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TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS, V

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Our aim is to prove

THEOREM. *Let L be a positive lattice of E-type such that $[L:\tilde{L}] < \infty$ and $\tilde{L}$ is indecomposable.*

(i) *If $L \cong L_1 \otimes L_2$ for positive lattices L_1, L_2, then L_1, L_2 are of E-type and $[L_1:\tilde{L}_1]$, $[L_2:\tilde{L}_2] < \infty$ and $\tilde{L}_1, \tilde{L}_2$ are indecomposable.*

(ii) *If L is indecomposable with respect to tensor product, then for each indecomposable positive lattice X we have*

(1) *$L \otimes X \cong L \otimes Y$ implies $X \cong Y$ for a positive lattice Y,*

(2) *If $X = \otimes^t L \otimes X'$ where X' is not divided by L, then $O(L \otimes X)$ is generated by $O(L)$, $O(X')$ and interchanges of L's, and*

(3) *$L \otimes X$ is indecomposable.*

We must explain notations and terminology. By a positive lattice we mean a lattice on positive definite quadratic space over the rational number field $\boldsymbol{Q}$. Let L be a positive lattice and put

$$m(L) = \min_{\substack{x \in L \\ x \neq 0}} Q(x)$$

where $Q(\ )$ is a quadratic form associated with L. Set $\mathfrak{M}(L) = \{x \in L \mid Q(x) = m(L)\}$. If $\mathfrak{M}(L \otimes M) = \mathfrak{M}(L) \otimes \mathfrak{M}(M)$ $(=\{x \otimes y \mid x \in \mathfrak{M}(L), y \in \mathfrak{M}(M)\})$ for any positive lattice M, then L is called of E-type. $\tilde{L}$ is a submodule of L spanned by $\mathfrak{M}(L)$. If $L \cong L_1 \otimes L_2$ implies $\operatorname{rank} L_1$ or $\operatorname{rank} L_2 = 1$, then we say that L is indecomposable with respect to tensor product. $O(L)$ denotes the orthogonal group of L. If a positive lattice X is isometric to $L \otimes K$ for a positive lattice K, then X is, by definition, divided by L. These notations and terminology will be used through this paper.

In § 1 we prove a theorem about weighted graphs. In § 2 we improve

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a result in [4] and in § 3 the above theorem is proved. In § 4 examples of a lattice L as in the theorem are given.

§ 1.

In this section we define a weighted graph and prove a fundamental theorem in this paper.

DEFINITION. Let A be a finite set and $[\ ,\ ]$ be a mapping from $A \times A$ into $\{t \mid 0 \leq t \leq 1\}$ such that

(i) $[a, a'] = 1$ if and only if $a = a'$, and

(ii) $[a, a'] = [a', a]$ for $a, a' \in A$.

Then we call $(A, [\ ,\ ])$ or simply A a weighted graph.

Let A be a weighted graph. A is called connected unless there exist subsets A_1, A_2 of A such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \phi$ and $[a_1, a_2] = 0$ for any $a_i \in A_i$ $(i = 1, 2)$. If $A = \cup A_i$ (disjoint) satisfies

(i) A_i is connected, and

(ii) $[a, b] = 0$ if $a \in A_i$, $b \in A_j$ and $i \neq j$,

then each A_i is called a connected component of A. Let A, B be weighted graphs. For $(a, b), (a', b') \in A \times B$ we define $[(a, b), (a', b')]$ by $[a, a'] \cdot [b, b']$. Then $A \times B$ becomes a weighted graph. If there exists a bijection σ from A on B such that $[a, a'] = [\sigma(a), \sigma(a')]$ $(a, a' \in A)$, then we say that A, B are isometric and write $\sigma: A \cong B$.

LEMMA 1. *Let A, B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^{n}$ and $\sigma: A \times B \cong A \times C$. Take any element $b \in B$ and fix it. Define $f_i \in A$, $c_i \in C$, $g_{ij} \in A$, $b_{ij} \in B$ by*

$$\sigma(e_i, b) = (f_i, c_i) \quad \text{and} \quad \sigma(g_{ij}, b_{ij}) = (f_i, c_j) .$$

Then we have $[e_i, e_j] = 0$ if $b_{ij} \neq b$.

Proof. Set $a_{ij} = [e_i, e_j]$. Then $a_{ij} = a_{ji}$ and

$$a_{ij} = [f_i, f_j][c_i, c_j] . \tag{0}$$

Fix any integer k $(1 \leq k \leq n)$ and define $e'_s \in A$, $b_s \in B$ by $\sigma(e'_s, b_s) = (f_k, c_s)$ $(1 \leq s \leq n)$. Put $S = \{s \mid b_s \neq b\}$. If $S \neq \phi$, then we take integers u, m such that

$$a_{um} = \max_{\substack{i:\, f_i = f_k \\ s \in S}} a_{is} \quad \text{and} \quad f_u = f_k\ , \quad m \in S\ .$$

If $a_{um} = 0$ can be shown, then the lemma will be proved. Assume $a_{um} \neq 0$ and put $b_m = b'$, $e'_m = e_p$. Then we have

$$\sigma(e_p, b') = (f_k, c_m) , \qquad b' \neq b .$$

Since $\sigma(e_i, b) = (f_i, c_i)$, we have

$$(1) \qquad a_{ip}[b, b'] = [f_i, f_k][c_i, c_m] ,$$

$$(2) \qquad a_{ip}[b, b'] = [c_i, c_m] \qquad \text{if } f_i = f_k ,$$

$$(3) \qquad a_{mp}[b, b'] = [f_m, f_k] .$$

Hence $f_u = f_k$ implies

$$\begin{aligned} a_{um} &= [f_u, f_m][c_u, c_m] && \text{by (0)} \\ &= [f_k, f_m][c_u, c_m] \\ &= a_{mp}a_{up}[b, b']^2 && \text{by (2), (3)} . \end{aligned}$$

$$(4) \qquad a_{um} = a_{mp}a_{up}[b, b']^2 .$$

Suppose $f_p = f_k$. Then $\sigma(e_p, b') = (f_k, c_m) = (f_p, c_m)$ implies $a_{um} \geq a_{pm}$. Hence we have

$$\begin{aligned} 0 < a_{um} &= a_{mp}a_{up}[b, b']^2 && \text{by (4)} \\ &\leq a_{um}a_{up}[b, b']^2 \\ &\leq a_{um} . \end{aligned}$$

This yields $a_{up}[b, b']^2 = 1$ and $[b, b'] = 1$. This contradicts $b \neq b'$. Therefore we get $f_p \neq f_k$. Suppose $p \in S$; then $a_{up} \leq a_{um}$ holds by definition. Hence we have

$$\begin{aligned} 0 < a_{um} &= a_{mp}a_{up}[b, b']^2 && \text{by (4)} \\ &\leq a_{mp}a_{um}[b, b']^2 \\ &\leq a_{um} . \end{aligned}$$

This implies $[b, b'] = 1$ and it contradicts $b \neq b'$. Hence we get $p \notin S$ and by definition of S there exists an integer t such that $\sigma(e_t, b) = (f_k, c_p)$. On the other hand $\sigma(e_t, b) = (f_t, c_t)$ holds. Hence we get $f_k = f_t$, $c_p = c_t$ and by (2)

$$a_{tp}[b, b'] = [c_t, c_m] ,$$

and by (1)

$$(5) \qquad [b, b'] = [f_p, f_k][c_p, c_m] .$$

From these follows

$$\begin{aligned}[c_p, c_m] &= [c_t, c_m] \\ &= a_{tp}[b, b'] \\ &= a_{tp}[f_p, f_k][c_p, c_m] \,.\end{aligned}$$

If $[c_p, c_m] \neq 0$, then $a_{tp}[f_p, f_k] = 1$ and this contradicts $f_p \neq f_k$. Hence we have $[c_p, c_m] = 0$ and $[b, b'] = 0$ by (5) and $[f_m, f_k] = 0$ by (3), and $a_{um} = [f_u, f_m][c_u, c_m] = [f_k, f_m][c_u, c_m] = 0$. This contradicts our assumption $a_{um} \neq 0$. Thus we have proved $a_{um} = 0$. Q.E.D.

THEOREM 1. *Let A, B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^n$ is connected and $\sigma: A \times B \cong A \times C$. Take any element $b \in B$ and put $\sigma(e_i, b) = (f_i, c_i)$. Then we* have

$$A \cong \{\sigma(e_i, b) \mid 1 \leq i \leq n\} = \{f_i \mid 1 \leq i \leq n\} \times \{c_i \mid 1 \leq i \leq n\} \,.$$

Proof. Put $C_i = \{c_k \mid k$ satisfies $f_k = f_i\}$ for $1 \leq i \leq n$, and denote by $\tilde{C}_i$ a connected component of C_i which contains c_i. Suppose $[e_i, e_j] = [f_i, f_j][c_i, c_j] \neq 0$. We will show $\tilde{C}_i = \tilde{C}_j$. Since $[e_i, e_j] \neq 0$, Lemma 1 implies that there exists an element $e_t \in A$ such that $\sigma(e_t, b) = (f_i, c_j)$. Hence we have $f_t = f_i$, $c_t = c_j$ since $\sigma(e_t, b) = (f_t, c_t)$. By definition of C_i we have $c_j = c_t \in C_i$ and hence $c_j \in \tilde{C}_i$ since $c_i \in \tilde{C}_i$ and $[c_j, c_i] \neq 0$. Thus we have proved $\tilde{C}_i \cap \tilde{C}_j \neq \phi$. Take any element $x \in \tilde{C}_i \cap \tilde{C}_j$; then there exists u such that $x = c_u$ and $f_u = f_i$ since $x \in C_i$. Take any $y \in \tilde{C}_j$ such that $[y, x] \neq 0$. Then y can be written $y = c_k$ with $f_k = f_j$. $[e_u, e_k] = [f_u, f_k][c_u, c_k] = [f_i, f_j][x, y] \neq 0$ yields that $\sigma(e_s, b) = (f_u, c_k)$ for some s. From $f_s = f_u = f_i$, $[c_s, c_u] = [c_k, c_u] \neq 0$ follows that $y = c_k = c_s \in \tilde{C}_i$ since $c_u = x \in \tilde{C}_i$, $c_s \in C_i$ and $\tilde{C}_i$ is a connected component of C_i. Thus we have shown that if $[x, y] \neq 0$ for $x \in \tilde{C}_i \cap \tilde{C}_j$, $y \in \tilde{C}_j$, then $y \in \tilde{C}_i$ holds. This implies $\tilde{C}_j \subset \tilde{C}_i$ and similarly $\tilde{C}_i \subset \tilde{C}_j$ and hence $\tilde{C}_i = \tilde{C}_j$ if $[e_i, e_j] \neq 0$. Since A is connected we get $\tilde{C}_1 = \cdots = \tilde{C}_n$. Take any s, t $(1 \leq s, t \leq n)$. From $c_t \in \tilde{C}_t = \tilde{C}_s \subset C_s$ follows that there exists i such that $c_t = c_i$ and $f_i = f_s$. Hence $(f_s, c_t) = (f_i, c_i) = \sigma(e_i, b)$ holds. Q.E.D.

§ 2.

Let L be an indecomposable positive lattice which satisfies the following condition (A′).

(A′) For any given positive lattices M, N and for any isometry σ from $L \otimes M$ on $L \otimes N$ which satisfies that $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0$, $n = 0$, there exists a finite subset $\{v_1, \cdots, v_m\}$ of L (depending on M, N, σ) such that

(1) each v_i is primitive in L and a submodule spanned by $\{v_1, \cdots, v_m\}$ of L is of finite index in L,

(2) putting $M_{v_i} = \{m \in M \mid \sigma(L \otimes m) \subset v_i \otimes N\}$,

$$N_{v_i} = \{n \in N \mid \sigma^{-1}(L \otimes n) \subset v_i \otimes M\} ,$$

we have $\operatorname{rank} M_{v_i} = \operatorname{rank} N_{v_i} = \operatorname{rank} M/\operatorname{rank} L$, and

(3) $\sigma(\boldsymbol{Q}(v_i \otimes M_{v_i})) = \boldsymbol{Q}(v_i \otimes N_{v_i})$.

Through this section the above L is fixed.

LEMMA 2. *Let $M, N, \sigma, v_i, M_{v_i}, N_{v_i}$ be those as in the condition* (A′). *Then M, N are isometric and they are divided by L, and $\sigma(L \otimes M_{v_i}) = v_i \otimes N$.*

Proof. By definition M_{v_i}, N_{v_i} are direct summands (as modules) of M, N respectively, and $\sigma(L \otimes M_{v_i}) \subset v_i \otimes N$, $\sigma^{-1}(L \otimes N_{v_i}) \subset v_i \otimes M$ and $\operatorname{rank} \sigma(L \otimes M_{v_i}) = \operatorname{rank}(v_i \otimes N)$, $\operatorname{rank} \sigma^{-1}(L \otimes N_{v_i}) = \operatorname{rank}(v_i \otimes M)$ imply $\sigma(L \otimes M_{v_i}) = v_i \otimes N$ and $\sigma^{-1}(L \otimes N_{v_i}) = v_i \otimes M$ since they are direct summands in $L \otimes N$, $L \otimes M$ respectively. This implies that M, N are divided by L. From (3) follows $\sigma(v_i \otimes M_{v_i}) = v_i \otimes N_{v_i}$ since they are direct summands of $L \otimes N$. Hence we can define an isometry $\mu_i : M_{v_i} \cong N_{v_i}$ by $\sigma(v_i \otimes m) = v_i \otimes \mu_i(m)$ for $m \in M_{v_i}$. For $m_i \in M_i$, $m_j \in M_j$ we show $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ where B stands for bilinear forms associated with quadratic modules.

$$\begin{aligned} B(v_i, v_j)B(m_i, m_j) &= B(v_i \otimes m_i, v_j \otimes m_j) \\ &= B(\sigma(v_i \otimes m_i), \sigma(v_j \otimes m_j)) \\ &= B(v_i \otimes \mu_i(m_i), v_j \otimes \mu_j(m_j)) \\ &= B(v_i, v_j)B(\mu_i(m_i), \mu_j(m_j)) . \end{aligned}$$

Hence we have $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ if $B(v_i, v_j) \neq 0$. Suppose $B(v_i, v_j) = 0$, then we have

$$\begin{aligned} B(L \otimes M_{v_i}, L \otimes M_{v_j}) &= B(\sigma(L \otimes M_{v_i}), \sigma(L \otimes M_{v_j})) \\ &= B(v_i \otimes N, v_j \otimes N) \\ &= 0 , \\ B(L \otimes \mu_i(M_{v_i}), L \otimes \mu_j(M_{v_j})) &= B(L \otimes N_{v_i}, L \otimes N_{v_j}) \\ &= B(\sigma^{-1}(L \otimes N_{v_i}), \sigma^{-1}(L \otimes N_{v_j})) \\ &= B(v_i \otimes M, v_j \otimes M) \\ &= 0 . \end{aligned}$$

Hence $B(M_{v_i}, M_{v_j}) = B(\mu_i(M_{v_i}), \mu_j(M_{v_j})) = 0$ follows. Thus we have proved $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ for $m_i \in M_{v_i}$, $m_j \in M_{v_j}$. By (1) we can choose a subset of $\{v_1, \cdots, v_m\}$, say $\{v_1, \cdots, v_n\}$, so that it is a basis of $\boldsymbol{Q}L$. Then $\sigma(L \otimes M_{v_i}) = v_i \otimes N$ implies that $\sum_{i=1}^n M_{v_i}$ is a direct sum and $[M: \sum_{i=1}^n M_{v_i}] < \infty$. Hence a linear mapping μ from $\boldsymbol{Q}M$ to $\boldsymbol{Q}N$ defined by $\mu(\sum_{i=1}^n m_i) = \sum_{i=1}^n \mu_i(m_i)$ $(m_i \in M_{v_i})$ becomes an isometry from $\boldsymbol{Q}M$ on $\boldsymbol{Q}N$. We have only to show $\mu(M) = N$. Take a basis $\{e_i\}$ of L and put

$$e_i = \sum_{j=1}^n a_{ij} v_j , \quad v_i = \sum_{j=1}^n b_{ij} e_j \quad (a_{ij}, b_{ij} \in \boldsymbol{Q}) .$$

$\sum_{k=1}^n b_{ik} a_{kj} = \delta_{ij}$ (Kronecker's delta) is obvious. Take any element $m = \sum_{i=1}^n m_i$ $(m_i \in \boldsymbol{Q}M_{v_i})$ of M and put $\sigma(v_j \otimes m_i) = v_i \otimes n_{ji}$ $(n_{ji} \in \boldsymbol{Q}N)$; then $n_{ii} = \mu(m_i)$ follows and

$$\begin{aligned}\sigma(e_k \otimes m) &= \sigma\left(\sum_{i,j} a_{kj} v_j \otimes m_i\right) \\ &= \sum_{i,j} a_{kj} v_i \otimes n_{ji} \\ &= \sum_t e_t \otimes \left(\sum_{i,j} a_{kj} b_{it} n_{ji}\right) .\end{aligned}$$

Since $\sigma(e_k \otimes m) \in L \otimes N$, we get $\sum_{i,j} a_{kj} b_{ik} n_{ji} \in N$. Summing up with respect to k, we have

$$\mu(m) = \sum \mu(m_i) = \sum n_{ii} \in N .$$

Thus $\mu(M) \subset N$ is proved. Since discriminants of M, N are equal, we have $\mu(M) = N$. Q.E.D.

LEMMA 3. *Let K, X, Y be positive lattices and assume that K is indecomposable and $\sigma: K \otimes X \cong K \otimes Y$. Then there exist submodules M_0, M of X and N_0, N of Y such that*

(i) *M_0, M, N_0, N are direct summands of X, Y respectively and $[X: M_0 \perp M]$, $[Y: N_0 \perp N] < \infty$, and*

$$\sigma(K \otimes M_0) = K \otimes N_0 , \qquad \sigma(K \otimes M) = K \otimes N ,$$

(ii) *if $\sigma(K \otimes m) = K \otimes n$ $(m \in M, n \in N)$, then $m = 0$ and $n = 0$, and*

(iii) *there exist orthogonal decompositions*

$$M_0 = \mathop{\perp}_{i=1}^{t} M_{0,i} , \qquad N_0 = \mathop{\perp}_{i=1}^{t} N_{0,i}$$

such that $\sigma(K \otimes M_{0,i}) = K \otimes N_{0,i}$ $(1 \leq i \leq t)$ and

$$\sigma|_{K \otimes M_{0,i}} = \alpha_i \otimes \beta_i$$

where $\alpha_i \in O(K)$, $\beta_i: M_{0,i} \cong N_{0,i}$.

Proof. Suppose that $m_1, \cdots, m_r$ are linearly independent elements of X so that there exist elements $n_i \in Y$ such that $\sigma(K \otimes m_i) = K \otimes n_i$. We may assume that r is maximal. Then we put $M = Z[m_1, \cdots, m_r]^{\perp}$, $N = Z[n_1, \cdots, n_r]^{\perp}$ and $M_0 = M^{\perp}$, $N_0 = N^{\perp}$. Clearly (i), (ii) are satisfied. (iii) follows from Lemma 1 in § 3 in [2]. Q.E.D.

LEMMA 4. *$L \otimes L$ is indecomposable and $O(L \otimes L)$ is generated by $O(L)$ and an interchange of L's.*

Proof. Take an isometry σ of $L \otimes L$. Suppose that there exist $x, y \in L$, $x \neq 0$ such that $\sigma(L \otimes x) = L \otimes y$. Supposing $K = X = Y = L$ in Lemma 3, a submodule of L corresponding to M of X in Lemma 3 is $\{0\}$ since its rank ($<$ rank L) is divided by rank L by Lemma 2. Hence we have $\sigma \in O(L) \otimes O(L)$ by Lemma 3. Suppose that there are no such elements x, y in L. Then, by Lemma 2, there is an element $v \in L$ such that

$$\sigma(L \otimes L_v) = v \otimes L ,$$

where $L_v = \{x \in L \mid \sigma(L \otimes x) \subset v \otimes L\}$. Since rank $L_v = 1$, there is an element u such that $L_v = Z[u]$. Then $\mu\sigma(L \otimes u) = L \otimes v$ holds where $\mu \in O(L \otimes L)$ is defined by $\mu(x \otimes y) = y \otimes x (x, y \in L)$. Hence $\mu\sigma \in O(L) \otimes O(L)$ follows as above. The indecomposability of $L \otimes L$ is proved quite similarly as in the proof of Lemma 4 in [4]. Q.E.D.

LEMMA 5. *$\otimes^m L$ is indecomposable provided that the orthogonal group $O(\otimes^m L)$ is generated by $O(L)$ and interchanges of L's and that $\otimes^{m-1} L$ is indecomposable.*

Proof. The proof is identical with that of Lemma 5 in [4].

THEOREM 2. *Let X be an indecomposable positive lattice. Then we have*

(i) *for any positive lattice Y, $L \otimes X \cong L \otimes Y$ implies $X \cong Y$,*

(ii) *if $X = \otimes^t L \otimes X'$ where X' is a positive lattice which is not divided by L, then $O(L \otimes X)$ is generated by $O(L)$, $O(X')$ and interchanges of L's,*

(iii) *$L \otimes X$ is indecomposable.*

Proof. We induct on rank X. In case of rank $X = 1$ our assertion is obvious. Suppose rank $X = k + 1$. Let Y be a positive lattice and $\sigma: L \otimes X \cong L \otimes Y$. Let M_0, M (resp. N_0, N) be submodules of X (resp. Y)

as in Lemma 3 for $K = L$. If $M = X$ (resp. $M_0 = X$), then $X \cong Y$ follows from Lemma 2 (resp. Lemma 3). Hence we may assume $M_0 \neq \{0\}$, $M \neq \{0\}$. Lemma 2 implies $M \cong N$. Hence we may assume $M = N = \perp K_i$ where K_i is indecomposable and suppose $K_i \cong \otimes^{r_i} L \otimes K_i'$ where K_i' is not divided by L. Since $\operatorname{rank} K_i \leq \operatorname{rank} M \leq k$, the inductive assumption implies that $L \otimes K_i$ is indecomposable and $O(L \otimes K_i)$ is generated by $O(L)$, $O(K_i')$ and interchanges of L's, identifying K_i and $\otimes^{r_i} L \otimes K_i'$. Hence, noting $L \otimes M \cong L \otimes N \cong \perp L \otimes K_i$, as in **2** in [4] for any basis $\{u_1, \cdots, u_n\}$ of L we have

$$\sigma(L \otimes M_{u_i}) = u_i \otimes N\,, \qquad \sigma^{-1}(L \otimes N_{u_i}) = u_i \otimes M\,,$$
$$\sigma(u_i \otimes M_{u_i}) = u_i \otimes N_{u_i}\,,$$

where

$$M_{u_i} = \{m \in M \mid \sigma(L \otimes m) \subset u_i \otimes N\}\,,$$
$$N_{u_i} = \{n \in N \mid \sigma^{-1}(L \otimes n) \subset u_i \otimes M\}\,.$$

Now $X = M_0 \perp M$, $Y = N_0 \perp N$ are proved quite similarly as in the proof of Theorem in § 1 in [3]. This is a contradiction since X is indecomposable. Thus (i) is proved. Let X be a positive lattice as in (ii). Assume that there exists an isometry $\sigma \in O(L \otimes X)$ which is not contained in a subgroup of $O(L \otimes X)$ generated by $O(L)$, $O(X')$ and interchanges of L's. Suppose that there exist $x, y \in X$ such that $\sigma(L \otimes x) = L \otimes y$. We define M_0, M, N_0, N as in Lemma 3 for $K = L$, $Y = X$. Then $X = M_0 \perp M$ holds as above. Since $M_0 \neq \{0\}$ and X is indecomposable, we have $X = M_0$ and Lemma 3 implies $\sigma \in O(L) \otimes O(X)$. If X is divided by L, that is, $t \geq 1$, then $O(X) = O(L \otimes (\otimes^{t-1} L \otimes X'))$ is generated by $O(L)$ and $O(X')$ and interchanges of L's since $\otimes^{t-1} L \otimes X'$ is indecomposable and $\operatorname{rank} \otimes^{t-1} L \otimes X' \leq k$. Thus σ is contained in a subgroup generated by $O(L)$, $O(X')$ and interchanges of L's in $O(L \otimes X)$. This is a contradiction. Therefore there exist no such elements x, y. Hence from Lemma 2 follows that $t \geq 1$ and there exists non-zero $v \in L$ such that $\sigma(L \otimes X_v) = v \otimes X$ where $X_v = \{x \in X \mid \sigma(L \otimes x) \subset v \otimes X\}$ by the assumption on L. Define $\mu_2 \in O(L \otimes X)$ by $\mu_2(x \otimes y \otimes z) = y \otimes x \otimes z$ $(x, y \in L,\ z \in \otimes^{t-1} L \otimes X')$; then $\mu_2\sigma(L \otimes X_v) = L \otimes v \otimes \otimes^{t-1} L \otimes X'$. If there exist $x \in X_v$, $y \in v \otimes \otimes^{t-1} L \otimes X'$ such that $x \neq 0$ and $\mu_2\sigma(L \otimes x) = L \otimes y$, then $\mu_2\sigma \in O(L \otimes X)$ must be contained in a subgroup generated by $O(L)$, $O(X')$ and interchanges of L's as above. This is also a contradiction. Repeating this operation we get, as in 1.6 in [4],

$$\mu_{t+1} \cdots \mu_2\sigma(L \otimes X_{v,\ldots,v'\ldots,}) = L \otimes v \otimes \cdots \otimes v'^{\cdots\prime} \otimes X' ,$$

where $\mu_j \in O(L \otimes X)$ is defined by

$$\begin{aligned} &\mu_j(x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_{t+1} \otimes y) \\ &\qquad = x_j \otimes \cdots \otimes x_1 \otimes \cdots \otimes x_{t+1} \otimes y \qquad (x_i \in L, y \in X') . \end{aligned}$$

If there exist $x \in X_{v,\ldots,v'\ldots,}$, $y \in v \otimes \cdots \otimes X'$ such that $x \neq 0$ and $\mu_{t+1} \cdots \mu_2\sigma(L \otimes x) = L \otimes y$, then $\mu_{t+1} \cdots \mu_2\sigma$ is contained in a subgroup generated by $O(L)$, $O(X')$ and interchanges of L's. This is a contradiction. Hence Lemma 2 yields that $v \otimes \cdots \otimes v'^{\cdots\prime} \otimes X'$ is divided by L. This contradicts the assumption on X'. Thus the proof of (ii) is completed. Let X be a positive lattice as in (ii). Then $O(L \otimes X) = O(\otimes^{t+1} L) \otimes O(X')$ has been proved as above. To complete the proof of (iii) we have only to show that $\otimes^{t+1} L$ is indecomposable by virtue of Lemma 3 in [4]. Since X is indecomposable, $\otimes^t L$ is also indecomposable. By virtue of (ii) $O(\otimes^{t+1} L)$ is generated by $O(L)$ and interchanges by L's. Hence Lemma 5 implies that $\otimes^{t+1} L$ is indecomposable. Q.E.D.

Remark. By (i), (iii) and Theorem in 105: 1 in [5] $L \otimes X \cong L \otimes Y$ implies $X \cong Y$ for any (not necessarily indecomposable) positive lattices X, Y.

§ 3.

Through this section we fix any positive lattice L of E-type such that $[L: \tilde{L}] < \infty$ and $\tilde{L}$ is indecomposable.

LEMMA 6. *Let M, N be positive lattices and assume $\sigma: L \otimes M \cong L \otimes N$. Then for each $m \in \mathfrak{M}(M)$ we have $\sigma(L \otimes m) = F \otimes G$, where F, G are submodules of L, N respectively and $m(F) = m(L)$, and $m(G) = m(N)$.*

Proof. Let X be a positive lattice. For $x, y \in \mathfrak{M}(X)/\pm$, we put $[x, y] = |B(x, y)|/m(X)$. Then $\mathfrak{M}(X)/\pm$ becomes a weighted graph and $\tilde{X}$ is indecomposable if and only if $\mathfrak{M}(X)/\pm$ is connected. Put $A = \mathfrak{M}(L)/\pm$, $B = \mathfrak{M}(M)/\pm$, $C = \mathfrak{M}(N)/\pm$. Since L is of E-type, we have $\mathfrak{M}(L \otimes M) = \mathfrak{M}(L) \otimes \mathfrak{M}(M)$, $\mathfrak{M}(L \otimes N) = \mathfrak{M}(L) \otimes \mathfrak{M}(N)$ and σ induces an isometry from $A \times B$ on $A \times C$. By Theorem 1 there exist subsets $F' \subset A$, $G' \subset C$ such that $\sigma(A, m) = (F', G')$. Denoting by F_0, G_0 submodules of L, N spanned by $F' \subset \mathfrak{M}(L)/\pm$, $G' \subset \mathfrak{M}(N)/\pm$ respectively, we have $\sigma(\tilde{L} \otimes m) = F_0 \otimes G_0$ and $m(F_0) = m(L)$, $m(G_0) = m(N)$. Put $F = \boldsymbol{Q}F_0 \cap L$, $G = \boldsymbol{Q}G_0 \cap N$; then

$[F : F_0]$, $[G : G_0] < \infty$, $m(F) = m(L)$, $m(G) = m(N)$ and $\sigma(L \otimes m)$, $F \otimes G$ are direct summands of $L \otimes N$. Hence $\sigma(L \otimes m) = F \otimes G$ follows. Q.E.D.

THEOREM 3. *If $L \cong L_1 \otimes L_2$ for positive lattices L_1, L_2, then L_1, L_2 are of E-type, $[L_1 : \tilde{L}_1]$, $[L_2 : \tilde{L}_2] < \infty$ and $\tilde{L}_1, \tilde{L}_2$ are indecomposable.*

Proof. Define $\sigma \in O(L_1 \otimes L_2 \otimes L_2)$ by $\sigma(x \otimes y \otimes z) = x \otimes z \otimes y$ $(x \in L_1, y, z \in L_2)$. For each $m \in \mathfrak{M}(L_2)$ $\sigma((L_1 \otimes L_2) \otimes m) = (L_1 \otimes m) \otimes L_2$ holds. Applying Lemma 6 in case of $M = N = L_2$, we have $m(L_1 \otimes m) = m(L)$. From Proposition 2 in [1] follows that $L_1 \otimes m$ is of E-type. Hence L_1 is of E-type. Similarly L_2 is of E-type. $\mathfrak{M}(L) = \mathfrak{M}(L_1) \otimes \mathfrak{M}(L_2)$ implies $[L_1 : \tilde{L}_1]$, $[L_2 : \tilde{L}_2] < \infty$ and $\tilde{L}_1, \tilde{L}_2$ are indecomposable since $[L : \tilde{L}] < \infty$ and $\tilde{L}$ is indecomposable. Q.E.D.

THEOREM 4. *Assume that L is indecomposable with respect to tensor product. Then L satisfies the condition* (A′) *in* § 2.

Proof. Suppose that L is decomposable and $L = L_1 \perp L_2$ $(L_1, L_2 \neq 0)$. Each $x \in \mathfrak{M}(L)$ is contained in L_1 or L_2. If $\mathfrak{M}(L) \cap L_1 = \phi$, then $\mathfrak{M}(L) \subset L_2$ and hence $\tilde{L} \subset L_2$ and $\operatorname{rank} L \leq \operatorname{rank} L_2$. This is a contradiction. Hence we have $\mathfrak{M}(L) \cap L_i \neq \phi$ $(i = 1, 2)$ and then $\tilde{L}$ spanned by $\mathfrak{M}(L)$ is decomposable. This contradicts our assumption on L. Thus L is indecomposable. Set $\mathfrak{M}(L) = \{\pm v_1, \cdots, \pm v_m\}$. We show that the condition (A′) is satisfied for the subset $\{v_1, \cdots, v_m\}$ of L by induction with respect to rank M. The first condition of (A′) follows from our assumption on L. Let M, N be positive lattices and suppose that for $\sigma : L \otimes M \cong L \otimes N$, $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0$, $n = 0$. Since L is of E-type, we have $\sigma(\mathfrak{M}(L) \otimes \mathfrak{M}(M)) = \mathfrak{M}(L) \otimes \mathfrak{M}(N)$ and hence $\sigma(\tilde{L} \otimes \tilde{M}) = \tilde{L} \otimes \tilde{N}$. Put $\tilde{M}^{\perp} = M'$, $\tilde{N}^{\perp} = N'$, $M'' = M'^{\perp}$ $(\neq \{0\})$, $N'' = N'^{\perp}$ $(\neq \{0\})$; then we have $[M : M' \perp M'']$, $[N : N' \perp N''] < \infty$, $\sigma(L \otimes M') = L \otimes N'$ and $\sigma(L \otimes M'') = L \otimes N''$ by virtue of $[L : \tilde{L}] < \infty$. Assume $M' \neq 0$; then the inductive assumption implies $\operatorname{rank} M'_{v_i} = \operatorname{rank} N'_{v_i} = \operatorname{rank} M' / \operatorname{rank} L$ and $\operatorname{rank} M''_{v_i} = \operatorname{rank} N''_{v_i} = \operatorname{rank} M'' / \operatorname{rank} L$, where

$$M'_{v_i} = \{m \in M' \mid \sigma(L \otimes m) \subset v_i \otimes N'\} ,$$

$$N'_{v_i} = \{n \in N' \mid \sigma^{-1}(L \otimes n) \subset v_i \otimes M'\} , \qquad \text{and}$$

M''_{v_i}, N''_{v_i} are defined similarly for M'', N''. Moreover M_{v_i}, N_{v_i} are defined similarly for M, N; then $M_{v_i} \supset M'_{v_i} \perp M''_{v_i}$ and $N_{v_i} \supset N'_{v_i} \perp N''_{v_i}$ are obvious. Hence $\operatorname{rank} M_{v_i} \geq \operatorname{rank} M / \operatorname{rank} L$ holds. Take any i $(1 \leq i \leq m)$ and a subset S of $\{v_1, \cdots, v_m\}$ such that S contains v_i and S is a basis of QL.

We may assume $i = 1$, $S = \{v_1, \cdots, v_n\}$ ($n = \operatorname{rank} L$). Then $\sigma(L \otimes M_{v_i}) \subset v_i \otimes N$ and $\sigma(L \otimes \sum_{i=1}^n M_{v_i}) \subset \sum_{i=1}^n v_i \otimes N$ imply that $\sum_{i=1}^n M_{v_i}$ is a direct sum. Thus we have $\operatorname{rank} M \geq \sum_{i=1}^n \operatorname{rank} M_{v_i} \geq \sum_{i=1}^n \operatorname{rank} M/\operatorname{rank} L = \operatorname{rank} M$ and hence $\operatorname{rank} M_{v_1} = \operatorname{rank} M/\operatorname{rank} L$. Hence $\operatorname{rank} M_{v_i} = \operatorname{rank} M/\operatorname{rank} L$ for each i and similarly $\operatorname{rank} N_{v_i} = \operatorname{rank} N/\operatorname{rank} L$ hold. From this follows that $\boldsymbol{Q}M_{v_i} = \boldsymbol{Q}M'_{v_i} \perp \boldsymbol{Q}M''_{v_i}$ and $\boldsymbol{Q}N_{v_i} = \boldsymbol{Q}N'_{v_i} \perp \boldsymbol{Q}N''_{v_i}$ and hence $\sigma(\boldsymbol{Q}(v_i \otimes M_{v_i})) = \sigma(\boldsymbol{Q}(v_i \otimes M'_{v_i}) \perp \boldsymbol{Q}(v_i \otimes M''_{v_i})) = \boldsymbol{Q}(v_i \otimes N'_{v_i}) \perp \boldsymbol{Q}(v_i \otimes N''_{v_i}) = \boldsymbol{Q}(v_i \otimes N_{v_i})$. Thus the condition (2), (3) are shown if $M' \neq 0$. Suppose $M' = 0$; then $[M: \tilde{M}]$, $[N: \tilde{N}] < \infty$ hold. For each $m \in \mathfrak{M}(M)$ Lemma 6 implies $\sigma(L \otimes m) = F \otimes G$ where F, G are submodules of L, N respectively and $m(F) = m(L)$. By the assumption on L we get $\operatorname{rank} F$ or $\operatorname{rank} G = 1$. $\operatorname{rank} G = 1$ implies $\sigma(L \otimes m) = L \otimes n$ for some $n \in N$ and it contradicts our assumption on σ. Hence we have $F = \boldsymbol{Z}[v]$ for $v \in \mathfrak{M}(L)$.

Thus for each $m \in \mathfrak{M}(M)$ there exists $v \in \mathfrak{M}(L)$ such that $\sigma(L \otimes m) \subset v \otimes N$.

Take any $v_i \in \mathfrak{M}(L)$ and fix it. For $n \in \mathfrak{M}(N)$ suppose $\sigma(v \otimes m) = v_i \otimes n$ for $v \in \mathfrak{M}(L)$, $m \in \mathfrak{M}(M)$. Since $\sigma(L \otimes m) \subset v_j \otimes N$ for $v_j \in \mathfrak{M}(L)$ as above, v_j must be equal to v_i and hence $\sigma(L \otimes m) \subset v_i \otimes N$, $m \in M_{v_i}$ and $m(M_{v_i}) = m(M)$. Therefore $v_i \otimes n = \sigma(v \otimes m) \in \sigma(L \otimes M_{v_i}) \subset v_i \otimes N$ holds for each $n \in \mathfrak{M}(N)$. Thus we get $v_i \otimes \tilde{N} \subset \sigma(L \otimes M_{v_i}) \subset v_i \otimes N$. From $[N: \tilde{N}] < \infty$ follows $\operatorname{rank} M_{v_i} = \operatorname{rank} N/\operatorname{rank} L$, $m(M_{v_i}) = m(M)$ and $[M_{v_i}: \tilde{M}_{v_i}] < \infty$. Similarly $\operatorname{rank} N_{v_i} = \operatorname{rank} M/\operatorname{rank} L$ follows.

For each $m \in \mathfrak{M}(M) \cap M_{v_i} = \mathfrak{M}(M_{v_i})$ we put $\sigma(v_i \otimes m) = v_i \otimes n$ ($n \in \mathfrak{M}(N)$). Then we have $\sigma^{-1}(L \otimes n) \subset v_i \otimes M$ by $\sigma^{-1}(v_i \otimes n) = v_i \otimes m$. Hence $n \in N_{v_i}$ follows. Conversely $n \in \mathfrak{M}(N) \cap N_{v_i} = \mathfrak{M}(N_{v_i})$ implies $\sigma^{-1}(v_i \otimes n) = v_i \otimes m$ for $m \in \mathfrak{M}(M)$ and $\sigma(L \otimes m) \subset v_i \otimes N$ by $\sigma(v_i \otimes m) = v_i \otimes n$. Hence we have $m \in M_{v_i}$ and $\sigma(v_i \otimes \mathfrak{M}(M_{v_i})) = v_i \otimes \mathfrak{M}(N_{v_i})$. $[M_{v_i}: \tilde{M}_{v_i}]$, $[N_{v_i}: \tilde{N}_{v_i}] < \infty$ yield $\sigma(\boldsymbol{Q}(v_i \otimes M_{v_i})) = \boldsymbol{Q}(v_i \otimes N_{v_i})$. This completes the proof of Theorem 4. Q.E.D.

Theorem 2, 3, 4 yield Theorem at the begining of this paper.

§4.

In this section we give examples of positive lattices in Theorem.

Proposition. *Let* $L = \boldsymbol{Z}[e_1, \cdots, e_n]$ *be a quadratic lattice and put* $ca_{ij} = B(e_i, e_j)$. *Assume that*

(0) $c, a_{ij} \in \boldsymbol{Q}$, $c > 0$,

(1) $a_{ii} = 1$ *and* $1 - \sum_{j \neq i} |a_{ij}| \geq 0$ *for* $i = 1, \cdots, n$,

(2) *for any non-empty subset S of* $\{1, 2, \cdots, n\}$

$$\#|S| - 1 \geq \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}| \,.$$

Then L is a positive lattice of E-type and $L = \tilde{L}$.

Proof. By scaling we may suppose $c = 1$ without loss of generality. Let M be a positive lattice with $m(M) = 1$. Take any non-zero element $x = \sum_{i=1}^{n} e_i \otimes u_i \in L \otimes M$. Put $b_{ij} = a_{ij}/|a_{ij}|$ if $a_{ij} \neq 0$, $= 0$ if $a_{ij} = 0$, and $S = \{i \,|\, u_i \neq 0\}$ $(\neq \phi)$. Then we have

$$\begin{aligned}
Q(x) &= \sum a_{ij} B(u_i, u_j) \\
&= \sum Q(u_i) + \frac{1}{2} \sum_{i \neq j} a_{ij}(2B(u_i, u_j)) \\
&= \sum Q(u_i) + \frac{1}{2} \sum_{i \neq j} |a_{ij}| (Q(b_{ij}u_i + u_j) - Q(u_i) - Q(u_j)) \\
&= \sum_{i \in S} \Big(1 - \sum_{\substack{j \in S \\ j \neq i}} |a_{ij}|\Big) Q(u_i) + \frac{1}{2} \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}| \, Q(b_{ij}u_i + u_j) \\
&\geq \#|S| - \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}| \\
&\geq 1 \,.
\end{aligned}$$

Hence L is positive and $m(L)m(M) \geq m(L \otimes M) \geq 1$. $m(L) \leq 1$, $m(\mathrm{M}) = 1$ imply $m(L \otimes M) = 1$ and $m(L) = 1$. If $Q(x) = 1$ and hence $x \in \mathfrak{M}(L \otimes M)$, then $b_{ij}u_i + u_j = 0$ and hence $u_i = \pm u_j$ for $i, j \in S$ with $i \neq j$, $a_{ij} \neq 0$. Suppose $S = S_1 \cup S_2$ and $a_{ij} = 0$ if $i \in S_1$, $j \in S_2$; then $x = (\sum_{i \in S_1} e_i \otimes u_i) + (\sum_{j \in S_2} e_j \otimes u_j)$ is an orthogonal sum and $x \in \mathfrak{M}(L \otimes M)$ implies that one of them must vanish. Thus we have S_1 or $S_2 = \phi$ and then $u_i = \pm u_j$ for $i, j \in S$. Therefore x should be $e \otimes u_i$ for $e \in L$, $i \in S$. By definition L becomes a lattice of E-type and $m(L) = 1$ implies $\mathfrak{M}(L) \supset \{e_i\}$ and hence $L = \tilde{L}$. Thus we complete the proof. Q.E.D.

Remark. If $a_{ii} = 1$, $|a_{ij}| < 1/n$ $(i \neq j)$, then the conditions (1), (2) are satisfied and $\mathfrak{M}(L) = \{\pm e_i \,|\, 1 \leq i \leq n\}$. In this case it is easy to see whether L is indecomposable or not. Suppose that L is indecomposable and $L = L_1 \otimes L_2$. Then from our theorem follows that L_1, L_2 are of E-type and $\mathfrak{M}(L) = \mathfrak{M}(L_1) \otimes \mathfrak{M}(L_2)$, $L_i = \tilde{L}_i$ and $|\mathfrak{M}(L_i)| = 2rkL_i$. Hence we can take minimal vectors as a basis of L_i, and then the matrix $(B(f_i, f_j))$ corresponding to L, where $\{\pm f_i\} = \{\pm e_i\}$, is a tensor product of matrices corresponding to L_i by their minimal vectors. Thus it is also easy to see whether L is indecomposable with respect to tensor product or not.

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