha(w)\partial + \sum_\beta g_\alpha^\beta(w)\partial_\beta$. Apply D to $[\partial_\alpha, \partial_\beta] = 0$, then we get that $\partial_\alpha f_\beta = \partial_\beta f_\alpha$ and $\partial_\alpha g^\gamma_\beta = \partial_\beta g^\gamma_\alpha$ for $1 \leqq \alpha, \beta, \gamma \leqq q$. Then there are unique functions f and $g^\gamma \in C^\infty(W)$ such that $f_\alpha = \partial_\alpha f$, $g^\gamma_\alpha = \partial_\alpha g^\gamma$ and $f(0) = g^\gamma(0) = 0$ for $1 \leqq \alpha, \gamma \leqq q$. Put $Z_1 = -f(w)\partial - \sum_{\alpha=1}^q g^\alpha(w)\partial_\alpha \in \mathscr{L}_\tau$, then $[Z_1, \partial_\alpha] = D(\partial_\alpha)$.

Let $D_1 = D - \operatorname{ad} Z_1$, then D_1 is a derivation of $\mathscr{L}_\tau$ and $D_1(\partial_\alpha) = 0$. Apply D_1 to $[\partial_\alpha, J] = \partial_\alpha$, then we get that $D_1(J)$ is with constant coefficients. Define the constants a and b^α $(1 \leqq \alpha \leqq q)$ as $D(J) = a\partial + \sum_\alpha b^\alpha\partial_\alpha$. Put $Z_2 = \sum_\alpha b^\alpha\partial_\alpha \in \mathscr{L}'$, then $D_1(J) = [Z_2, J] + a\partial$ and $[Z_2, \partial_\alpha] = 0$.

Let $D_2 = D_1 - \operatorname{ad} Z_2$, then we get that $D_2(\partial_\alpha) = 0$, $D_2(J) = a\partial$, and $D_2 \in \mathscr{D}er(\mathscr{L}_\tau)$. Apply D_2 to $[\partial, \partial_\alpha] = [\partial, J] = 0$, then easily we get that $D_2(\partial)$ can be written as $D_2(\partial) = c\partial$ for some constant $c \in \boldsymbol{R}$. Put $Z_3 = -cx\partial = -cI \in \mathscr{T}_{c\tau}$, then we get that $[Z_3, \partial] = c\partial = D_2(\partial)$ and $[Z_3, \partial_\alpha] = [Z_3, J] = 0$.

Thus we get a vector field $Z = Z_1 + Z_2 + Z_3$ satisfying the conditions of the lemma.

To prove the uniqueness of Z, it is enough to show that vector fields Z on V must be in $\mathfrak{z}$ if $[Z, \partial] = [Z, \partial_\alpha] = 0$ and $[Z, J] \in \mathfrak{z}$. Q.E.D.

6.5. *Outer derivations.*

DEFINITION 6.9. Let $\mathfrak{G}$ be a Lie subalgebra of $\mathfrak{A}(V)$.

(i) A derivation D of $\mathfrak{G}$ is called *natural outer*, if there exists a vector field $Z \notin \mathfrak{G}$ on V such that $D = \operatorname{ad} Z$ on $\mathfrak{G}$, and there are no such vector fields in $\mathfrak{G}$.

(ii) A derivation D of $\mathfrak{G}$ is called *properly outer*, if there are no vector fields Z on V such that $D = \operatorname{ad} Z$ on $\mathfrak{G}$.

Let $a \in \boldsymbol{R}$. Define a linear map D^a of $\mathscr{L}_\tau$ to itself, as $D^a(f\partial_\alpha) = a(\partial_\alpha f)\partial$ and $D^a(f\partial) = 0$ for any function $f \in C^\infty(W)$.

LEMMA 6.10. (i) *D^a is a derivation of $\mathscr{L}_\tau$, and*

(ii) *D^a is properly outer for $a \neq 0$.*

Proof. (i) Let f and $g \in C^\infty(W)$. Since $[f\partial, g\partial] = 0$, $[f\partial_\alpha, g\partial] = f(\partial_\alpha g)\partial$ and $[f\partial_\alpha, g\partial_\beta] = f(\partial_\alpha g)\partial_\beta - g(\partial_\beta f)\partial_\alpha$, we can easily check the derivation property of D^a.

(ii) Let Z be a vector field on V such that $D^a = \operatorname{ad} Z$ on $\mathscr{L}_\tau$. Since $D^a(\partial) = D^a(\partial_\alpha) = D^a(w_\alpha\partial) = 0$, then we get easily that Z is in the center $\mathfrak{z}$. However, this contradicts that $[Z, J] = D^a(J) = qa\partial \neq 0$. Q.E.D.

LEMMA 6.11. *Let D be a derivation of $\mathscr{L}_\tau$. Assume that $D(\partial) = D(\partial_\alpha) = 0$ $(1 \leqq \alpha \leqq q)$ and $D(J) = qa\partial \in \mathfrak{z}$, then $D = D^a$ on $\mathscr{L}_\tau \cap \tilde{\mathfrak{A}}$.*

Proof. Since $D^a(\partial) = D^a(\partial_\alpha) = 0$ and $D^a(J) = qa\partial = D(J)$, so $D_1 = D - D^a$ is a derivation of $\mathscr{L}_\tau$ such that $D_1(\partial) = D_1(\partial_\alpha) = D_1(J) = 0$. Then by Lemma 6.2, we get that $D_1 = 0$, that is, $D = D^a$ on $\mathscr{L}_\tau \cap \mathfrak{A}$. Q.E.D.

6.6.

PROPOSITION 6.12. *Let D be a derivation of $\mathscr{L}_\tau$. Then there exists a vector field Z on V and a constant $a \in \boldsymbol{R}$ such that $D = \mathrm{ad}\, Z + D^a$ on $\mathscr{L}_\tau$. Moreover, Z is in $\mathscr{L}_\tau + \boldsymbol{R}\cdot I$, and is unique modulo $\mathfrak{z}$.*

Proof. Let $D \in \mathscr{D}er(\mathscr{L}_\tau)$. Then by Lemmata 6.8 and 6.11, we get a vector field Z and a constant a such that $D = \mathrm{ad}\, Z + D^a$ on $\mathscr{L}_\tau \cap \mathfrak{A}$. Hence by Lemma 6.5, we get similarly as Proposition 6.4 that $D = \mathrm{ad}\, Z + D^a$ on $\mathscr{L}_\tau$. Q.E.D.

THEOREM 6.13. *The derivation algebra $\mathscr{D}er(\mathscr{L}_\tau)$ has the 2-dimensional subspace of outer derivations, and the 1-dimensional one of natural outer derivations. Hence,*

$$H^1(\mathscr{L}_\tau; \mathscr{L}_\tau) \cong \boldsymbol{R} \oplus \boldsymbol{R} .$$

6.7. Since $\mathscr{T}_\tau$ is abelian and $\mathscr{T}_\tau = C^\infty(W)$, then all non trivial derivations of $\mathscr{T}_\tau$ are outer, and the derivation algebra $\mathscr{D}er(\mathscr{T}_\tau)$ is naturally isomorphic to the space $\mathscr{L}in(C^\infty(W), C^\infty(W))$ of all linear maps of $C^\infty(W)$ to itself.

LEMMA 6.14. *Let D be a derivation of $\mathscr{T}_\tau$. Then there exists a vector field Z on V such that $D(X) = [Z, X]$ for $X = \partial$ and $w_\alpha\partial$ $(1 \leqq \alpha \leqq q)$. Moreover, Z is in $\mathscr{L}_{c\tau}$ and is unique modulo $\mathscr{T}_\tau$.*

Proof. Let $D \in \mathscr{D}er(\mathscr{T}_\tau)$. Define the functions $f, f_\alpha \in C^\infty(W)$ such that $D(\partial) = f(w)\partial$ and $D(w_\alpha\partial) = (f_\alpha(w) + w_\alpha f(w))\partial$ for $1 \leqq \alpha \leqq q$. Put

$$Z = -xf(w)\partial + \sum_\alpha f_\alpha(w)\partial_\alpha \in \mathscr{L}_{c\tau} ,$$

then we get

$$\begin{cases} [Z, \partial] = f(w)\partial = D(\partial) , \\ [Z, w_\alpha\partial] = (f_\alpha(w) + w_\alpha f(w))\partial = D(w_\alpha\partial) . \end{cases}$$

Let $D_1 = D - \mathrm{ad}\, Z$, then D_1 is a derivation of $\mathscr{T}_\tau$ and $D_1(\partial) = D_1(w_\alpha\partial) = 0$. Let a vector field X on V which satisfies $[X, \partial] = [X, w_\alpha\partial] = 0$ $(1 \leqq \alpha \leqq q)$, then easily we get that $X \in \mathscr{T}_\tau$. Hence, Z is unique modulo $\mathscr{T}_\tau$. Q.E.D.

Here we summarize results for $\mathscr{D}er(\mathscr{T}_\tau)$;

THEOREM 6.15. (i) *All derivations of $\mathscr{T}_\tau$ are outer, and the derivation algebra $\mathscr{D}er(\mathscr{T}_\tau)$ is isomorphic to the linear space $\mathscr{L}in(C^\infty(W), C^\infty(W))$. Hence,*

$$H^1(\mathscr{T}_\tau, \mathscr{T}_\tau) \cong \mathscr{L}in(C^\infty(W), C^\infty(W)) .$$

(ii) *The space of natural outer derivations of $\mathscr{T}_\tau$ is isomorphic to $C^\infty(W)\cdot I + \mathscr{L}'$.*

(iii) *Any linear map ϕ of $C^\infty(W)^{(1)}$ to $C^\infty(W)$ defines a properly outer derivation D_ϕ of $\mathscr{T}_\tau$ such that $D_\phi(\partial) = D_\phi(w_\alpha\partial) = 0$, and $D_\phi(f\partial) = \phi(f)\partial$ for $f \in C^\infty(W)^{(1)}$, where $C^\infty(W)^{(1)}$ is the subspace of $C^\infty(W)$ consisting of functions whose 1-jets vanish at the origin.*

6.8.

Remark. Since any one of $\mathscr{T}_\tau, \mathscr{T}_{c\tau}, \mathscr{L}_\tau$ and $\mathscr{L}_{c\tau}$ does not satisfy the property (A), I don't know the way to localize their derivations in general. However in some examples of $(M, \mathscr{F}, \tau)$, we can determine the structure of these Lie algebras and their derivation algebras. I will discuss such examples elsewhere.

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