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RINGS OF CONVERGENT POWER SERIES AND WEIERSTRASS PREPARATION THEOREM

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§0.

Let B be a B-ring with a nonarchimedean valuation $|\ |$, i.e., B is an integral domain satisfying the following conditions: (i) B is bounded ($|a| \leqq 1$ for every $a \in B$), (ii) the boundary $\partial(B) = \{a \in B; |a| = 1\}$ forms a multiplicative group. Let Z_+ denote the set of all nonnegative integers. Let $n \in Z_+$. Let $x_1, \cdots, x_n$ be n variables over B. We denote by $A_n = B\langle x_1, \cdots, x_n\rangle$ the set of all elements which can be written in the form

$$\sum_\nu a_\nu x^\nu ,$$

where $a_\nu \in B$ for all $\nu \in Z_+^n$ and $|a_\nu| \to 0$ as $\nu_1 + \cdots + \nu_n \to \infty$. We define a norm $\|\ \|$ on A_n: For $g = \sum a_\nu x^\nu \in A_n$, let $\|g\| = \max\{|a_\nu|\}$. Let m be the maximal ideal of B and $k = B/m$ be the residue field. Let τ be the canonical mapping of B onto k. Then τ can be extended to an epimorphism from A_n to a polynomial ring $k[x_1, \cdots, x_n]$ in the usual manner. We assume, throughout this paper, the B-ring B is complete. We shall identify $A_{n-1}\langle x_n\rangle$ with A_n so that each element g of A_n has an expression $\sum g_i x_n^i$, where $g_i \in A_{n-1}$ for all $i \in Z_+$ and $\|g_i\| \to 0$ as $i \to \infty$. For any $s \in Z_+$, let P_s denote the set of all polynomials of $A_{n-1}[X_n]$ of degree $< s$. One can see several properties on a B-ring in [2], [4].

In this paper, we shall prove Weierstrass Preparation Theorem for A_n. We shall obtain Weierstrass Form Theorem and Scherung Theorem for A_n also.

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§1.

To consider Weierstrass Preparation Theorem we need some information on unit elements of A_n. We prove

PROPOSITION 1.1. *Let* $g = \sum a_\nu x^\nu \in A_n$. *Then* g *is a unit element of* A_n *if and only if*

$$\begin{cases} |a_{0,\dots,0}| = 1\,, \\ |a_\nu| < 1 \text{ for each } \nu \neq (0, \cdots, 0)\,. \end{cases} \tag{1.1}$$

Proof. Let g be a unit element of A_n then there exists an element u of A_n such that $gu = 1$. It follows

$$1 = \tau(gu) = \tau(g)\tau(u) \in k[x_1, \cdots, x_n]\,.$$

Hence (1.1) holds. Conversely, suppose g satisfies (1.1). Then it can be seen that the inverse element of g is given by

$$g^{-1} = (a_{0,\dots,0})^{-1}[1 + \sum_1^\infty (g'')^i]\,, \tag{1.2}$$

where

$$-g'' = (a_{0,\dots,0})^{-1} \sum_{\nu \neq (0,\cdots,0)} a_\nu x^\nu\,.$$

With this the proof is complete.

From this proposition, we have the followings:

Remark 1.2. If $n \geqq 1$, then A_n is not a quasi-local ring.

Proof. For a contradiction, we assume A_n is a quasi-local ring. Then it follows the set M of all nonunit elements of A_n forms a maximal ideal. By Proposition 1.1, for instance, x_1 and $1 + x_1$ belong to M. Then 1 belongs to M, a contradiction.

Let $g = \sum g_i x_n^i \in A_{n-1}\langle x_n \rangle$. Let $s \in Z_+$. We say that g is general (allgemein) in x_n of order s if g_s is a unit element of A_{n-1} and $\|g_i\| < 1$ for all $i > s$.

Remark 1.3. $g \in A_n$ is general in x_n of order $s \geqq 0$ if and only if

$$\tau(g) = \tau(g_0) + \tau(g_1)x_n + \cdots + \tau(g_s)x_n^s \tag{1.3}$$

for which $\tau(g_i) \in k[x_1, \cdots, x_{n-1}]$ for each $i = 0, \cdots, s-1$, and $\tau(g_s) \in k^* = k - \{0\}$.

Proof. By Proposition 1.1, it is clear g_s is a unit element of A_{n-1} if and only if $\tau(g_s)$ is in k^*. It is easy to verify the other conditions on the coefficients of g.

§ 2.

In this section we shall show Weierstrass Form for A_n, which is a generalization of the result of Grauert-Remmert [1].

THEOREM 2.1 (Weierstrass Form for A_n). *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then for each $f \in A_n$ there exists a unique pair $q \in A_n$, $r \in P_s$ satisfying*

$$f = qg + r \,. \tag{2.1}$$

Further, we have

$$\|f\| = \max \{\|q\|, \|r\|\} \,. \tag{2.2}$$

In order to prove this theorem we need the following lemmas. Lemma 2.3 is established for $K\langle x_1, \cdots, x_n\rangle$ (Satz 2.1 of [1]). But in our case it cannot be assumed that for a nonzero element f of $K\langle x_1, \cdots, x_n\rangle$ there exists a nonzero element a in K satisfying $\|af\| = 1$, because we take an arbitrary B-ring B as a coefficient ring. So, we prove at first Lemma 2.2 analogous to Theorem 3.20 in [3].

LEMMA 2.2. *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then for $q \in A_n$ and $r \in P_s$ we have*

$$\|qg + r\| \geqq \|q\| \,. \tag{2.3}$$

Proof. Let $q = \sum b_\nu x^\nu \in A_n$. Let $\mu = (\mu_1, \cdots, \mu_n) \in Z_+^n$ be the heighest indexterm of ν such that $\|q\| = |b_\nu|$. If $qg = \sum c_\nu x^\nu \in A_n$ then $\|q\| = \|qg\| = |c_{\mu'}|$, where $\mu' = (\mu_1, \cdots, \mu_{n-1}, \mu_n + s)$. If $r \in A_n$ such that the coefficient of $x^{\mu'}$ vanishes, then $\|qg + r\| \geqq \|q\|$. In particular this is true for all $r \in P_s$ and now (2.3) follows.

LEMMA 2.3. *Let $g \in A_{n-1}[x_n]$ be of degree s and the leading coefficient be a unit element of A_{n-1}. Then for each $f \in A_n$ there exists a pair $q \in A_n$, $r \in P_s$ satisfying*

$$f = qg + r \,. \tag{2.4}$$

Proof. Let $f = \sum f_i x_n^i \in A_{n-1}\langle x_n\rangle$. It follows for each $i \geqq 0$ there exist

q_i and $r_i \in A_{n-1}[x_n]$ such that r_i is of degree $< s$ and $f_i x_n^i = q_i g + r_i$. Then (2.3) implies

$$\|f_i\| = \max\{\|q_i\|, \|r_i\|\} . \tag{2.5}$$

Let $r = \sum_0^\infty r_i$ and $q = \sum_0^\infty q_i$. Then we see that $q \in A_n$ and $r \in P_s$. With these q and r we obtain the equation (2.4).

Proof of Theorem 2.1. If $r \in gA_n \cap P_s$ then by Lemma 2.2, $0 = \|r - r\| \geqq \|r\|$. Therefore we have

$$gA_n \cap P_s = 0 ,$$

which shows the uniqueness of a pair q, r of (2.1).

We next prove

$$A_n = gA_n + P_s ,$$

by using Grauert-Remmert's method in [1]. In fact, let $g = \sum g_i x_n^i \in A_{n-1}\langle x_n \rangle$ and let $g = g^{(1)} + g^{(2)}$, where $g^{(1)} = \sum_0^s g_i x_n^i$. Then we have $\delta = \|g^{(2)}\| < 1$. We define a set of elements f_j, q_j and r_j of A_n in the following way: Let $f_0 = f = q_0 g^{(1)} + r_0$, where $r_0 \in P_s$. For $j \in Z_+$ we put $f_{j+1} = f_j - q_j g - r_j = q_{j+1} g^{(1)} + r_{j+1}$, where $r_{j+1} \in P_s$. This procedure is possible by Lemma 2.3. Then it follows

$$f_{j+1} = -q_j g^{(2)}$$

whence, by (2.5)

$$\|f_{j+1}\| = \delta\|q_j\| \leqq \delta\|f_j\| ,$$

therefore we have

$$\|f_{j+1}\| \leqq \delta\|f_j\| .$$

By induction on $j \geqq 0$, we have

$$\|f_j\| \leqq \delta^j\|f\|, \ \|q_j\| \leqq \delta^j\|f\|$$

and

$$\|r_j\| \leqq \delta^j\|f\| .$$

Putting $q = \sum_0^\infty q_j$ and $r = \sum_0^\infty r_j$, we have $q \in A_n$ and $r \in P_s$ satisfying (2.7) as required.

By the definition it is clear $\|q\| \leqq \|f\|$. Then we see $\|r\| = \|f - qg\| \leqq \max\{\|f\|, \|q\|\} = \|f\|$. Therefore we have $\|f\| \geqq \max\{\|q\|, \|r\|\}$. This proves half of (2.2) and the other half is obvious. Thus our theorem is completely proved.

§ 3.

THEOREM 3.1 (Weierstrass Preparation Theorem for A_n). *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then there exist uniquely $u, a_0, \cdots, a_{s-2}$ and a_{s-1} satisfying the following conditions: u is a unit element of A_n, $a_0, \cdots, a_{s-1}$ are in A_{n-1} and*

$$g = u(x_n^s + a_{s-1}x_n^{s-1} + \cdots + a_1 x_n + a_0) . \tag{3.1}$$

Proof. By Theorem 2.1 there exists a unique pair $q \in A_n$, $r \in P_s$ satisfying

$$x_n^s = qg + r .$$

Applying Theorem 2.1 again, this time with $x_n^s - r$ instead of g, we obtain a unique pair $q' \in A_n$, $r' \in P_s$ satisfying

$$g = q'(x_n^s - r) + r' .$$

Then

$$g = q'qg + r' ,$$

therefore we must have $q'q = 1$ and $r' = 0$. In particular we have

$$g = q'(x_n^s - r) . \tag{3.2}$$

Put $-a_0, -a_1, \cdots, -a_{s-1}$ as the coefficients of r and $u = q'$. The uniqueness follows from the choice of r, which shows our assertion.

§ 4.

In this section we prove Scherung Theorem for A_n.

THEOREM 4.1. *Suppose the residue field k of B is infinite. Let f be in A_n and $\|f\| = 1$. Then there exists a B-automorphism σ of A_n such that $\sigma(f)$ is general in x_n.*

Proof. Let $f = \sum_0^\infty f_j$, where each f_j is the j-th homogeneous part of f. Then there exists f_s such that $\|f_s\| = 1$ and $\|f_j\| < 1$ for all $j > s$. Let $\tau(f_s) = \bar{f}_s$. Then $\bar{f}_s$ is a nonzero element of $k[x_1, \cdots, x_n]$. If $n = 1$ then the assertion is clear. Assume $n \geqq 2$. By our assumption that k is infinite, we can choose an element $(\bar{a}_1, \cdots, \bar{a}_{n-1}, \bar{a}_n) \in k^n$ satisfying

$$f_s(\bar{a}_1, \cdots, \bar{a}_n) \in k^* ,$$

where $\bar{a}_j = \tau(a_j)$ for $a_j \in B$, $j = 1, \cdots, n$. Here we may assume $|a_n| = 1$. Put $b = f_s(a_1, \cdots, a_n)$. Then $|b| = 1$. We define a B-algebra endomorphism σ such as

$$\sigma(x_j) = x_j + a_j x_n, \qquad j = 1, \cdots, n-1 ,$$
$$\sigma(x_n) = x_n .$$

Then σ^{-1} is given by

$$\sigma^{-1}(x_j) = x_j - a_j x_n, \qquad j = 1, \cdots, n-1 ,$$
$$\sigma^{-1}(x_n) = x_n .$$

It can be seen by easy calculations

$$\sigma(f) = \sum_0^\infty f_i^* x_n^i ,$$

where each $f_i^* \in A_{n-1}$ and $\|f_i^*\| < 1$ for all $i > s$. In particular f_s^* is a unit element of A_{n-1}, for the constant term is equal to b and the norm of the part of terms of degree $\geqq 1$ is less than 1. Therefore $\sigma(f)$ is general in x_n of order s. Thus σ is the B-automorphism to be desired.

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