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DUALITY BETWEEN $D(X)$ AND $D(\hat{X})$ WITH ITS APPLICATION TO PICARD SHEAVES

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Introduction

As is well known, for a real vector space V, the Fourier transformation

$$\hat{f}(\alpha) = \int_V f(v) e^{2\pi i \langle v, \alpha \rangle} dv \qquad \alpha \in V^\vee$$

gives an isometry between $L^2(V)$ and $L^2(V^\vee)$, where $V^\vee$ is the dual vector space of V and $\langle\ ,\ \rangle\colon V \times V^\vee \to \boldsymbol{R}$ is the canonical pairing.

In this article, we shall show that an analogy holds for abelian varieties and sheaves of modules on them: Let X be an abelian variety, $\hat{X}$ its dual abelian variety and $\mathscr{P}$ the normalized Poincaré bundle on $X \times \hat{X}$. Define the functor $\hat{\mathscr{S}}$ of $\mathscr{O}_X$-modules M into the category of $\mathscr{O}_{\hat{X}}$-modules by

$$\hat{\mathscr{S}}(M) = \pi_{\hat{X},*}(\mathscr{P} \otimes \pi_X^* M) .$$

Then the derived functor $\boldsymbol{R}\hat{\mathscr{S}}$ of $\hat{\mathscr{S}}$ gives an equivalence of categories between two derived categories $\boldsymbol{D}(X)$ and $\boldsymbol{D}(\hat{X})$ (Theorem 2.2).

In § 3, we shall investigate the relations between our functor $\boldsymbol{R}\hat{\mathscr{S}}$ and other functors, translation, tensoring of line bundles, direct (inverse) image by an isogeny, etc. The result (3.14) that if X is principally polarized then $\boldsymbol{D}(X)$ has a natural action of $\mathrm{SL}(2, Z)$ seems to be significant.

In §§ 4 and 5, we shall apply the duality between $\boldsymbol{D}(X)$ and $\boldsymbol{D}(\hat{X})$ to the study of Picard sheaves. We shall compute the cohomology of Picard sheaves (Proposition 4.4), determine the moduli of deformations of them (Theorem 4.8) and give a characterization of them in the case of $\dim X = 2$ (Theorem 5.4). Other applications of the duality will be treated elsewhere.

After the original paper was written, the author learned by a letter from G. Kempf that Proposition 3.11 and some results in § 4 had also been proved independently by him.

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NOTATIONS. We denote by k a fixed algebraically closed field and mean by a *scheme* a scheme of finite type over k. For the product variety $X\times Y\times Z$, π_X (or p_1) and $\pi_{X,Y}$ (or p_{12}) are the projections of $X\times Y\times Z$ to X and $X\times Y$, respectively. For a coherent sheaf F on a variety X, $r(F)$ denotes the rank of F at the generic point of X. $F^\vee$ denotes $\mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

§1. Preliminary

Let X and Y be schemes and F an $\mathcal{O}_{X\times Y}$-module. We define the functor $\mathcal{S}_{X\to Y,F}$ from the category Mod (X) of $\mathcal{O}_X$-modules into Mod (Y) by

$$\mathcal{S}_{X\to Y,F}(?) = \pi_{Y,*}(F\otimes \pi_X^*?)\ , \tag{1.1}$$

where ? is an $\mathcal{O}_X$-module or an $\mathcal{O}_X$-homomorphism.

EXAMPLE 1.2. Let Γ_f be the graph of a morphism $f\colon X\to Y$ and F the structure sheaf $\mathcal{O}_{\Gamma_f}$ of Γ_f.
Then $\mathcal{S}_{X\to Y,F} = f_*$ and $\mathcal{S}_{Y\to X,F} = f^*$.

We denote by $\boldsymbol{D}(X)$ the derived category of Mod (X) and by $\boldsymbol{D}_{qc}(X)$ (resp. $\boldsymbol{D}_c(X)$) the full subcategory of $\boldsymbol{D}(X)$ consisting of the complexes whose i-th cohomologies are quasi-coherent (resp. coherent) for all i. $\boldsymbol{D}^-(X)$ (resp. $\boldsymbol{D}^b(X)$) is the full subcategory of $\boldsymbol{D}(X)$ consisting of the complexes bounded above (resp. bounded on both sides) and $\boldsymbol{D}^-_{qc}(X) = \boldsymbol{D}^-(X)\cap \boldsymbol{D}_{qc}(X)$, $\boldsymbol{D}^b_c(X) = \boldsymbol{D}^b(X)\cap \boldsymbol{D}_c(X)$, etc.

For an object F of $\boldsymbol{D}^-(X\times Y)$, we define the functor $\boldsymbol{R}\mathcal{S}_{X\to Y,F}$ from $\boldsymbol{D}^-(X)$ into $\boldsymbol{D}^-(Y)$ by

$$\boldsymbol{R}\mathcal{S}_{X\to Y,F}(?) = \boldsymbol{R}\pi_{Y,*}(F\overset{L}{\underset{=}{\otimes}} \pi_X^*?)\ . \tag{1.4}$$

If F is an $\mathcal{O}_X$-flat module, then $\boldsymbol{R}\mathcal{S}_{X\to Y,F}$ is the derived functor of $\mathcal{S}_{X\to Y,F}$. To consider the derived functors has the following advantage:

PROPOSITION 1.3. *Let Z be a scheme and G an object of $\boldsymbol{D}^-(X\times Y)$. Then there is a natural isomorphism of functors*:

$$\boldsymbol{R}\mathcal{S}_{Y\to Z,G}\circ \boldsymbol{R}\mathcal{S}_{X\to Y,F} \cong \boldsymbol{R}\mathcal{S}_{X\to Z,H}\ ,$$

where $H = \boldsymbol{R}\pi_{X,Z,*}(\pi_{X,Y}^*F\overset{L}{\underset{=}{\otimes}} \pi_{Y,Z}^*G)$.

Proof. We use (1) the commutativity of $\boldsymbol{R}$ and the composition of functors, (2) the projection formula and (3) the base change theorem. (See

[2] Proposition 5.1, 5.3, 5.6, 5.12)
Let ? be an object or morphism in $\boldsymbol{D}^-(X)$.

$$\begin{aligned}
&\boldsymbol{R}\mathscr{S}_{Y\to Z,G}(\boldsymbol{R}\mathscr{S}_{X\to Y,F}(?)) \\
&\cong \boldsymbol{R}\pi_{Z,*}(G \overset{L}{\underset{=}{\otimes}} \pi_Y^*(\boldsymbol{R}\pi_{Y,*}(F \overset{L}{\underset{=}{\otimes}} \pi_X^* ?))) \\
&\cong \boldsymbol{R}\pi_{Z,*}(G \overset{L}{\underset{=}{\otimes}} \boldsymbol{R}\pi_{Y,Z,*}(\pi_{X,Y}^*(F \overset{L}{\underset{=}{\otimes}} \pi_X^* ?))) && (3) \\
&\cong \boldsymbol{R}\pi_{Z,*}\boldsymbol{R}\pi_{Y,Z,*}(\pi_{Y,Z}^* G \overset{L}{\underset{=}{\otimes}} \pi_{X,Y}^* F \overset{L}{\underset{=}{\otimes}} \pi_X^* ?) && (2) \\
&\cong \boldsymbol{R}\pi_{Z,*}\boldsymbol{R}\pi_{X,Z,*}(\pi_{Y,Z}^* G \overset{L}{\underset{=}{\otimes}} \pi_{X,Y}^* F \overset{L}{\underset{=}{\otimes}} \pi_{X,Z}^* \pi_X^* ?) && (1) \\
&\cong \boldsymbol{R}\pi_{Z,*}(H \overset{L}{\underset{=}{\otimes}} \pi_X^* ?) = \boldsymbol{R}\mathscr{S}_{X\to Z,H}(?) && (2)
\end{aligned}$$

q.e.d.

PROPOSITION 1.4. (1) *If F has finite Tor-dimension as a complex of $\mathcal{O}_X$-modules, then we can extend the domain of definition of $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ to*

$$\boldsymbol{R}\mathscr{S}_{X\to Y,F}\colon \boldsymbol{D}(X) \longrightarrow \boldsymbol{D}(Y)$$

and $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}^b(X)$ into $\boldsymbol{D}^b(Y)$.

(2) *If F belongs to $\boldsymbol{D}_{qc}^-(X\times Y)$, then $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}_{qc}^-(X)$ into $\boldsymbol{D}_{qc}^-(Y)$.*

(3) *If X is proper and $F\in \boldsymbol{D}_c^-(X\times Y)$, then $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}_c^-(X)$ into $\boldsymbol{D}_c^-(Y)$.*

Proof. For (1), see [2] Proposition 4.2 and Corollary 4.3. (2) and (3) follow from [EGA] III 1.4.10 and 3.2.1, respectively. q.e.d.

§ 2. Fourier functor

Let X be an abelian variety of dimension g (the business is similar for a complex torus) and $\hat{X}$ its dual abelian variety. Let $\mathscr{P}$ be the normalized Poincaré bundle on $X\times\hat{X}$. Here "normalized" means that both $\mathscr{P}|_{X\times\hat{0}}$ and $\mathscr{P}|_{0\times\hat{X}}$ are trivial. For $\hat{x}\in\hat{X}$ (resp. $x\in X$), $P_{\hat{x}}$ (resp. P_x) denotes $\mathscr{P}|_{X\times\hat{x}}$ (resp. $\mathscr{P}|_{x\times\hat{X}}$). We put $\mathscr{S}=\mathscr{S}_{\hat{X}\to X,\mathscr{P}}$ and $\hat{\mathscr{S}}=\mathscr{S}_{X\to\hat{X},\mathscr{P}}$. Since $\hat{X}$ is complete and $\mathscr{P}$ is $\mathcal{O}_{\hat{X}}$-flat, we have by Proposition 1.4,

PROPOSITION 2.1. *The derived functor $\boldsymbol{R}\mathscr{S}\colon \boldsymbol{D}(\hat{X})\to\boldsymbol{D}(X)$ of $\mathscr{S}$ can be defined. It maps $\boldsymbol{D}^b(\hat{X})$, $\boldsymbol{D}_{qc}^-(\hat{X})$ and $\boldsymbol{D}_c^-(\hat{X})$ into $\boldsymbol{D}^b(X)$, $\boldsymbol{D}_{qc}^-(X)$ and $\boldsymbol{D}_c^-(X)$ respectively.*

The following theorem is fundamental:

THEOREM 2.2. *There are isomorphisms of functors:*

$$\boldsymbol{R}\mathscr{S}\circ\boldsymbol{R}\hat{\mathscr{S}}\cong(-1_X)^*[-g]$$

and

$$\boldsymbol{R}\hat{\mathscr{S}}\circ\boldsymbol{R}\mathscr{S}\cong(-1_{\hat{X}})^*[-g]\,,$$

where $[-g]$ *denotes "shift the complex g places to the right". In other words,* $\boldsymbol{R}\mathscr{S}$ *gives an equivalence of categories between* $\boldsymbol{D}(\hat{X})$ *and* $\boldsymbol{D}(X)$, *and its quasi-inverse is* $(-1_{\hat{X}})^*\circ\boldsymbol{R}\hat{\mathscr{S}}[g]$.

Proof. It suffices to show that $(\boldsymbol{R}\mathscr{S}|_{D^-(\hat{X})})\circ(\boldsymbol{R}\hat{\mathscr{S}}|_{D^-(X)})=(-1_X)^*[-g]$. By (1.3) the left side is isomorphic to $\boldsymbol{R}\mathscr{S}_{X\to X,H}$ with $H=\boldsymbol{R}p_{12,*}(p_{13}^*\mathscr{P}\otimes p_{23}^*\mathscr{P})$, where p_{ij} are projections of $X\times X\times\hat{X}$. Since $p_{13}^*\mathscr{P}\otimes p_{23}^*\mathscr{P}\cong(m\times 1)^*\mathscr{P}$ (which is easily verified by the seesaw principle), $H\cong\boldsymbol{R}p_{12,*}(m\times 1)^*\mathscr{P}\cong m^*\boldsymbol{R}p_{1,*}\mathscr{P}$. As was shown in the course of the proof of the theorem in [6] § 13, $R^ip_{1,*}\mathscr{P}=0$ for every $i\neq g$ and $R^gp_{1,*}\mathscr{P}\cong k(0)$, i.e., $\boldsymbol{R}p_{1,*}\mathscr{P}\cong k(0)[-g]$. Hence H is isomorphic to $\mathcal{O}_E[-g]$, where E is the graph of -1_X: $X\to X$. Therefore $\boldsymbol{R}\mathscr{S}_{X\to X,H}\cong(-1_X)^*[-g]$ (see Example 1.2). q.e.d.

In order to apply the theorem, we need

DEFINITION 2.3. We say that W.I.T. (weak index theorem) holds for a coherent sheaf F on X if $R^i\hat{\mathscr{S}}(F)=0$ for all but one i. This i is denoted by $i(F)$ and called the index of F. We denote the coherent sheaf $R^{i(F)}\hat{\mathscr{S}}(F)$ on $\hat{X}$ by $\hat{F}$ and call it the Fourier transform of F.

We say that I.T. (index theorem) holds for F if $H^i(X,F\otimes P)=0$ for all $P\in\operatorname{Pic}^\circ X$ and all but one i.

Since $(\mathscr{P}\otimes\pi_{\hat{X}}^*F)|_{X\times\hat{x}}\cong P_{\hat{x}}\otimes F$, we see by virtue of the base change theorem, that I.T. implies W.I.T. and $\hat{F}$ is locally free if I.T. holds for F. We always identify $\mathcal{O}_X$-module F with the complex consisting of F in degree 0, and 0 elsewhere. Hence if W.I.T. holds for F, then $\boldsymbol{R}\mathscr{S}(F)$ is isomorphic to $\hat{F}[-i(F)]$. Hence we have

COROLLARY 2.4. *If W.I.T. holds for F, then so does for* $\hat{F}$ *and* $i(\hat{F})=g-i(F)$. *Moreover* $\hat{\hat{F}}$ *is isomorphic to* $(-1_X)^*F$.

COROLLARY 2.5. *Assume that W.I.T. holds for F and G. Then* $\operatorname{Ext}^i_{\mathcal{O}_X}(F,G)\cong\operatorname{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{G})$ *for every integer i, where* $\mu=i(F)-i(G)$. *Especially, we have an isomorphism* $\operatorname{Ext}^i_{\mathcal{O}_X}(F,F)\simeq\operatorname{Ext}^i_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{F})$ *for every i.*

$$\begin{aligned} \textit{Proof.}\quad \mathrm{Ext}^i_{\mathcal{O}_X}(F, G) &\cong \mathrm{Hom}_{D(X)}(F, G[i]) \\ &\cong \mathrm{Hom}_{D(\hat{X})}(\boldsymbol{R}\mathcal{S}(F), \boldsymbol{R}\mathcal{S}(G)[i]) \\ &\cong \mathrm{Hom}_{D(\hat{X})}(\hat{F}[-i(F)], \hat{G}[i - i(G)]) \\ &\cong \mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{G}) \end{aligned}$$ q.e.d.

EXAMPLE 2.6. Let $k(\hat{x})$ be the one dimensional sky-scraper sheaf supported by $\hat{x} \in \hat{X}$. Since $H^i(X, k(\hat{x}) \otimes P) = 0$ for every $i > 0$ and $P \in \mathrm{Pic}^\circ \hat{X}$, I.T. holds for $k(\hat{x})$, $i(k(\hat{x})) = 0$ and $\widehat{k(\hat{x})} \simeq P_{\hat{x}}$. Hence by Corollary 2.4, W.I.T. holds for $P_{\hat{x}}$, $i(P_{\hat{x}}) = g$ and $\widehat{P_{\hat{x}}} \simeq k(-x)$. Note that I.T. does not hold for $P_{\hat{x}}$.

Combining the above with Corollary 2.5, we have

PROPOSITION 2.7. *Assume that W.I.T. holds for a coherent sheaf F on X. Then we have*

$$H^i(X, F \otimes P_{\hat{x}}) \cong \mathrm{Ext}^{g-i(F)+i}_{\mathcal{O}_{\hat{X}}}(k(\hat{x}), \hat{F})$$

and

$$\mathrm{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong H^{i-i(F)}(\hat{X}, \hat{F} \otimes P_{-x}) .$$

Proof. By Corollary 2.4, it suffices to show the first isomorphism. Since $P_{\hat{x}}$ is locally free, $H^i(X, F \otimes P_{\hat{x}})$ is isomorphic to $\mathrm{Ext}^i_{\mathcal{O}_X}(P_{-\hat{x}}, F)$. Hence by Corollary 2.5, it is isomorphic to $\mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{P}_{-\hat{x}}, \hat{F}) \cong \mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(k(\hat{x}), \hat{F})$, where $\mu = i(P_{-\hat{x}}) - i(F) = g - i(F)$. q.e.d.

COROLLARY 2.8. *The Euler-Poincaré characteristic of F is equal to $(-1)^{i(F)}r(F)$.*

$$\begin{aligned} \textit{Proof.}\quad \chi(X, F) &= \sum_i (-1)^i h^i(X, F) \\ &= \sum_i (-1)^i \dim \mathrm{Ext}^{i+g-i(F)}_{\mathcal{O}_X}(k(\hat{x}), \hat{F}) \\ &= (-1)^{i(F)} r(\hat{F}) . \end{aligned}$$ q.e.d.

EXAMPLE 2.9 ([4] § 4). A vector bundle U on X is said to be unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong \mathcal{O}_X$ for $i = 1, 2, \cdots, n$. Since the functor $R^i\mathcal{S}$ is semi-exact for all i, W.I.T. holds for U, $i(U) = g$ and the coherent sheaf $\hat{U}$ is supported by $\hat{0} \in \hat{X}$. Hence $R^g\mathcal{S}$ gives an equivalence of the categories

(Unipotent vector bundles on X) and (Coherent sheaves on $\hat{X}$ supported by $\hat{0}$) = (Artinian B-modules), where B is the local ring $\mathcal{O}_{\hat{X},\hat{0}}$ of $\hat{X}$ at $\hat{0}$. Moreover we have by Proposition 2.7.

$$H^i(X, U) \cong \mathrm{Ext}^i_B(k(\hat{0}), \hat{U}) .$$

§ 3. Relations between $\boldsymbol{R}\mathcal{S}$ and other functors

The properties of the Poincaré bundle $\mathcal{P}$ give relations between $\mathcal{S}$ and other functors. From this we obtain by the universal property of $\boldsymbol{R}\mathcal{S}$, relations between $\boldsymbol{R}\mathcal{S}$ and other functors. For example, from the isomorphism $T^*_{(0,\hat{x})}\mathcal{P} \cong \mathcal{P} \otimes \pi^*_X P_{\hat{x}}$, we obtain the isomorphism of functors $\mathcal{S} \circ T^*_{\hat{x}} \cong (\otimes P_{-\hat{x}}) \circ \mathcal{S}$ because $\mathcal{S}(T^*_{\hat{x}}?) = \pi_{X,*}(\mathcal{P} \otimes T^*_{(0,\hat{x})}\pi^*_{\hat{X}}?) \cong \pi_{X,*}T^*_{(0,\hat{x})}(T^*_{(0,-\hat{x})}\mathcal{P} \otimes \pi^*_{\hat{X}}?) \cong \pi_{X,*}(\mathcal{P} \otimes \pi^*_X P_{-\hat{x}} \otimes \pi^*_{\hat{X}}?) \cong \mathcal{S}(?) \otimes P_{-\hat{x}}$.
Hence we have

(3.1) (Exchange of translations and $\otimes$ Pic°)

$$\boldsymbol{R}\mathcal{S} \circ T^*_{\hat{x}} \cong (\otimes P_{-\hat{x}}) \circ \boldsymbol{R}\mathcal{S}$$
$$\boldsymbol{R}\mathcal{S} \circ (\otimes P_x) \cong T^*_x \circ \boldsymbol{R}\mathcal{S} .$$

EXAMPLE 3.2. W.I.T. holds for every homogeneous vector bundle H on X. The index $i(H)$ is equal to g and $\hat{H}$ is a coherent sheaf supported by a finite set of points. Hence $R^g\hat{\mathcal{S}}$ gives an equivalence of categories between $\boldsymbol{H}_X$ = (Homogeneous vector bundles on X) and $\boldsymbol{C}^f_{\hat{X}}$ = (Coherent sheaves on $\hat{X}$ supported by a finite set of points).

Proof. If a coherent sheaf M on $\hat{X}$ is supported by a finite set of points, then $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^\circ \hat{X}$ and hence $\mathcal{S}(M)$ is a homogeneous vector bundle by (3.1). Therefore it suffices to show the first statement. Put $M_i = R^i\hat{\mathcal{S}}(H)$. Since $T^*_x H \cong H$ for all $x \in X$, $M_i \otimes P \cong M_i$ for all $P \in \mathrm{Pic}^\circ \hat{X}$ by (3.1). Hence by the lemma (3.3) below M_i is supported by a finite set of points. By Theorem 2.2, there is a spectral sequence whose E_2 term is $R\mathcal{S}^j(M_i)$ and which converges to zero when $i + j \neq g$. Since $R\mathcal{S}^j(M_i) = 0$ if $j \neq 0$, the spectral sequence degenerates and M_i is zero for every $i \neq g$. q.e.d.

LEMMA 3.3. *Let M be a coherent sheaf on an abelian variety $\hat{X}$. If $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^\circ \hat{X}$, then* Supp M *is finite.*

Proof. Suppose that $\dim \mathrm{Supp}\, M \geq 1$. Take a curve C contained in Supp M and let $\tilde{C}$ be its normalization. Put $N = M \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{O}_{\tilde{C}}$ and $L = N/$ "the torsion part of N". Then N is a vector bundle on $\tilde{C}$ and $N \otimes f^*P$

$\cong N$ for all $P \in \operatorname{Pic}^\circ \hat{X}$, where f is the natural morphism $\tilde{C} \to C \subsetneq X$. Therefore, taking the determinant of both sides, we see that $(f^*P)^{\otimes r(N)}$ is trivial for all $P \in \operatorname{Pic}^\circ \hat{X}$. This is a contradiction because the morphism $f^*\colon \operatorname{Pic}^\circ \hat{X} \to \operatorname{Pic}^\circ \tilde{C}$ is not zero. q.e.d.

Combining Example 2.9 and 3.2, we have

THEOREM (Matsushima, Morimoto, Miyanishi, Mukai). *A vector bundle F on X is homogeneous if and only if F is isomorphic to $\bigoplus_{i=1}^n P_i \otimes U_i$ for some $P_1, \cdots, P_n \in \operatorname{Pic}^\circ X$ and unipotent vector bundles $U_1, \cdots, U_n$.*

Let Y be an abelian variety, $\varphi\colon Y \to X$ an isogeny and $\hat{\varphi}\colon \hat{X} \to \hat{Y}$ the dual isogeny of φ.

(3.4) (Exchange of the direct image and the inverse image)

$$\varphi^* \circ \boldsymbol{R}\mathscr{S}_X \cong \boldsymbol{R}\mathscr{S}_Y \circ \hat{\varphi}_*$$
$$\varphi_* \circ \boldsymbol{R}\mathscr{S}_Y \cong \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* .$$

Proof. The second isomorphism is obtained from the first in the following manner. Replacing φ by $\hat{\varphi}$ in the first isomorphism, we have $\hat{\varphi}^* \circ \boldsymbol{R}\hat{\mathscr{S}}_Y \cong \boldsymbol{R}\hat{\mathscr{S}}_X \circ \varphi_*$. By Theorem 2.2,

$$\begin{aligned}
\varphi_* \circ \boldsymbol{R}\mathscr{S}_Y &\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \boldsymbol{R}\hat{\mathscr{S}}_X \circ \varphi_* \circ \boldsymbol{R}\mathscr{S}_Y[g] \\
&\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \circ \boldsymbol{R}\hat{\mathscr{S}}_Y \circ \boldsymbol{R}\mathscr{S}_Y[g] \\
&\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \circ (-1_{\hat{Y}})^* \\
&\cong \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* .
\end{aligned}$$

Hence it suffices to show $\varphi^* \circ \mathscr{S}_X \cong \mathscr{S}_Y \circ \hat{\varphi}_*$. By the definition of $\hat{\varphi}$, $(\varphi \times 1)^*\mathscr{P}_X \cong (1 \times \hat{\varphi})^*\mathscr{P}_Y$. Hence we have

$$\begin{aligned}
\varphi^*\mathscr{S}_X(?) &= \varphi^*\pi_{X,*}(\mathscr{P}_X \otimes \pi_{\hat{X}}{}^*?) \\
&\cong \pi_{Y,*}((\varphi \times 1)^*\mathscr{P}_X \otimes \pi_{\hat{X}}{}^*?) \\
&\cong \pi_{Y,*}(1 \times \hat{\varphi})_*((1 \times \hat{\varphi})^*\mathscr{P}_Y \otimes \pi_{\hat{X}}^*?) \\
&\cong \pi_{Y,*}(\mathscr{P}_Y \otimes (1 \times \hat{\varphi})_*\pi_{\hat{X}}^*?) \\
&\cong \mathscr{S}_Y(\hat{\varphi}_*?) .
\end{aligned}$$

$$\begin{array}{ccccc}
 & & Y & \xrightarrow{\ \varphi\ } & X \\
 & & \uparrow{\scriptstyle \pi_X} & & \uparrow{\scriptstyle \pi_X} \\
\hat{X} & \xleftarrow{\ \pi_{\hat{X}}\ } & Y \times \hat{X} & \xrightarrow{\ \varphi \times 1\ } & X \times \hat{X} \\
{\scriptstyle \hat{\varphi}}\downarrow & & {\scriptstyle 1 \times \hat{\varphi}}\downarrow & & \downarrow \\
\hat{Y} & \xleftarrow[\ \pi_{\hat{Y}}\ ]{} & Y \times \hat{Y} & \longrightarrow & X \times \hat{Y}
\end{array}$$

q.e.d.

Remark 3.5. The second isomorphism can be also proved in the same way as the first by the isomorphism $(1 \times \hat{\varphi})_* \mathscr{P}_X \cong (\varphi \times 1)_* \mathscr{P}_Y$ which was proved in [7].

EXAMPLE 3.6. If H is a homogeneous vector bundle on X (resp. Y), so is $\varphi^* H$ (resp. $\varphi_* H$). Moreover the following diagram is (quasi-)commutative.

$$\begin{array}{ccc} \boldsymbol{C}^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}_X} & \boldsymbol{H}_X \\ {\scriptstyle \hat{\varphi}_*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \boldsymbol{C}^f_{\hat{Y}} & \xrightarrow[\mathscr{S}_Y]{\sim} & \boldsymbol{H}_Y \end{array} \qquad\qquad \begin{array}{ccc} \boldsymbol{C}^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}_X} & \boldsymbol{H}_X \\ {\scriptstyle \hat{\varphi}^*}\uparrow & & \uparrow{\scriptstyle \varphi_*} \\ \boldsymbol{C}^f_{\hat{Y}} & \xrightarrow[\mathscr{S}_Y]{\sim} & \boldsymbol{H}_Y \end{array}$$

Now we investigate other properties of the Fourier functor $\boldsymbol{R}\mathscr{S}$. Let $m \colon X \times X \to X$ be the group law of X. For $\mathcal{O}_X$-modules M and N, we define the Pontrjagin product $M * N$ of M and N by $M * N = m_*(p_1^* M \otimes p_2^* N)$. $*$ is a bifunctor from $\mathrm{Mod}(X) \times \mathrm{Mod}(X)$ into $\mathrm{Mod}(X)$. We denote its derived functor by $\overset{R}{\underline{*}}$.

(3.7) (Exchange of the Pontrjagin product and the tensor product)

$$\boldsymbol{R}\mathscr{S}\left(F \overset{R}{\underline{*}} ?\right) \cong \boldsymbol{R}\mathscr{S}(F) \overset{L}{\underline{\otimes}} \boldsymbol{R}\mathscr{S}(?)$$

$$\boldsymbol{R}\mathscr{S}\left(F \overset{L}{\underline{\otimes}} ?\right) \cong \boldsymbol{R}\mathscr{S}(F) \overset{R}{\underline{*}} \boldsymbol{R}\mathscr{S}(?)\ [g]$$

where $F \in \boldsymbol{D}(\hat{X})$ and ? is an object or a morphism in $\boldsymbol{D}(\hat{X})$.

Proof. It suffices to show the first isomorphism. We use the isomorphism $(1 \times m)^* \mathscr{P} \cong p_{12}^* \mathscr{P} \otimes p_{13}^* \mathscr{P}$, where p_{ij}'s are projections of $X \times \hat{X} \times \hat{X}$.

$$\begin{aligned} \boldsymbol{R}\mathscr{S}(F \overset{R}{\underline{*}} ?) &\cong \boldsymbol{R}\pi_{X,*}(\mathscr{P} \otimes \pi_{\hat{X}}^*(\boldsymbol{R}m_*(p_1^* F \otimes p_2^* ?))) \\ &\cong \boldsymbol{R}\pi_{X,*}(\mathscr{P} \otimes \boldsymbol{R}(1 \times m)_* p_{23}^*(p_1^* F \otimes p_2^* ?)) \\ &\cong \boldsymbol{R}\pi_{X,*}\boldsymbol{R}(1 \times m)_*((1 \times m)^* \mathscr{P} \otimes p_2^* F \otimes p_3^* ?) \\ &\cong \boldsymbol{R}p_{1,*}(p_{12}^* \mathscr{P} \otimes p_{13}^* \mathscr{P} \otimes p_2^* F \otimes p_3^* ?) \\ &\cong \boldsymbol{R}p_{1,*}(p_{12}^*(\mathscr{P} \otimes \pi_{\hat{X}}^* F) \otimes p_{13}^*(\mathscr{P} \otimes \pi_{\hat{X}}^* ?)) \\ &\cong \boldsymbol{R}\mathscr{S}(F) \overset{L}{\underline{\otimes}} \boldsymbol{R}\mathscr{S}(?) \end{aligned}$$

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times \hat{X} & \xleftarrow{1 \times m} & X \times \hat{X} \times \hat{X} \\ & & \downarrow{\scriptstyle \pi_{\hat{X}}} & & \downarrow{\scriptstyle p_{23}} \\ & & \hat{X} & \xleftarrow{m} & \hat{X} \times \hat{X} \end{array}$$

q.e.d.

Let Δ_X be the dualizing functor. Since the canonical module of X is trivial, $\Delta_X(?) = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(?, \mathcal{O}_X)\,[g]$.

(3.8) (Skew commutativity of $\boldsymbol{R}\mathcal{S}$ and Δ)

$$\Delta_X \circ \boldsymbol{R}\mathcal{S} \cong ((-1_X)^* \circ \boldsymbol{R}\mathcal{S} \circ \Delta_{\hat{X}})\,[g]\,.$$

Proof. We use the isomorphism $\mathcal{P}^{-1} \cong ((-1_X) \times 1_{\hat{X}})^*\mathcal{P}$ and the Grothendieck duality.

$$\begin{aligned}
\Delta_X(\boldsymbol{R}\mathcal{S}(?)) &= \Delta_X \boldsymbol{R}\pi_{X,*}(\mathcal{P} \otimes \pi_{\hat{X}}^*?) \\
&\cong \boldsymbol{R}\pi_{X,*}\Delta_{X\times\hat{X}}(\mathcal{P} \otimes \pi_{\hat{X}}{}^*?) \\
&\cong \boldsymbol{R}\pi_{X,*}(\mathcal{P}^{-1} \otimes \pi_{\hat{X}}^*\Delta_{\hat{X}}?)\,[g] \\
&\cong \boldsymbol{R}\pi_{X,*}(((-1_X) \times 1_{\hat{X}})^*\mathcal{P} \otimes \pi_{\hat{X}}^*\Delta_{\hat{X}}?)\,[g] \\
&\cong (-1_X)^*\boldsymbol{R}\mathcal{S}(\Delta_{\hat{X}}?)\,[g]
\end{aligned}$$

q.e.d.

EXAMPLE 3.9. Let U and V be unipotent vector bundles on X. As we saw in Example 2.9, $\hat{U}$ and $\hat{V}$ are artinian B-modules. $U \otimes V$ and $U^\vee$ are also unipotent vector bundles. $\widehat{U \otimes V}$ is isomorphic to $\hat{U} * \hat{V}$ and $\widehat{U^\vee}$ is isomorphic to $(-1_B)^*\Delta(\hat{U})$. $\hat{U} * \hat{V}$ is $\hat{U} \otimes_k \hat{V}$ regarded as a $\hat{B}$-modules via the co-multiplication μ: $\hat{B} \to \hat{B} \hat{\otimes} \hat{B}$ of the formal group $\hat{B}$. -1_B is an automorphism of B induced by $-1_{\hat{X}}$: $\hat{X} \to \hat{X}$ and Δ is the dualizing functor of Mod (B).

Next we investigate the relation between $\boldsymbol{R}\hat{\mathcal{S}}$ and $\otimes N$ for a line bundle N on X. In the rest of this section we always assume that N is nondegenerate, i.e., $\chi(N) \neq 0$. Hence ϕ_N ([6] p. 59, p. 131) is an isogeny.

(3.10) $\quad ?\overset{R}{\underset{=}{*}} N \cong (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^*)\,(?)$

where ? is an object or a morphism in $\boldsymbol{D}(X)$.

Proof. Consider the isomorphism ψ: $X \times X \to X \times X$ such that $\psi(x, y) = (x, x+y)$. The morphisms p_1, p_2 and m is sent by ψ to p_1, μ and p_2, respectively, where μ: $X \times X \to X$, $\mu(x, y) = y - x$. Hence $?*N = m_*(p_1^*? \otimes p_2^*N)$ is isomorphic to $p_{2,*}(p_1^*? \otimes \mu^*N)$. By the definition of the morphism ϕ_N: $X \to \hat{X}$, we have $m^*N \cong p_1^*N \otimes p_2^*N \otimes (1 \times \phi_N)^*\mathcal{P}$ and hence $\mu^*N \cong p_1^*(-1_X)^*N \otimes p_2^*N \otimes (-1_X \times \phi_N)^*\mathcal{P}$. Therefore the functor $?*N$ is isomorphic to $(\otimes N) \circ \mathcal{S}_{X\to X,\,(-1_X\times\phi_N)^*\mathcal{P}} \circ (\otimes (-1_X)^*N)$. By our assumption on N, ϕ_N is an isogeny, hence a flat morphism. Hence $\mathcal{S}_{X\to X,\,(-1_X\times\phi_N)^*\mathcal{P}} = \phi_N^* \circ \hat{\mathcal{S}} \circ (-1_X)^*$.

q.e.d.

Since I.T. holds for N ([6] § 16), $\hat{N}$ is a vector bundle on $\hat{X}$. $\hat{N}$ is simple, i.e., $\mathrm{End}_{\mathcal{O}_{\hat{X}}}(\hat{N}) \cong k$ by Corollary 2.5.

PROPOSITION 3.11. (1) $\phi_N^* \hat{N} \cong (N^{-1})^{\oplus|\chi(N)|}$

(2) $\hat{N}^{\oplus|\chi(N)|} \cong \phi_{N,*} N^{-1}$

(3) *If* $|\chi(N)| = 1$, *e.g., N is a principal polarization of X, then* $\hat{N} \cong (\phi_N^{-1})^* N^{-1}$.

(4) *There is an isogeny* $\pi: X \to Y$ *of degree* $|\chi(N)|$ *and a line bundle L on Y such that* $N \cong \pi^* L$. *Since* $\operatorname{Ker}(\pi) \subset K(N)$, *there is an isogeny* $\tau: Y \to \hat{X}$ *such that* $\tau \circ \pi = \phi_N$. *Then* $\hat{N}$ *is isomorphic to* $\tau_* L^{-1}$.

Proof. (1) is obtained from (3.10) by putting $? = \mathcal{O}_X$, because then the left side is $\mathcal{O}_X \overset{R}{\underset{=}{*}} N \cong \boldsymbol{R}p_{2,*}(p_1^* N) \cong \mathcal{O}_X \otimes_k H^i(X, N)\,[-i]$ and the right side is $N \otimes \phi_N^* \hat{N}\,[-i]$, where $i = i(F)$. Replacing N by N^{-1} in (1), we have $N^{\oplus|\chi(N)|} \cong (-\phi_N)^* \widehat{N^{-1}}$. Operating $\wedge$ on both sides, we have (2) because $\hat{N}^{\oplus|\chi(N)|} \cong (-\phi_N)_*(-1_X)^* N^{-1} \cong \phi_{N,*} N^{-1}$ by (3.4). Since $\deg \phi_N = |\chi(N)|^2$, ϕ_N is an isomorphism if $|\chi(N)| = 1$. Hence (3) is a special case of (1) or (2). For the first half of (4), see [6] § 23. It suffices to show the last statements. Since $|\chi(L)| = 1$, we have by (3), $\hat{N} \cong \widehat{\pi^* L} \cong \hat{\pi}_* \hat{L} \cong \hat{\pi}_* \phi_{L,*} L^{-1}$. On the other hand, since $N \cong \pi^* L$, we have $\phi_N = \hat{\pi} \circ \phi_L \circ \pi$. Since $\phi_N = \tau \circ \pi$ and π is an isogeny, we have $\tau = \hat{\pi} \circ \phi_L$. Hence $\hat{N} \cong \tau_* L^{-1}$. q.e.d.

(3.10) gives us an interesting relation between two functors $\boldsymbol{R}\mathcal{S}$ and $\otimes N$.

(3.12) $(\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\ [g + i(N)] \cong (\otimes \mathcal{O}_X^{\oplus|\chi(N)|}) \circ \phi_N^* \circ \phi_{N,*}$.

Especially, when the group scheme $K(N)$ is discrete, e.g., when $\chi(N)$ is prime to the characteristic exponent p of the ground field, then we have

(3.12′) $(\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\,[g + i(N)] \cong (\bigoplus_{x \in K(N)} T_x^*)^{\oplus|\chi(N)|}$.

Proof. First operate $\boldsymbol{R}\mathcal{S}$ on both sides of (3.10). By (3.7), we have

$$\boldsymbol{R}\mathcal{S}(?) \overset{L}{\underset{=}{\otimes}} \boldsymbol{R}\mathcal{S}(N) \cong (\boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^*)\,(?)$$

i.e.,

$$\otimes \hat{N} \circ \boldsymbol{R}\hat{\mathcal{S}}[-i(N)] \cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^*\ .$$

Operating $(-1_X)^* \circ \boldsymbol{R}\mathcal{S} \circ \phi_{N,*}$ from the right, we have

$$\begin{aligned} \otimes \hat{N} \circ \varphi_{N,*}[-g - i(N)] &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \boldsymbol{R}\mathcal{S} \circ \phi_{N,*} \\ &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \\ &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^2\ . \end{aligned}$$

Hence $\otimes N \circ \phi_N^* \circ \otimes \hat{N} \circ \phi_{N,*} \cong (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\,[g + i(N)]$. By (1) of Proposition 3.11, we have $\phi_N^*\hat{N} \cong (N^{-1})^{\oplus|\chi(N)|}$ and hence $\phi_N^* \circ \otimes \hat{N} \cong (\otimes\, \phi_N^*\hat{N}) \circ \phi_N^* \cong (\otimes\, (N^{-1})^{\oplus|\chi(N)|}) \circ \phi_{N,*}$, which proves our assertion. q.e.d.

In the case (X, L) is a principally polarized abelian variety, $\hat{X}$ is identified with X by the isomorphism $\phi_L\colon X \to \hat{X}$. Hence $\boldsymbol{R}\mathcal{S}$ is considered to be an automorphism of $\boldsymbol{D}(X)$. We summarize the results derived in this section for this case.

THEOREM 3.13. *Let (X, L) be a principally polarized abelian variety of dimension g. Then we have*

(1) $(\boldsymbol{R}\mathcal{S})^2 \cong (-1_X)^*[-g]$,

(2) $\boldsymbol{R}\mathcal{S} \circ \otimes P_x \cong T_x^* \circ \boldsymbol{R}\mathcal{S}$ *for* $x \in X$,

(3) $\boldsymbol{R}\mathcal{S} \circ \varphi \cong \hat{\varphi} \circ \boldsymbol{R}\mathcal{S}$ *for an isogeny* $\varphi\colon X \to X$.

(4) $\boldsymbol{R}\mathcal{S} \circ \Delta \cong ((-1_X)^* \circ \Delta \circ \boldsymbol{R}\mathcal{S})[g]$, *where Δ is the dualizing functor of* $\boldsymbol{D}(X)$,

(5) $\hat{L} \cong L^{-1}$ *and* $\widehat{L^{-1}} \cong (-1_X)^*L$,

(6) $(\otimes L \circ \boldsymbol{R}\mathcal{S})^3 \cong [-g]$.

(1) and (6) implies that the relation modulo the shift [ ] between two automorphisms $\boldsymbol{R}\mathcal{S}$ and $\otimes L$ is same as the relation between the generators $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of SL$(2, \boldsymbol{Z})$. In other words,

(3.14) if X is principally polarized, then SL$(2, \boldsymbol{Z})$ acts on $\boldsymbol{D}(X)$ modulo the shift.

Remark 3.15. The relation between automorphisms of $\boldsymbol{D}(X)$ and semi-homogeneous vector bundles on X will be discussed in [5]. Some applications of (3.14) to the vector bundles on an abelian surface will be treated in a forthcoming paper.

§ 4. Picard sheaves

In this section as an application of Fourier functor, we calculate the cohomology of Picard sheaves and determine the moduli of deformations of Picard sheaves.

Let C be a nonsingular complete curve of genus ≥ 2. We fix a point c of C and put $\xi_n = \mathcal{O}_C(n(c))$. We identify C with the subvariety $\{(x)-(c) \mid x \in C\}$ of the Jacobian variety $X = J(C)$ and also identify a sheaf on C with a sheaf on X supported by C. The subvariety $W_i = \overbrace{C + \cdots + C}^{i}$ of X is said to be the distinguished subvariety of dimension i, for $0 \leq i \leq g-1$.

W_{g-1} is a divisor of X and (X, L) is a principally polarized abelian variety of dimension g, where $L = \mathcal{O}_X(W_{g-1})$. We denote the canonical point of (X, L) by κ, that is, $\kappa - W_{g-1} = W_{g-1}$.

DEFINITION 4.1. The sheaf $F_n = R^1\mathcal{S}(\xi_n)$ is called a Picard sheaf of rank $g - n - 1$.

Our definition of F_n is same as that in [8], because a normalized Poincaré bundle $\mathcal{L}$ on $C \times X$ is isomorphic to $\mathcal{P}|_{C\times X}$. Replacing c by another point $c' \in C$, we get another Picard sheaf F'_n.

PROPOSITION 4.2. $F'_n \cong T^*_{n(c'-c)}F_n \otimes P_{c-c'}$

Proof.
$$\begin{aligned} F'_n &= R^1\mathcal{S}(T^*_{c'-c}\xi'_n) \\ &\cong R^1\mathcal{S}(\xi'_n) \otimes P_{c-c'} \\ &\cong R^1\mathcal{S}(\xi_n \otimes P_{nc'-nc}) \otimes P_{c-c'} \\ &\cong T^*_{n(c'-c)}F_n \otimes P_{c-c'}\,. \end{aligned}$$
q.e.d.

We summarize some fundamental properties of F_n.

THEOREM 4.2 (See [8].)

(1) *F_n is zero for $n > 2g-2$.* Supp F_n *is $\kappa - W_{2g-2-n}$ for $g - 1 \leqq n \leqq 2g-2$.* Supp F_n *is X and the rank of F_n at the generic point of X is $g - n - 1$ for $n < g - 1$. F_n is locally free for $n < 0$.*

(2) *The i-th Chern class $c_i(F_n)$ is rationally equivalent to W_{g-i} for $i \leqq g - 1$. Especially,* $\det F_n \cong L$ *for $n \leq g - 1$.*

(3) *The projective fibre space $\boldsymbol{P}(\alpha^*F_n)$ associated with α^*F_n is isomorphic to the $(2g-2-n)$-th symmetric product* $\mathrm{Sym}^{2g-2-n}(C)$. *Where α is the automorphism of X for which $\alpha(x) = \kappa - x$.*

By the following proposition, we can apply the theory of Fourier functor to Picard sheaves.

PROPOSITION 4.3. (1) *For $n \leq g - 1$, F_n is $\hat{\xi}_n$, W.I.T. holds for F_n, $i(F_n) = g - 1$ and $\hat{F}_n \cong (-1_X)^*\xi_n$.*

(2) *For $n \geq g - 1$, F_n is isomorphic to $\alpha^* \mathcal{E}xt^1_{\mathcal{O}_X}(F_{2g-2-n}, \mathcal{O}_X)$ and $\mathcal{S}(\xi_n) \cong \alpha^* \mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n}, \mathcal{O}_X)$.*

(3) *$\mathcal{E}xt^i_{\mathcal{O}_X}(F_n, \mathcal{O}_X)$ is zero for $i \geq 2$, $n \geq g - 1$.*

Proof. Since $\dim \mathrm{Supp}\, \xi_n = 1$, $R^i\mathcal{S}(\xi_n)$ is zero for $i > 1$. On the other hand, $\mathcal{S}(\xi_n)$ is zero for $n < g$ ([8] § 3). Hence, when $n < g$, W.I.T. holds for

ξ_n and $i(\xi_n)=1$. Therefore (1) follows from Corollary 2.4. Since $\Delta(\mathcal{O}_C)$ is isomorphic to $K_C[1]\cong\xi_{2g-2}\otimes P_\kappa[1]$, ξ_n is isomorphic to $\Delta(\xi_{2g-2-n}\otimes P_\kappa)[-1]$. Hence, by (3.8), we have

$$\begin{aligned}\boldsymbol{R}\mathcal{S}(\xi_n) &\cong \boldsymbol{R}\mathcal{S}(\Delta(\xi_{2g-2-n}\otimes P_\kappa)[-1]) \\ &\cong ((-1_X)^*\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}\otimes P_\kappa))[-g-1] \\ &\cong (-1_X)^*T_\kappa^*(\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}))[-g-1] \\ &\cong \alpha^*(\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}))[-g-1]\ .\end{aligned}$$

When $n\geq g-1$, $\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n})$ is isomorphic to $F_n[-1]$ by (1). Hence we have

$$\begin{aligned}\boldsymbol{R}\mathcal{S}(\xi_n) &\cong \alpha^*(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n}[-1],\mathcal{O}_X)[g])[-g-1] \\ &\cong \alpha^*\boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n},\mathcal{O}_X)\ .\end{aligned}$$

Therefore, $R^i\mathcal{S}(\xi_n)$ is isomorphic to $\alpha^*\,\mathcal{E}xt^i_{\mathcal{O}_X}(F_{2g-2-n},\mathcal{O}_X)$, which shows (2) and (3). q.e.d.

Applying the result in § 3 and § 4, we have the following three propositions.

PROPOSITION 4.4 (Cohomology of Picard sheaf). *Assume that* $n\leq g-1$.

(1) $h^g(X,F_n\otimes P_x)=0$ *for all* $x\in X$. *When* $0\leq i\leq g-1$, *we have*

$$h^i(X,\ F_n\otimes P_X)=\begin{cases}\dbinom{g-1}{i} & \text{if } -x\in C\\ 0 & \text{if } -x\notin C\end{cases}$$

(2) $h^i(X,F_n\otimes L^{-1}\otimes P_x)=h^{i-g+1}(C,\xi_{n+g}\otimes P_{\kappa+x})$ *for all* $x\in X$

(3) $h^i(X,F_n\otimes L\otimes P_x)=\begin{cases}2g-n-1 & \text{for } i=0\\ 0 & \text{for } i>0\end{cases}$.

Proof. By Proposition 2.7, $H^i(X,F_n\otimes P_x)$ is isomorphic to $\mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k(x),(-1_X)^*\xi_n)$, which shows (1). By Corollary 2.5 and (5) of Theorem 3.13, $H^i(X,F_n\otimes L^{-1}\otimes P_x)\cong\mathrm{Ext}^i_{\mathcal{O}_X}(L\otimes P_{-x},F_n)$ is isomorphic to $\mathrm{Ext}^{i-g+1}_{\mathcal{O}_X}(\widehat{L\otimes P_{-x}},\hat{F}_n)\cong\mathrm{Ext}^{i-g+1}_{\mathcal{O}_X}(L^{-1}\otimes P_x,(-1_X)^*\xi_n)$. Since $L|_C\cong\xi_g$, we have $H^i(X,F_n\otimes L^{-1}\otimes P_x)\cong H^{i-g+1}(X,L\otimes P_{-x}\otimes(-1_X)^*\xi_n)\cong H^{i-g+1}(C,\xi_n\otimes(-1_X)^*L|_C\otimes P_x)\cong H^{i-g+1}(C,\xi_{n+g}\otimes P_{\kappa+x})$, which shows (2). In a similar manner, we have $H^i(X,F_n\otimes L\otimes P_x)\cong\mathrm{Ext}^{i+1}_{\mathcal{O}_X}((-1_x)^*(L\otimes P_x),(-1_X)^*\xi_n)\cong H^{i+1}(C,\xi_{n-g}\otimes P_{-x})$. Since $\deg\xi_{n-g}=n-g<0$, we have by Riemann-Roch theorem, $h^0(C,\xi_{n-g}\otimes P_{-x})=0$ and $h^1(C,\xi_{n-g}\otimes P_{-x})=2g-n-1$. Hence we have proved (3). q.e.d.

PROPOSITION 4.5 (Local property of Picard sheaf).

$$\mathrm{Tor}_i^{\mathcal{O}_X}(F_n, k(x)) \cong \begin{cases} H^1(C, \xi_n \otimes P_x) & i = 0 \\ H^0(C, \xi_n \otimes P_x) & i = 1 \\ \mathrm{Tor}_{i-2}^{\mathcal{O}_X}(\mathcal{S}(\xi_n), k(x)) & i \geq 2 \end{cases}$$

Proof. Assume that $n \leq g-1$. Then we have by Proposition 2.7, $\mathrm{Ext}^i_{\mathcal{O}_X}(k(x), F_n) \cong H^{i-g+1}(X, (-1_X)^*\xi_n \otimes P_{-x})$. Hence by the duality theorem, $\mathrm{Tor}^i_{\mathcal{O}_X}(F_n, k(x))$ is isomorphic to $\mathrm{Ext}^{g-i}_{\mathcal{O}_X}(k(x), F_n) \cong H^{1-i}(C, \xi_n \otimes P_x)$, which proves our assertion for $n \leq g-1$ because $\mathcal{S}(\xi_n)$ is zero for $n \leq g-1$. By what we have shown, the minimal resolution of $F_n \otimes \mathcal{O}_{X,x}$ is

$$0 \longleftarrow F_n \otimes \mathcal{O}_{X,x} \longleftarrow \mathcal{O}_{X,x} \underset{k}{\otimes} H^1(C, \xi_n \otimes P_x) \longleftarrow \mathcal{O}_{X,x} \otimes H^0(C, \xi_n \otimes P_x) \longleftarrow 0.$$

By (2) of Proposition 4.3, the sequence

(4.6) $0 \leftarrow F_{2g-2-n} \otimes \mathcal{O}_{X,\alpha(x)} \leftarrow \mathcal{O}_{X,\alpha(x)} \otimes H^0(C, \xi_n \otimes P_x)^\vee \leftarrow \mathcal{O}_{X,\alpha(x)} \otimes H^1(C, \xi_n \otimes P_x)^\vee \leftarrow \mathcal{S}(\xi_{2g-2-n}) \otimes \mathcal{O}_{X,\alpha(x)} \leftarrow 0$ is exact.

It is easy to see that the left three terms of (4.6) is the minimal resolution of $F_{2g-2-n} \otimes \mathcal{O}_{X,\alpha(x)}$. Hence $\mathrm{Tor}_i^{\mathcal{O}_X}(F_{2g-2-n}, k(\alpha(x)))$ is isomorphic to $H^i(C, \xi_n \otimes P_x)^\vee \cong H^{1-i}(C, K_C \otimes \xi_{-n} \otimes P_{-x}) \cong H^{1-i}(C, \xi_{2g-2-n} \otimes P_{\alpha(x)})$ for $i = 0, 1$ and isomorphic to $\mathrm{Tor}_{i-2}^{\mathcal{O}_X}(\mathcal{S}(\xi_{2g-2-n}), k(\alpha(x)))$. Hence our assertion has been proved for $n \geq g-1$, too. q.e.d.

PROPOSITION 4.7. *Assume that $n \leq g-1$. Then I.T. holds for $F_n \otimes L$, its index is zero and $\widehat{F_n \otimes L} \cong \alpha^* F_{n-g} \otimes L^{-1}$.*

Proof. The first half has been proved in (3) of Proposition 4.4. By (6) of Theorem 3.13, we have $(\otimes L \circ \boldsymbol{R}\mathcal{S} \circ \otimes L)(\widehat{F_n \otimes L}) = (\otimes L \circ \boldsymbol{R}\mathcal{S})^3(\xi_n)[1] \cong \xi_n[1-g]$. Hence $\widehat{F_n \otimes L}$ is isomorphic to $(\otimes L^{-1} \circ \boldsymbol{R}\mathcal{S}^{-1} \circ \otimes L^{-1})(\xi_n)[1-g] \cong ((-1_X)^* \boldsymbol{R}\mathcal{S}(\xi_n \otimes L^{-1}) \circ \otimes L^{-1})[1] \cong ((-1_X)^* \boldsymbol{R}\mathcal{S}(\xi_{n-g} \otimes P_\kappa) \otimes L^{-1})[-1] \cong \alpha^* F_{n-g} \otimes L^{-1}$. q.e.d.

Next we consider the moduli of deformations of Picard sheaves. Define the functor $\mathcal{S}pl_X$ from the category of schemes (of finite type over k) into the category of sets by

$\mathcal{S}pl_X(T) = \{E \mid E$ is a T-flat coherent $\mathcal{O}_{X \times T}$-module and $E_t = E|_{X \times t}$ is simple for every $t \in T\}/_\sim$,

for every scheme T, where $E \sim E'$ if and only if $E \cong E' \otimes_{\mathcal{O}_T} L$ for some line bundle L on T, and $\mathcal{S}pl_X(f)\colon \mathcal{S}pl_X(T') \to \mathcal{S}pl_X(T)$ is the usual pull back for every morphism $f\colon T \to T'$. For every simple coherent sheaf F

on X, $\mathscr{S}pl_X^F$ denotes the connected component of $\mathscr{S}pl_X$ containing F. The following is the main theorem in this section.

THEOREM 4.8. *Assume that* $n \leq g-1$ *and* $(*)$ $g(C)=2$ *or* C *is not hyperelliptic. Then* $\mathscr{S}pl_X^{F_n}$ *is represented by* $X \times X$ *and the coherent sheaf* $\tilde{F}_n = p_{12}^* m^* F_n \otimes p_{13}^* \mathscr{P}$ *on* $X \times (X \times X)$.

Let $A_F\colon X \times \hat{X} \to \mathscr{S}pl_X^F$ be the morphism of functors such that $A_F(f, g) = T_f^* F_T \otimes P_g$ for every scheme T and T-valued point (f, g) of $X \times \hat{X}$, where we always identify a scheme S and the contravariant functor h_S on the category of schemes for which $h_S(T)$ is the set of T-valued points of S, i.e., morphisms from T to S. Theorem 4.8 says that A_F is an isomorphisms for $F=F_n$ $(n \leq g-1)$ under the assumption $(*)$. The following three lemmas are essential for the proof of the theorem.

LEMMA 4.9. *Picard sheaf* F_n $(n \leq g-1)$ *is simple and we have*

$$\begin{aligned} \dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(F_n, F_n) &= 3g-2 && \textit{if } C \textit{ is hyperelliptic} \\ &= 2g && \textit{otherwise}\,. \end{aligned}$$

Proof. By Corollary 2.5 and Proposition 4.3, it suffices to show the equality for $\dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n)$. Since there is a spectral sequence

$$H^i(X, \mathscr{E}xt^j_{\mathcal{O}_X}(\xi_n, \xi_n)) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{O}_X}(\xi_n, \xi_n)\,,$$

we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n)) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n) \longrightarrow H^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n)) \\ &\longrightarrow H^2(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n)) \longrightarrow 0\,. \end{aligned}$$

Since $\mathscr{E}nd_{\mathcal{O}_X}(\xi_n)$ is isomorphic to $\mathcal{O}_C$, $H^2(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n))$ is zero and we have

$$\begin{aligned} \dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n) &= h^1(C, \mathcal{O}_C) + h^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n)) \\ &= g + h^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n))\,. \end{aligned}$$

SUBLEMMA. Let ξ be a line bundle on a subscheme C of X. Then there is a canonical isomorphism $\varphi\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \overset{\sim}{\rightarrow} \mathscr{E}xt^i_{\mathcal{O}_X}(\xi, \xi)$ for every i.

Since $\mathscr{E}xt$ commutes with localizations, it suffices to give the canonical isomorphism in the case X is affine and $\xi \cong \mathcal{O}_C$. Let $f\colon \mathcal{O}_C \overset{\sim}{\rightarrow} \xi$ be an isomorphism. Since $\mathscr{E}xt^i_{\mathcal{O}_X}(*, *)$ is a bifunctor, we have two isomorphisms

$$\begin{aligned} f_a &= \mathscr{E}xt^i_{\mathcal{O}_X}(\mathrm{id}, f)\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \overset{\sim}{\rightarrow} \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \xi) \\ f_b &= \mathscr{E}xt^i_{\mathcal{O}_X}(f, \mathrm{id})\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\xi, \xi) \overset{\sim}{\rightarrow} \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \xi)\,. \end{aligned}$$

Put $\varphi = f_b^{-1} \circ f_a$. If $g\colon \mathcal{O}_C \rightrightarrows \xi$ is another isomorphism, then there is a unit $\bar{u}$ of $\mathcal{O}_C$ such that $g = f \circ (\times \bar{u})$. There is an affine neighbourhood Y of C and a unit u of $\mathcal{O}_Y$ whose image by the natural homomorphism $\mathcal{O}_Y \to \mathcal{O}_C$ is $\bar{u}$. Since $(g_b^{-1} \circ g_a)|_Y = ((\times u) \circ f_b|_Y)^{-1} \circ (f_a|_Y \circ (\times u)) = \varphi|_Y$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \otimes \mathcal{O}_{X,x}$ is zero for every $x \notin Y$, φ does not depend on the choice of the isomorphism f. This proves the sublemma.

By this sublemma, we have only to compute the dimension of

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C)) \cong H^0(C, N_{C/X}) .$$

There is a natural exact sequence

$$0 \longrightarrow (N_{C/X})^\vee \longrightarrow \Omega_X \otimes \mathcal{O}_C \longrightarrow K_C \longrightarrow 0 .$$

Since Ω_X is trivial, tensoring K_C, we have the exact sequence

$$0 \longrightarrow (N_{C/X})^\vee \otimes K_C \longrightarrow K_C^{\oplus g} \longrightarrow K_C^{\otimes 2} \longrightarrow 0 .$$

In the long exact sequence

$$\begin{aligned} H^0(K_C)^{\oplus g} \xrightarrow{\alpha} H^0(K_C^{\otimes 2}) &\longrightarrow H^1((N_{C/X})^\vee \otimes K_C) \\ &\longrightarrow H^1(K_C)^{\oplus g} \longrightarrow H^1(K_C^{\otimes 2}) \longrightarrow 0 , \end{aligned}$$

the map α is just the natural map $H^0(K_C) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2})$. By Riemann-Roch theorem, we have $h^0(N_{C/X}) = h^1((N_{C/X})^\vee \otimes K_C) = \dim \operatorname{Coker} \alpha + gh^1(K_C) - h^1(K_C^{\otimes 2}) = \dim \operatorname{Coker} \alpha + g$. In the case C is hyperelliptic, $\dim \operatorname{Coker} \alpha$ is $g - 2$ and otherwise α is surjective by a theorem due to Noether, [3] p. 502, which completes our proof. q.e.d.

LEMMA 4.10. *If $n \leq g - 1$ and $T_x^* F_n \otimes P_y \cong T_{x'}^* F_n \otimes P_{y'}$ for x, x', y, $y' \in X$,* then $x = x'$ and $y = y'$.

Proof. The assumption implies that $P_x \otimes T_{-y}^* \xi_n \cong P_{x'} \otimes T_{-y'}^* \xi_n$ by (3.1). Since $\operatorname{Supp} \xi_n = C$, y equals to y' and since $\operatorname{Pic}^\circ X \to \operatorname{Pic}^\circ C$ is injective, x is equal to x'. q.e.d.

We denote the tangential map of A_F at $(0, \hat{0})$ by α_F. Since the tangent spaces of X at 0, of $\hat{X}$ at $\hat{0}$ and of $\mathcal{S}pl_X$ at F are identified with $H^0(X, T_X)$, $H^1(X, \mathcal{O}_X)$ and $\operatorname{Ext}^1_{\mathcal{O}_X}(F, F)$, respectively, α_F is a k-linear map from $H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X)$ into $\operatorname{Ext}^1_{\mathcal{O}_X}(F, F)$.

LEMMA 4.11. *α_{F_n} is injective for the Picard sheaf F_n ($n \leq g - 1$).*

Assume that W.I.T. holds for F. By (3.1), we have $\widehat{T_x^* F \otimes P_y} \cong T_y^* \hat{F}$

$\otimes P_{-x}$. This is easily extended to scheme valued points and we have $T_f^* \widehat{F_S \otimes P_g} \cong T_g^* \hat{F}_S \otimes P_{-f}$ for every scheme S and S-valued point (f, g) of $X \times \hat{X}$. As a special case $S = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$, we have

PROPOSITION 4.12. *Assume that W.I.T. holds for a coherent sheaf F on X. Then the diagram*

$$\begin{array}{ccc} \alpha_F\colon H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) \\ \downarrow\wr\, j & & \downarrow\wr\, R\mathcal{S} \\ \alpha_{\hat{F}}\colon H^1(X, \mathcal{O}_X) \oplus H^0(X, T_X) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{F}) \\ \|\wr & & \\ H^0(\hat{X}, T_{\hat{X}}) \oplus H^1(\hat{X}, \mathcal{O}_{\hat{X}}) & & \end{array}$$

is commutative, where $j(a, b) = (b, -a)$.

By this proposition, the injectivity of α_{F_n} is equivalent to that of α_{ξ_n}. Let

$$0 \longrightarrow H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F)) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) \xrightarrow{\ \varepsilon\ } H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(F, F))$$

be the exact sequence obtained from the local-global spectral sequence with respect to Ext. The following proposition is easily verified.

PROPOSITION 4.13. (1) *$\alpha_F(H^1(X, \mathcal{O}_X))$ is contained in $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F))$.*

(2) *The diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & T_{X\times\hat{X},(0,\hat{0})} & \longrightarrow & H^0(X, T_X) & \longrightarrow & 0 \\ & & \downarrow \beta_F & & \downarrow \alpha_F & & \downarrow \varepsilon\circ\gamma_F & & \\ 0 & \longrightarrow & H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F)) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) & \xrightarrow{\ \varepsilon\ } & H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(F, F)) & & \end{array}$$

is commutative, where β_F and γ_F are the restrictions of α_F to $H^1(X, \mathcal{O}_X)$ and $H^0(X, T_X)$, respectively.

(3) *β_F is equal to $H^1(i)$, where i is the natural homomorphism from $\mathcal{O}_X$ into $\mathcal{E}nd_{\mathcal{O}_X}(F)$.*

(4) *$H^0(X, T_X)$ is the set of derivations of $\mathcal{O}_X$. For $D \in \operatorname{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$, $\gamma_F(D)$ is the extension class of*

$$0 \longrightarrow F \longrightarrow F_D \longrightarrow F \longrightarrow 0\,,$$

where F_D is $F \oplus F$ as a sheaf of abelian groups and regarded as an $\mathcal{O}_X$-module by $a(m, m') = (am + D(a)m', am')$ for every $a \in \mathcal{O}_X$ and $(m, m') \in F \oplus F$.

(5) *$\varepsilon \circ \gamma_F$ is equal to $H^0(\tilde{\gamma}_F)$, where $\tilde{\gamma}_F$ is an $\mathcal{O}_X$-homomorphism from T_X into $\mathcal{E}xt^1_{\mathcal{O}_X}(F, F)$ such that $\tilde{\gamma}_F|_U$ is equal to γ_{F_U}: $\mathrm{Der}_k(\mathcal{O}_U, \mathcal{O}_U) \to \mathrm{Ext}^1_{\mathcal{O}_U}(F_U, F_U)$ for every affine open subset U of X.*

(6) *If Y is a subscheme of X and F is a line bundle on Y, then $\tilde{\gamma}_F$ is the composition of the natural morphisms $T_X \to T_X \otimes \mathcal{O}_Y$ and $T_X \otimes \mathcal{O}_Y \to N_{Y/X} \cong \mathcal{E}xt^1_{\mathcal{O}_X}(F, F)$.*

In the case F is ξ_n, a line bundle on C, $\beta_F = H^1[\mathcal{O}_X \to \mathcal{O}_C]$ is an isomorphism, $H^0[T_X \to T_X \otimes \mathcal{O}_C]$ is also an isomorphism and by the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_X \otimes \mathcal{O}_C \longrightarrow N_{C/X} \longrightarrow 0\,,$$

$H^0[T_X \otimes \mathcal{O}_C \to N_{C/X}]$ is injective. Hence by (6) of Proposition 4.13, $\varepsilon \circ \gamma_F = H^0(\tilde{\gamma}_F)$ is an injection. Therefore by the diagram (2) of the proposition, α_F is an injection. This completes the proof of Lemma 4.11.

For the proof of Theorem 4.8, we need the following general facts about the flat deformation of a simple coherent sheaf.

(4.14) (Relative representability of $\mathcal{S}pl$) Let f: $V \to S$ be a proper integral morphism and F and G coherent $\mathcal{O}_V$-modules. Assume that F is S-flat and $F \otimes k(s)$ is simple for every $s \in S$. Then there exists a subscheme W of S such that for every morphism α: $T \to S$, F_T is isomorphic to $G_T \otimes_{\mathcal{O}_T} L$ with some line bundle L on T if and only if α factors through the inclusion $W \hookrightarrow S$. We call W the maximal subscheme over which F and G are isomorphic to each other.

(4.15) (Pro-representability of $\mathcal{S}pl$) Let F be a simple coherent $\mathcal{O}_X$-module. The functor $\mathcal{D}$ on artinian local rings A over k such that

$\mathcal{D}(A) = \{E | E$ is an A-flat coherent $\mathcal{O}_{X_A}$-module such that $E \otimes_A A/m$ is isomorphic to $F\}/\mathrm{isom}$.

is representable by a complete local ring R whose Zariski tangent space t_R is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F)$. We call R the local moduli of F.

(4.16) (Jumping never happens) Let E be an element of $\mathcal{S}pl_X(T)$. If $E|_{X \times t} \cong F$ for every closed point t of an open dence subset U of T, then $E|_{X \times t} \cong F$ for every $t \in T$.

The proofs are not so difficult and those of (4.14) and (4.15) are similar to the case of simple vector bundles. The stronger fact that the étale sheafification of $\mathcal{S}pl_{V/S}$ is representable by an algebraic space has been proved in [1]. Since the fact does not make our business so easy, we prove our theorem directly by (4.14), (4.15) and (4.16).

Step I. The functor A_{F_n} is injective.

Let f and g be two morphism from T to $X \times X$ such that $A_{F_n} \circ f = A_{F_n} \circ g$. Since $X \times X$ is a group scheme and A_{F_n} is an $X \times X$-morphism with respect to the natural action of $X \times X$ to $\mathcal{S}pl_X^{F_n}$, we may assume that g is the constant map to $(0, 0)$. Let $\Phi(F_n)$ be the maximal subscheme of $X \times X$ over which $\tilde{F}_n$ and $p_1^* F_n$ on $X \times (X \times X)$ are isomorphic to each other. Since $A_{F_n} \circ f$ is the constant map to F_n by our assumption, f factors the inclusion $\Phi(F_n) \subsetneq X \times X$. By Lemma 4.10, $\Phi(F_n)$ is supported by the origin $(0, 0)$ and by Lemma 4.11, the tangent space of $\Phi(F_n)$ is zero. Hence $\Phi(F_n)$ is $(0, 0)$ and f is zero. (It is easily seen that $\Phi(F_n)$ is a group subscheme of $X \times X$. Hence Lemma 4.11 is not necessary for the proof of our assertion in the case $\operatorname{char} k = 0$.)

Step II. A_{F_n} is an open immersion.

A_{F_n} induces the homomorphism $f\colon R \to Q$ of complete local rings, where $(R, \mathfrak{m})$ is the local moduli of F_n and $(Q, \mathfrak{n})$ is the completion of $\mathcal{O}_{X \times X, (0,0)}$. Since A_{F_n} is injective, the fibre $Q/\mathfrak{m}Q$ of f is isomorphic to $Q/\mathfrak{n}$. Hence f is a surjection. By Lemma 4.9, we have

$$2g = \dim Q \leq \dim R \leq \dim t_R = 2g \, .$$

Hence $\dim R = 2g$, R is regular and f is a bijection. For every morphism $g\colon T \to \mathcal{S}pl_X^{F_n}$, by virtue of (4.14), $T \times_{\mathcal{S}pl_X^{F_n}} (X \times X)$ is representable by a scheme U. By what we have shown, $\hat{\mathcal{O}}_{T, h(u)} \to \hat{\mathcal{O}}_{U, u}$ is an isomorphism for every $u \in U$.

$$\begin{array}{ccc} X \times X & \xrightarrow{A_{F_n}} & \mathcal{S}pl_X^{F_n} \\ \uparrow{\scriptstyle \ell} & & {\scriptstyle g}\uparrow \\ U & \xrightarrow[h]{} & T \end{array} \qquad \text{cartesian}$$

Hence h is étale. By Step I, h is an open immersion.

Step III. A_{F_n} is a closed immersion.

In the above situation, we have to show that U is a union of connected components of T. Hence we may assume that T is irreducible and it suffices to prove that the set of k-rational points $U(k)$ of U is empty or equal to $T(k)$. Hence we may also assume that T is reduced. Assume that $U(k) \neq \phi$. Since $X \times X$ is an abelian variety, every rational map from T to $X \times X$ is a morphism. Hence there is a morphism $e = (e_1, e_2)\colon T \to X \times X$ whose restriction to U is equal to ℓ. Let $\mu\colon X \times X \times \mathcal{S}pl_X^{F_n} \to \mathcal{S}pl_X^{F_n}$ be the natural action of $X \times X$ on $\mathcal{S}pl_X^{F_n}$. Put $c = [T \xrightarrow{(-e,g)} X \times X \times \mathcal{S}pl_X^{F_n} \xrightarrow{\mu} \mathcal{S}pl_X^{F_n}]$. Then $c(U(k)) = \{F_n\}$ and hence by virtue of (4.16), we have $c(T(k)) = \{F_n\}$, that is, $g(a) = T^*_{e_1(a)}F_n \otimes P_{e_2(a)}$ for every $a \in T(k)$. Hence $U(k)$ is equal to $T(k)$.

Step IV. A_{F_n} is an isomorphism.

It suffices to show that $A_{F_n}(k)\colon (X \times X)(k) \to \mathcal{S}pl_X^{F_n}(k)$ is a surjection. By the definition, $\mathcal{S}pl_X^{F_n}$ is connected. Hence, for every $F \in \mathcal{S}pl_X^{F_n}(k)$, there exist a connected scheme T and a morphism $g\colon T \to \mathcal{S}pl_X^{F_n}$ such that $g(T(k))$ contains both F and F_n. By what we have shown in Step II and Step III, g factor through A_{F_n}. Hence F is contained in $\operatorname{Im} A_{F_n}(k)$.

We have completed the proof of Theorem 4.8.

Remark 4.17. Even if the condition (*) does not hold, $A_{F_n}(k)$ is bijective for $n \leq g-1$. But if C is hyperelliptic and $g(C) \geq 3$, then the dimension of the tangent space of $\mathcal{S}pl_X^{F_n}$ is greater than $2g$, hence $\mathcal{S}pl_X^{F_n}$ is not reduced.

§ 5. A characterization of Picard sheaf

In this section we give a characterization of the Picard sheaf in the case $g(C) = 2$.

Let ξ_n be the same as in the beginning of § 4. There is a natural exact sequence

$$0 \longrightarrow \xi_{n-1} \longrightarrow \xi_n \longrightarrow k(0) \longrightarrow 0\,.$$

This gives the exact sequence

$$0 \longrightarrow \mathcal{S}(\xi_{n-1}) \longrightarrow \mathcal{S}(\xi_n) \longrightarrow \mathcal{O}_X \xrightarrow{f} F_{n-1} \longrightarrow F_n \longrightarrow 0\,.$$

If $n \leq g-1$, then $\mathcal{S}(\xi_n)$ is zero ([8] § 3). Hence, for $n \leq g-1$, we have the exact sequence

$$(5.1) \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{f} F_{n-1} \longrightarrow F_n \longrightarrow 0\,.$$

By (1) of Proposition 4.4, both $\dim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F_n) = h^0(F_n)$ and $\dim \mathrm{Ext}^1_{\mathcal{O}_X}(F_n, \mathcal{O}_X) = h^{g-1}(F_n)$ is equal to 1 for $n \leq g-1$. Hence we have

LEMMA 5.2. *Assume that $n \leq g-1$. Then f is the unique* (*up to constant multiplications*) *nonzero homomorphism from $\mathcal{O}_X$ into F_n and* (5.1) *is the unique nontrivial extension of F_n by $\mathcal{O}_X$.*

We denote the set $\{T_x^* F_n \otimes P_y \mid x, y \in X\}$ by Φ_n.
The above lemma is generalized for members of $\mathrm{Pic}^\circ X$ and Φ_n.

PROPOSITION 5.3. *Assume that $n \leq g-1$.*

(1) *Every nonzero homomorphism f from $P_x \in \mathrm{Pic}^\circ X$ to $F \in \Phi_{n-1}$ is injective and* Coker f *is isomorphic to a member of Φ_n.*

(2) *If $P_x \in \mathrm{Pic}^\circ X$, $F \in \Phi_n$ and the exact sequence*

$$0 \longrightarrow P_x \longrightarrow F' \longrightarrow F \longrightarrow 0$$

does not split, then F' is isomorphic to a member of Φ_{n-1}.

Proof. We prove only (2), because (1) can be proved in a quite similar manner. First we may assume that $F = F_n$. Since $\mathrm{Ext}^1_{\mathcal{O}_X}(F_n, P_x) \neq 0$, we have by (1) of Proposition 4.4, that x belongs to C and $\dim \mathrm{Ext}^1_{\mathcal{O}_X}(F_n, P_x)$ is equal to 1. Since $x \in C$, there is a surjection $\xi_n \to k(x)$ and we have the non-splitting exact sequence

$$0 \longrightarrow \xi_{n-1} \otimes P_x \longrightarrow \xi_n \longrightarrow k(x) \longrightarrow 0 .$$

Operating $\boldsymbol{R}\mathcal{S}$, we have the exact sequence

$$0 \longrightarrow P_x \longrightarrow T_x^* F_{n-1} \longrightarrow F_n \longrightarrow 0 .$$

Since this does not split, F' is isomorphic to $T_x^* F_{n-1}$. q.e.d.

For every nontorsion coherent sheaf F on X, let $\mu(F)$ denote the rational number $r(F)^{-1} \deg(\det F)|_C$. Umemura has showed that F_n is μ-stable for $n \leq g-1$ in the case $g(C) = 2$ ([9]). The following theorem says that the converse is also true.

THEOREM 5.4. *Assume that $g(C) = 2$ and F is a torsion free coherent sheaf with $r(F) = r \geqq 1$, $\det F$ algebraically equivalent to $\mathcal{O}_X(C)$ and $\chi(F)$ zero. Then the following conditions are equivalent to one another*:

1) *F is μ-stable, i.e., $\mu(E) < \mu(F)$ for every $E \subsetneqq F$* with *$r(E) < r$.*

1′) *F is μ-semi-stable, i.e., $\mu(E) \leq \mu(F)$ for every $E \subseteq F$.*

2) $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ *is zero for every* $P \in \mathrm{Pic}^\circ X$. *If H is a homogeneous vector bundle with $r(H) < r$ contained in F, then the quotient F/H is torsion free.*

3) $F \cong T_x^* F_{1-r} \otimes P$ *for some* $x \in X$ *and* $P \in \mathrm{Pic}^\circ X$.

Proof. Obviously 1) implies 1′). Assume that F is μ-semi-stable and H is a homogeneous vector bundle with $r(F) < r$ contained in F. Since $\mu(F) = 2/r$ is greater than $\mu(P)=0$, $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is zero for every $P \in \mathrm{Pic}^\circ X$. Let $f\colon F \to F/H$ be the projection and T the torsion part of F/H. Then $H' = f^{-1}(T)$ contains H and $r(H')$ is equal to $r(H)$. We have a nonzero homomorphism $\det H \to \det H'$. Hence $\det H' \cong \det H \otimes \mathcal{O}_X(D)$ for some divisor $D \geq 0$. Since $\det H \in \mathrm{Pic}^\circ X$ and F is μ-semi-stable, we have

$$\frac{(\mathcal{O}_X(D).\,\mathcal{O}_X(C))}{r(H)} = \frac{(\det H'.\,\mathcal{O}_X(C))}{r(H')} \leq \mu(F) = \frac{2}{r}\,.$$

Since $D \geq 0$, $(\mathcal{O}_X(D).\,\mathcal{O}_X(C))$ is not less than zero and different from one ([9] Lemma 3.5). Hence by the inequality above $(\mathcal{O}_X(D).\,\mathcal{O}_X(C))$ is zero. Hence $D = 0$ and $\det H \to \det H'$ is an isomorphism. Since H is locally free, H' is isomorphic to H. Therefore T is zero. Hence 1′) implies 2). 3) implies 1), because if F is μ-stable, so is $T_x^* F \otimes P$ for every $x \in X$ and $P \in \mathrm{Pic}\, X$. Hence we have only to show that 2) implies 3). We prove it by induction on r.

Case $r = 1$. $\mathrm{Sym}^2 C \to X$ is the blowing up whose center is the canonical point κ. Hence, by (3) of Proposition 4.2, F_0 is isomorphic to $N \otimes \mathfrak{m}_{X,0}$ with some line bundle N, where $\mathfrak{m}_{X,0}$ is the maximal ideal of $\mathcal{O}_X$ at 0. By (2) of Proposition 4.2, N is isomorphic to $\mathcal{O}_X(C)$. Since $r(F) = 1$ and F is torsion free, F is contained in $\det F$. By the assumption, $\det F \cong \mathcal{O}_X(C) \otimes P$ for some $P \in \mathrm{Pic}^\circ X$. Since length $(\det F/F) = \chi(\det F) - \chi(F) - 1$, $\det F/F$ is isomorphic to the one dimensional sky-scraper sheaf $k(x)$ supported by a point $x \in X$. Hence F is isomorphic to $\det F \otimes \mathfrak{m}_{X,x} \cong T_x^* F_0 \otimes P \otimes P_{-x}$.

Case $r \geq 2$. We need the following easy but useful lemma.

LEMMA 5.5. *Let F be a nonzero coherent sheaf on an abelian surface. If $\chi(F)$ is zero, then* $\mathrm{Hom}_{\mathcal{O}_X}(P, F)$ *or* $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ *is not zero for some* $P \in \mathrm{Pic}^\circ X$.

Assume the contrary. Since $\dim \mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is equal to $h^2(F \otimes P^{-1})$

by virtue of the duality theorem and since $\chi(F \otimes P^{-1})$ is zero, $h^1(F \otimes P^{-1})$ is zero for all $P \in \mathrm{Pic}^\circ X$. Hence $R^i\mathscr{S}(F)$ is zero for every i. This means that $\boldsymbol{R}\hat{\mathscr{S}}(F)$ is zero. Therefore by virtue of Theorem 2.2, F is zero. This shows Lemma 5.5.

By the assumption and the above lemma, $\mathrm{Hom}_{\mathcal{O}_X}(P, F)$ is not zero for some $P \in \mathrm{Pic}^\circ X$. Let $f: P \to F$ be a nonzero homomorphism. Since F is torsion free, f is injective. Since P is homogeneous, $F' = \mathrm{Coker}\, f$ is torsion free. We have the exact sequence

$$0 \longrightarrow P \xrightarrow{f} F \xrightarrow{g} F' \longrightarrow 0 .$$

Since $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is zero, this exact sequence does not split. Hence by (2) of Proposition 5.3, it suffices to show $F' \cong T_x^* F_{2-r} \otimes Q$ for some $x \in X$ and $Q \in \mathrm{Pic}^\circ X$. F' is torsion free, $\det F' = \det F \otimes P^{-1}$ is algebraically equivalent to $\mathcal{O}_X(C)$ and $\chi(F') = \chi(F) - \chi(P)$ is equal to zero. By induction hypothesis, we have only to show that 2) holds for F'. Obviously $\mathrm{Hom}_{\mathcal{O}_X}(F', Q)$ is zero for every $Q \in \mathrm{Pic}^\circ X$. Let H' be a homogeneous vector bundle contained in F'. $H = g^{-1}(H')$ is an extension of H' by P. Hence by the theorem after Lemma 3.3, H is also homogeneous. By the assumption on F, $F'/H' \cong F/H$ is torsion free. Hence 2) holds for F', which completes the proof of Theorem 5.4. q.e.d.

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