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ON A DECOMPOSITION OF SPACES OF CUSP FORMS AND TRACE FORMULA OF HECKE OPERATORS

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Introduction

For a positive integer N, put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \pmod{N} \right\}.$$

For a positive integer κ and a Dirichlet character ψ modulo N, let $S_\kappa(N, \psi)$ denote the space of holomorphic cusp forms for $\Gamma_0(N)$ of weight κ and character ψ. For a positive integer n prime to N, the Hecke operator T_n is defined on $S_\kappa(N, \psi)$, and in the case where $\kappa \geq 2$, an explicit formula for the trace $\operatorname{tr} T_n$ of T_n is known by Eichler [6] and Hijikata [8]. But for higher levels, in particular, when N contains a power of a prime as a factor, this formula is not suitable for numerical computations. It is natural to ask a decomposition of $S_\kappa(N, \psi)$ stable under the action of Hecke operators and a formula for $\operatorname{tr} T_n$ on each subspace. In fact, when ψ is the trivial character ψ_1, Yamauchi [18] gave a decomposition of $S_\kappa(N, \psi_1)$ and a formula for $\operatorname{tr} T_n$ on each subspace by means of the normalizers of $\Gamma_0(N)$. In the case where $N = p^\nu$ with a prime p, $S_\kappa(p^\nu, \psi_1)$ is divided into two subspaces by this decomposition. When $\nu \geq 2$, in Saito-Yamauchi [11] another decomposition of $S_\kappa(p^\nu, \psi_1)$ into four subspaces and the formulas for $\operatorname{tr} T_n$ on these subspaces were given by using the normalizer $W = \begin{pmatrix} 0 & -1 \\ p^\nu & 0 \end{pmatrix}$ of $\Gamma_0(p^\nu)$ and the twisting operator R_ε for ε the quadratic residue symbol modulo p. In this paper, we shall generalize these results. In § 1, we define an operator U_χ on $S_\kappa(N, \psi)$ for a character χ which satisfies a certain condition. This operator is a generalization of $R_\varepsilon W R_\varepsilon W$ in [11]. In a similar way as in [11], we can give a formula for $\operatorname{tr} U_\chi T_n$ and also for $\operatorname{tr} U_\chi W T_n$ with a normalizer W of $\Gamma_0(N)$ when ψ is trivial (§ 2. Th. 2.5. and Th. 2.9.). In § 3, we shall prove a multiplicative property of U_χ. This

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property makes it possible to define a decomposition of $S_\kappa(N, \psi)$ into subspaces. This decomposition is finer than the one given in [11] even in the case where $N = p^3$ and is trivial. The trace of T_n on each subspace is given by a linear combination of tr $U_\chi T_n$ and tr $U_\chi W T_n$. In § 4, we give a numerical example for $N = 11^3$, $\kappa = 2$ and the trivial ψ. In this example, we find a congruence between a cusp form associated with a Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$ and a certain primitive cusp form modulo a prime ideal $\mathfrak{p}$ with the norm 99527. By means of a result of Shimura [16], this prime ideal can be related to the special values of certain L-functions of $\boldsymbol{Q}$ and $\boldsymbol{Q}(\sqrt{-11})$. We can observe such a congruence also in the examples of Doi-Yamauchi [3] for $N = 7^3$ and [11] for $N = 11^3$. These observations were done under the influence of Doi-Ohta [4] and Doi-Hida [5]. In the Appendix, we give more examples for $N = 13^3$, 19^3 under the condition that $\kappa = 2$ and ψ is trivial.

Notation

The symbols $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote respectively the ring of rational integers, the rational number field, the real number field, and the complex number field. For a prime p, $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. For a prime p, v_p denotes the additive valuation of $\boldsymbol{Q}_p$ normalized as $v_p(p) = 1$. For an associative ring S with an identity element, we denote by $S^\times$ the group of all invertible elements of S, and by $M_n(S)$ the ring of all square matrices of size n with coefficients in S. We put $GL_n(S) = M_n(S)^\times$. For subsets S_{ij} of S, $1 \leq i, j \leq n$, (S_{ij}) denotes the subsets $\{(s_{ij}) \in M_n(S) \mid s_{ij} \in S_{ij}\}$. For a group G and its subgroup H, we denote by $\underset{H}{\sim}$ the conjugacy with respect to H, i.e., $g \underset{H}{\sim} g'$ if and only if $h^{-1}gh = g'$ with $h \in H$, and for a subset X of G, we denote by $X/\underset{H}{\sim}$ a complete system of representatives of X with respect to H. Finally, for a finite dimensional vector space V over $\boldsymbol{C}$ and a linear operator T on V, tr $T \mid V$ denotes the trace of T on V.

§ 1. The operator U_χ

Let $\mathfrak{H}$ denote the complex upper half plane $\{z \in \boldsymbol{C} \mid \operatorname{Im}(z) > 0\}$ and $GL_2(\boldsymbol{R})^+ = \{\gamma \in GL_2(\boldsymbol{R}) \mid \det \gamma > 0\}$. Let κ be a positive integer. For a complex-valued function $f(z)$ on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\boldsymbol{R})^+$, we define a function $f \mid [\gamma]_\kappa$ on $\mathfrak{H}$ by

$$(f|[\gamma]_\kappa)(z) = (\det \gamma)^{\kappa/2}(cz + d)^{-\kappa} f(\gamma(z)) ,$$

where $\gamma(z) = (az + b)/(cz + d)$ for $z \in \mathfrak{H}$. For a positive integer N and a Dirichlet character ψ modulo N such that $\psi(-1) = (-1)^\kappa$, let $G_\kappa(N, \psi)$ denote the vector space of holomorphic modular forms $f(z)$ satisfying

$$f|[\gamma]_\kappa = \psi(d)f \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) .$$

We denote by $S_\kappa(N, \psi)$ the subspace of $G_\kappa(N, \psi)$ consisting of cusp forms and by $S_\kappa^0(N, \psi)$ the space of new forms in $S_\kappa(N,\psi)$. For the trivial character ψ_1, we put $S_\kappa(N) = S_\kappa(N, \psi_1)$ and $S_\kappa^0(N) = S_\kappa^0(N, \psi_1)$. For a positive integer n prime to N, the Hecke operator T_n on $S_\kappa(N, \psi)$ is defined in the usual way by

$$f|T_n = n^{\kappa/2-1} \sum_{\substack{ad=n \\ b \bmod d}} \psi(a) f \left| \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]_\kappa \right. .$$

For a Dirichlet character χ, we denote by $\mathfrak{f}_\chi$ the conductor of χ. Let χ be a primitive character with $\mathfrak{f}_\chi = c$. Then for $f \in S_\kappa(N, \psi)$, the twisting operator R_χ is defined as follows;

$$f|R_\chi = \frac{1}{\mathfrak{g}(\bar\chi)} \sum_{i \bmod c} \bar\chi(i) f \left| \left[\begin{pmatrix} 1 & i/c \\ 0 & 1 \end{pmatrix} \right]_\kappa \right. ,$$

where $\mathfrak{g}(\bar\chi)$ is the Gauss sum for $\bar\chi$. Then it is known (c.f. [13]) that $f|R_\chi$ belongs to $S_\kappa(N', \psi\chi^2)$, where N' is the least common multiple of N, $\mathfrak{f}_\psi \mathfrak{f}_\chi$ and $\mathfrak{f}_\chi^2$. For a positive divisor M of N such that $(M, N/M) = 1$, we choose and fix an element γ_M of $SL_2(Z)$ which satisfies

$$\gamma_M \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\bmod M^4) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\bmod (N/M)^4) \end{cases}$$

and put

$$\eta_M = \gamma_M \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} .$$

For $M = N$ and $M = 1$, we take respectively

$$\eta_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} , \qquad \eta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

For a positive divisor M of N, we denote by $\tilde{M}$ the divisor of N such that the sets of primes which divide M and $\tilde{M}$ are the same and $(\tilde{M}, N/\tilde{M}) = 1$. For a positive divisor M of N, we put $\eta_M = \eta_{\tilde{M}}$, and define the operator W_M by

$$f|W_M = f|[\eta_M]_\kappa .$$

Let χ be a character modulo N, and M a divisor of N such that $(M, N/M) = 1$. Then χ can be expressed as $\chi = \chi_M \chi_{N/M}$, where χ_M (resp. $\chi_{N/M}$) is a character modulo M (resp. N/M). For a positive divisor M' of N, we put $\chi_{M'} = \chi_{\tilde{M}'}$. In this notation, it is known that $f|W_M$ is contained in $S_\kappa(N, \bar{\psi}_M \psi_{N/M})$. These operators T_n, R_χ, and W_M satisfy the following properties.

LEMMA 1.1. *Let χ be a primitive character, and M a positive divisor of N such that $(M, N/M) = 1$. Then for $f \in S_\kappa(N, \psi)$, one has*

(1) *If n is a positive integer prime to $N\mathfrak{f}_\chi$, then*

$$f|T_n R_\chi = \bar{\chi}(n) f|R_\chi T_n$$
$$f|T_n W_M = \psi_M(n) f|W_M T_n .$$

(2) *Suppose $(M, \mathfrak{f}_\chi) = 1$. Then*

$$f|R_\chi W_M = \bar{\chi}(M) f|W_M R_\chi .$$

(3) *Let M' be a positive divisor of N such that $(M', N/M') = 1$ and $(M, M') = 1$. Then*

$$f|W_M W_{M'} = \bar{\psi}_{M'}(M) f|W_{MM'}$$
$$f|W_M W_M = \psi_M(-1) \bar{\psi}_{N/M}(M) f .$$

These properties of T_n, R_χ, and W_M are given in Atkin-Li [1] and can be verified easily by straightforward computations.

Now we give a definition of the operator U_χ, which is essential to our decomposition of $S_\kappa(N, \psi)$. Let χ be a primitive character with the conductor $\mathfrak{f}_\chi = M$. We assume

$$\mathfrak{f}_\chi^2 | N \quad \text{and} \quad \mathfrak{f}_\chi \mathfrak{f}_\psi | N . \tag{1.1}$$

For such a character χ, we define the operator U_χ by

$$U_\chi = R_\chi W_M R_\chi W_M .$$

For the trivial character χ_1, we define $U_{\chi_1} =$ the identity map. Then U_χ induces a map

$$U_\chi\colon S_\kappa(N, \psi) \longrightarrow S_\kappa(N, \psi)\ .$$

Furthermore, U_χ satisfies the following properties.

PROPOSITION 1.2. *The notation being as above, let* $f \in S_\kappa(N, \psi)$.

(1) *If* n *is a positive integer prime to* N, *then*

$$f \mid T_n U_\chi = f \mid U_\chi T_n\ .$$

(2) *Let* χ' *be a primitive character which satisfies the condition* (1.1). *Suppose* $(\mathfrak{f}_\chi, \mathfrak{f}_{\chi'}) = 1$. *Then*

$$f \mid U_\chi U_{\chi'} = \bar{\psi}_M \bar{\chi}(M') \bar{\psi}_{M'} \bar{\chi}'(M) f \mid U_{\chi\chi'}\ ,$$

where $M = \mathfrak{f}_\chi$ *and* $M' = \mathfrak{f}_{\chi'}$.

(3) *If* ψ *is the trivial character, then for a positive divisor* L *of* N *prime to* $\mathfrak{f}_\chi$, *it holds*

$$f \mid U_\chi W_L = f \mid W_L U_\chi\ .$$

Proof. Let $M = \mathfrak{f}_\chi$, then by (1) of Lemma 1.1, we see

$$\begin{aligned} f \mid T_n U_\chi &= f \mid T_n R_\chi W_M R_\chi W_M \\ &= \bar{\chi}(n) f \mid R_\chi T_n W_M R_\chi W_M \\ &= \chi(n) \psi_M(n) f \mid R_\chi W_M T_n R_\chi W_M \\ &= f \mid R_\chi W_M R_\chi W_M T_n\ . \end{aligned}$$

The assertions (3) and (3) can be proved in a similar way by using Lemma 1.1, and we omit the proof.

For $M = \mathfrak{f}_\chi$, let $\tilde{M}$ be as above, and put

$$\tilde{U}_\chi = \psi_{\tilde{M}}(-1) \psi_{N/\tilde{M}}(\tilde{M}) \chi(N/\tilde{M}) U_\chi\ .$$

Then the assertion (2) of the above proposition is equivalent to the following.

COROLLARY 1.3. *If* $\mathfrak{f}_\chi$ *is prime to* $\mathfrak{f}_{\chi'}$, *then*

$$\tilde{U}_\chi \tilde{U}_{\chi'} = \tilde{U}_{\chi\chi'}\ .$$

PROPOSITION 1.4. *The notation being as above, then the following assertions hold.*

(1) *If* f *is a primitive form in* $S^0_\kappa(N, \psi)$, *then* f *is an eigen-function for* U_χ. *In particular,* U_χ *induces a map*

$$U_\chi\colon S^0_\kappa(N, \psi) \longrightarrow S^0_\kappa(N, \psi)\ .$$

(2) *Suppose* $v_p(\mathfrak{f}_\chi\mathfrak{f}_\psi) < v_p(N)$ *and* $v_p(\mathfrak{f}_\chi^2) < v_p(N)$ *for a prime divisor* p *of* $\mathfrak{f}_\chi$. *If* g *belong to* $S_\kappa(N/p, \psi)$, *then*

$$g|U_\chi = 0$$

(3) *Let* f *be a primitive form in* $S_\kappa^0(N, \psi)$. *If* $f|U_\chi = 0$ *for a character* χ *with* $\mathfrak{f}_\chi = p^\mu$, *where* p *is a prime divisor of* N, *then it holds* $v_p(\mathfrak{f}_\chi\mathfrak{f}_\psi) = v_p(N)$ *or* $v_p(\mathfrak{f}_\chi^2) = v_p(N)$, *and there exists* $g \in S_\kappa(N/p, \psi\chi^2)$ *such that* $f = g|R_{\bar{\chi}}$.

(4) *If* ψ *is the trivial character* ψ_1 *and* $f \in S_\kappa^0(N, \psi_1)$, *then for any divisor* L *of* N, *it holds*

$$f|U_\chi W_L = f|W_L U_\chi .$$

Proof. The assertions (1) and (4) easily follows from Prop. 1.2. We shall prove (2) and (3). To prove (2), we may assume g is a primitive form. From the assumption, it follows $g|R_\chi \in S_\kappa(N/p, \psi\chi^2)$. Put $\eta'_M = \gamma_M\begin{pmatrix} M/p & 0 \\ 0 & 1\end{pmatrix}$, then $g|R_\chi[\eta'_M]_\kappa$ belongs to $S_\kappa(N/p, \bar{\psi}_M\psi_{N/M}\chi^2)$. Hence $g|R_\chi W_M = g'(pz)$ for $g' \in S_\kappa(N/p, \bar{\psi}_M\psi_{N/M}\chi^2)$, and $g|R_\chi W_M R_\chi = 0$. This proves the assertion (2). Now we prove (3). By the assumption on χ, we have $v_p(N) \geq 2$ and $v_p(\mathfrak{f}_\psi) < v_p(N)$. Hence the p-th Fourier coefficient a_p of f vanishes, and $f|R_\chi R_{\bar{\chi}} = f$. If $f|R_\chi$ is a primitive form in $S_\kappa^0(N, \psi\chi^2)$, then $f|R_\chi W_M$ is also a non-zero constant multiple of a primitive form, and $f|R_\chi W_M R_\chi W_M \neq 0$. Hence if $f|U_\chi = 0$, then $f|U_\chi$ is not a primitive form in $S_\kappa^0(N, \psi\chi^2)$, and there exist g, $h \in S_\kappa(N/p, \psi\chi^2)$ such that $(f|R_\chi)(z) = g(z) + h(pz)$. Then we have $f = f|R_\chi R_{\bar{\chi}} = g|R_{\bar{\chi}}$. Now we show that $f|R_\chi$ is a primitive form in $S_\kappa^0(N, \psi\chi^2)$ if $v_p(\mathfrak{f}_\psi\mathfrak{f}_\chi) < v_p(N)$ and $v_p(\mathfrak{f}_\chi^2) < v_p(N)$. Otherwise $f|R_\chi$ can be written as $f|R_\chi = g'(z) + h'(pz)$ with g', $h' \in S_\kappa(N/p, \psi\chi^2)$. Then $f = f|R_\chi R_{\bar{\chi}} \in S_\kappa(N/p, \psi)$, because $v_p(N/p) \geq v_p(\mathfrak{f}_\psi\mathfrak{f}_\chi)$ and $v_p(N/p) \geq v_p(\mathfrak{f}_\chi^2)$. This contradicts to our assumption that $f \in S_\kappa^0(N, \psi)$.

§ 2. Formula for tr $U_\chi T_n$ and tr $U_\chi W_L T_n$

Let N and ψ be as in § 1. For a primitive character χ which satisfies the condition (1.1), we defined an operator U_χ: $S_\kappa(N, \psi) \longrightarrow S_\kappa(N, \psi)$ in § 1. We shall give a formula for tr $U_\chi T_n | S_\kappa(N, \psi)$. For $M = \mathfrak{f}_\chi$, we write $N = N_1N_2$, where $N_1 = \tilde{M}$ and $N_2 = N/\tilde{M}$. We put

$$R(N) = \begin{pmatrix} Z & Z \\ NZ & Z \end{pmatrix}$$

and for each prime p

$$U_p = (R(N) \otimes \boldsymbol{Z}_p)^\times .$$

For the archimedean prime ∞, we put $U_\infty = GL_2(\boldsymbol{R})^+$. We denote by U the subgroup $\prod_v U_v$ of $GL_2(\boldsymbol{Q}_A)$, where v runs through all places of $\boldsymbol{Q}$. Let p be a prime divisor of N and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_p$. We define

$$\tilde{\psi}_p(\gamma) = \psi_p(d) ,$$

and for $\gamma \in \prod_{p|N} U_p \times \prod_{p\nmid N} GL_2(\boldsymbol{Q}_p) \times U_\infty$

$$\tilde{\psi}(\gamma) = \prod_{p|N} \tilde{\psi}_p(\gamma_p) ,$$

where γ_p is the p-th component of γ. For a prime which divides N_1, we define a subset $\Xi_p(U_\chi)$ of $M_2(\boldsymbol{Z}_p)$ by

$$\Xi_p(U_\chi) = \left\{ g \in \begin{pmatrix} p^{\nu+2\mu}\boldsymbol{Z}_p & p^{\nu+\mu}\boldsymbol{Z}_p^\times \\ p^{2\nu+\mu}\boldsymbol{Z}_p^\times & p^{\nu+2\mu}\boldsymbol{Z}_p^\times \end{pmatrix} \middle| v_p(\det g) = 2\nu + 4\mu \right\} ,$$

where $\nu = v_p(N)$ and $\mu = v_p(\mathfrak{f}_\chi)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Xi_p(U_\chi)$, we put

$$\tilde{\chi}_p(g) = \chi_p(-bc/p^{3\nu+2\mu})\tilde{\psi}_p(-d/p^{\nu+2\mu}) . \tag{2.1}$$

Then for γ, $\gamma' \in U_p$ and $g \in \Xi_p(U_\chi)$, we see

$$\tilde{\chi}_p(\gamma g \gamma') = \tilde{\psi}_p(\gamma\gamma')^{-1}\chi_p(\det(\gamma\gamma'))\tilde{\chi}_p(g) , \tag{2.2}$$

and in particular for $\gamma' = \gamma^{-1}$,

$$\tilde{\chi}_p(\gamma g \gamma^{-1}) = \tilde{\chi}_p(g) . \tag{2.3}$$

For $g \in \prod_{p|N_1} \Xi_p(U_\chi) \times \prod_{p|N_2} U_p \times \prod_{p\nmid N} GL_2(\boldsymbol{Q}_p) \times U_\infty$, put

$$\tilde{\chi}(g) = \prod_{p|N_1} \tilde{\chi}_p(g_p) \prod_{p|N_2} \tilde{\psi}_p(g_p)^{-1} ,$$

where g_p denotes the p-th component of g. Then by (2.2), we see for γ, $\gamma' \in \prod_{p|N} U_p \times \prod_{p\nmid N} GL_2(\boldsymbol{Q}) \times U_\infty$,

$$\tilde{\chi}(\gamma g \gamma') = \tilde{\psi}(\gamma\gamma')^{-1} \prod_{p|N_1} \chi_p(\det(\gamma_p\gamma'_p))\tilde{\chi}(g) , \tag{2.4}$$

and in particular, if γ, $\gamma' \in \Gamma_0(N)$, then

$$\tilde{\chi}(\gamma g \gamma') = \tilde{\psi}(\gamma\gamma')^{-1}\tilde{\chi}(g) . \tag{2.5}$$

For rational integers i, j, put

$$\alpha_{ij} = \begin{pmatrix} M & i \\ 0 & M \end{pmatrix} \eta_M \begin{pmatrix} M & j \\ 0 & M \end{pmatrix} \eta_M ,$$

where $M = \mathfrak{f}_\chi$. For a positive integer n prime to N, let $\Xi(T_n) = \prod_p \Xi_p(T_n) \times U_\infty$, where

$$\Xi_p(T_n) = \{g \in R(N) \otimes Z_p \mid v_p(\det g) = v_p(n)\},$$

and let $\Xi(T_n) \cap GL_2(\boldsymbol{Q}) = \bigcup_{k=1}^{d} \Gamma_0(N)\beta_k$ be a disjoint union.

LEMMA 2.1. *The notation being as above, let p be a prime divisor of $\mathfrak{f}_\chi$ and $\nu = v_p(N)$, $\mu = v_p(\mathfrak{f}_\chi)$. Then for $g = \begin{pmatrix} p^{\nu+2\mu}a & p^{\nu+\mu}b \\ p^{2\nu+\mu}c & p^{\nu+2\mu}d \end{pmatrix}$ and $g' = \begin{pmatrix} p^{\nu+2\mu}a' & p^{\nu+\mu}b' \\ p^{2\nu+\mu}c' & p^{\nu+2\mu}d' \end{pmatrix}$ in $\Xi_p(U_\chi)$, $U_p g = U_p g'$ if and only if $a/b \equiv a'/b'$ modulo p^μ and $c/d \equiv c'/d'$ modulo p^μ. If this is the case, $\tilde{\psi}_p(gg'^{-1}) = \psi_p(a'd\text{-}p^{\nu-2\mu}b'c)$.*

This can be verified easily by a direct calculation, and we omit the proof.

LEMMA 2.2. *The notation being as above, let $\Xi(U_\chi T_n) = \prod_{p|N_1} \Xi_p(U_\chi) \times \prod_{p\nmid N_1} \Xi_p(T_n) \times U_\infty$. Then the union*

$$\Xi(U_\chi T_n) \cap GL_2(\boldsymbol{Q}) = \bigcup_{ij} \bigcup_{k=1}^{d} \Gamma_0(N)\alpha_{ij}\beta_k$$

is disjoint, where i and j runs through a complete system of representatives of $(Z/\mathfrak{f}_\chi Z)^\times$.

Proof. Since $U \cap GL_2(\boldsymbol{Q}) = \Gamma_0(N)$ and $\alpha_{ij}\beta_k \in GL_2(\boldsymbol{Q})$, it is enough to prove the union $\Xi(U_\chi T_n) = \bigcup_{ij} \bigcup_k U\alpha_{ij}\beta_k$ is disjoint. We note the union $\prod_{p\nmid N_1} \Xi_p(T_n) = \bigcup_k \prod_{p\nmid N_1} U_p\beta_k$ is disjoint and $\alpha_{ij} \in \prod_{p\nmid N_1} U_p$, $\beta_k \in \prod_{p|N_1} U_p$. Hence the proof can be reduced to showing the union $\prod_{p|N_1} \Xi_p(U_\chi) = \bigcup_{ij} \prod_{p|N_1} U_p\alpha_{ij}$ is disjoint. Let $M = \mathfrak{f}_\chi$ and $\tilde{M} = N_1$, then

$$\alpha_{ij} \equiv \begin{cases} \begin{pmatrix} ij\tilde{M}^2 - \tilde{M}M^2 & -i\tilde{M}M \\ j\tilde{M}^2M & -\tilde{M}M^2 \end{pmatrix} & (\mathrm{mod}\ \tilde{M}^4) \\ \begin{pmatrix} \tilde{M}^2M^2 & j\tilde{M}M + iM \\ 0 & M^2 \end{pmatrix} & (\mathrm{mod}\ (N/\tilde{M})^4) \end{cases},$$

and by the definition of $\Xi_p(U_\chi)$, $\alpha_{ij} \in \prod_{p|N_1} \Xi_p(U_\chi)$. By Lemma 2.1, for integers i, j, i', j' prime to N_1, we see

$$U_p\alpha_{ij} = U_p\alpha_{i'j'} \Longleftrightarrow i \equiv i',\ j \equiv j' \qquad (\mathrm{mod}\ p^\mu).$$

Hence the right side of the union is disjoint. We show $\prod_{p|N_1} \Xi_p(U_\chi) \subset \bigcup_{ij} \prod_{p|N_1} U_p\alpha_{ij}$. For a prime p which divides N_1, let $g = \begin{pmatrix} p^{\nu+2\mu}a & p^{\nu+\mu}b \\ p^{2\nu+\mu}c & p^{\nu+2\mu}d \end{pmatrix}$

$\in \Xi_p(U_\chi)$. If we put $\tilde{M} = p^\nu \tilde{M}'$, $M = p^\mu M'$ and take two integers i, j which satisfy

$$\begin{cases} (ij\tilde{M}'^2 - M'M'^2)b \equiv -i\tilde{M}'M'a \\ j\tilde{M}'^2M'd \equiv -\tilde{M}'M'^2c \end{cases} \pmod{p^\mu},$$

then by Lemma 2.1, we have $U_p g = U_p \alpha_{ij}$. Such i and j are determined uniquely modulo p^μ, because $ad - bc \not\equiv 0 \pmod{p}$. Our assertion follows from this.

As a corollary of this Lemma, we obtain

COROLLARY 2.3. *The notation being as above, let $f \in S_\kappa(N, \psi)$. Then it holds*

$$f|U_\chi T_n = C \sum_{g \in \Gamma_0(N) \backslash \Xi(U_\chi T_n) \cap GL_2(\mathbf{Q})} \tilde{\chi}(g) f|[g]_\kappa$$

$$C = \frac{\chi\psi(n)}{\mathfrak{g}(\bar{\chi})^2} \prod_{p|N_1} \chi_p(A_p)\psi_p(B_p) \prod_{p|N_2} \psi_p(M^2),$$

where g runs through a complete system of representatives of the left cosets of $\Xi(U_\chi T_n) \cap GL_2(\mathbf{Q})$ by $\Gamma_0(N)$ and for a prime divisor p of N_1, $A_p = \tilde{M}^3 M^2/p^{3\nu+2\mu}$ and $B_p = \tilde{M}M^2/p^{\nu+2\mu}$ with $\nu = v_p(N)$ and $\mu = v_p(\mathfrak{f}_\chi)$.

Proof. We note the right hand side is independent of the choice of the representatives because of (2.5). We may assume β_k is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Since we have

$$\alpha_{ij}\beta_k \equiv \begin{cases} \begin{pmatrix} a(ij\tilde{M}^2 - \tilde{M}M^2) & b(ij\tilde{M}^2 - \tilde{M}M^2) - id\tilde{M}M \\ aj\tilde{M}^2M & bj\tilde{M}^2M - d\tilde{M}M^2 \end{pmatrix} & \pmod{\tilde{M}^4} \\ \begin{pmatrix} a\tilde{M}^2M^2 & b\tilde{M}^2M^2 + d(j\tilde{M}M + iM) \\ 0 & dM^2 \end{pmatrix} & \pmod{(N/\tilde{M})^4} \end{cases}$$

we see $\tilde{\chi}(\alpha_{ij}\beta_k) = \bar{\chi}(ij)\psi(a)C^{-1}$. By the definition of U_χ and T_n, we obtain our corollary.

By means of Eichler-Selberg's trace formula (c.f. [6], [8], [10], [12]) and a result of Hijikata [8], we can express trace of $U_\chi T_n$ on $S_\kappa(N, \psi)$ in an explicit way. Let us introduce some notations. For two rational integers s, n, put $\Phi(X) = X^2 - sX + n$, $K(\Phi) = \mathbf{Q}[X]/(\Phi(X))$, and denote by $\tilde{X}$ the class containing X. For a prime p, let $\nu = v_p(N)$ and $K(\Phi)_p = K(\Phi) \otimes \mathbf{Q}_p$. If we define $R_p(\nu) = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^\nu \mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix}$, then $R(N) \otimes \mathbf{Z}_p = R_p(\nu)$. For α in $GL_2(\mathbf{Q}_p)$ or $GL_2(\mathbf{R})$, we denote by $f_\alpha(X)$ the minimal polynomial of α. For a $\mathbf{Z}_p$-order Λ_p of $K(\Phi)_p$, we define

$$C_p(\nu, \Phi, \Lambda_p) = \{\alpha \in R_p(\nu) \mid f_\alpha = \Phi,\ \varphi_\alpha(\Lambda_p) = \boldsymbol{Q}_p[\alpha] \cap R_p(\nu)\}\,,$$

where φ_α denotes the isomorphism from $K(\Phi)_p$ to $\boldsymbol{Q}_p[\alpha]$ such that $\varphi_\alpha(\tilde{X}) = \alpha$. For Λ_p which contains $\boldsymbol{Z}_p[\tilde{X}]$, we define also the following sets;

$$\Omega_p(\nu, \Phi, \Lambda_p) = \{\xi \in \boldsymbol{Z}_p \mid \Phi(\xi) \equiv 0 \pmod{P^{\nu+2\rho}}\}$$

$$\Omega'_p(\nu, \Phi, \Lambda_p) = \begin{cases} \{\eta \in \boldsymbol{Z}_p \mid \Phi(\eta) \equiv 0 \pmod{p^{\nu+2\rho+1}}\}\,, \\ \qquad \text{if } p^{-2\rho}(s^2 - 4n) \equiv 0 \pmod{p} \text{ and } \nu > 0 \\ \phi\,, \qquad \text{otherwise}\,, \end{cases}$$

where ρ is the non-negative integer such that $[\Lambda_p\colon \boldsymbol{Z}_p[\tilde{X}]] = p^\rho$. We denote by $\tilde{\Omega}_p(\nu, \Phi, \Lambda_p)$ (resp. $\tilde{\Omega}'_p(\nu, \Phi, \Lambda_p)$) a complete system of representatives of $\Omega_p(\nu, \Phi, \Lambda_p)$ (resp. $\Omega'_p(\nu, \Phi, \Lambda_p)$) modulo $p^{\nu+2\rho}$. For $\xi \in \Omega_p(\nu, \Phi, \Lambda_p)$ and $\eta \in \Omega_p(\nu, \Phi, \Lambda_p)$ we define elements $\varphi_\xi(\tilde{X})$ and $\varphi'_\eta(\tilde{X})$ in $C_p(\nu, \Phi, \Lambda_p)$ by

$$\varphi_\xi(\tilde{X}) = \begin{pmatrix} \xi & p^\rho \\ -p^{-\rho}\Phi(\xi) & s - \xi \end{pmatrix}$$

$$\varphi'_\eta(\tilde{X}) = \begin{pmatrix} s - \eta & -p^{-\nu-\rho}\Phi(\eta) \\ p^{\nu+\rho} & \eta \end{pmatrix}.$$

We define a map

$$\varphi\colon \Omega_p(\nu, \Phi, \Lambda_p) \cup \Omega'_p(\nu, \Phi, \Lambda_p) \longrightarrow C_p(\nu, \Phi, \Lambda_p)$$

by $\varphi(\xi) = \varphi_\xi(\tilde{X})$ for $\xi \in \Omega_p(\nu, \Phi, \Lambda_p)$ and $\varphi(\eta) = \varphi'_\eta(\tilde{X})$ for $\eta \in \Omega'_p(\nu, \Phi, \Lambda_p)$. In these notations, we have

LEMMA 2.4. *The notation being as above, let* $\Phi(X) = X^2 - sX + N^2\mathfrak{f}_\chi^4 n$ *and for a prime* p, *let* Λ_p *a* $\boldsymbol{Z}_p$*-order of* $K(\Phi)_p$ *such that* $\Lambda_p \supset \boldsymbol{Z}_p[\tilde{X}]$. *Then the followings hold.*

(1) *If* p *does not divide* N, *then* φ *induces a bijective map*

$$\varphi\colon \Omega_p(0, \Phi, \Lambda_p) \longrightarrow C_p(0, \Phi, \Lambda_p) \cap \Xi_p(T_n)/\underset{\widetilde{U_p}}{}\,,$$

and $|\tilde{\Omega}_p(0, \Phi, \Lambda_p)| = 1$.

(2) *If* p *divides* N_2, *then* φ *induces a bijective map*

$$\varphi\colon \Omega_p(\nu, \Phi, \Lambda_p) \cup \Omega'_p(\nu, \Phi, \Lambda_p) \longrightarrow C_p(\nu, \Phi, \Lambda_p) \cap U_p/\underset{\widetilde{U_p}}{}\,,$$

where $\nu = v_p(N)$.

(3) *If* p *divides* N_1, *then* $C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi) \neq \phi$ *only if* $s \equiv 0 \pmod{p^{\nu+2\mu}}$ *and* $\rho = \nu + \mu$, *and for* Φ *with* $s \equiv 0 \pmod{p^{\nu+2\mu}}$ *and* Λ_p *with* $\rho = \nu + \mu$, φ *induces a bijective map*

$$\varphi\colon \tilde{\Omega}_p \longrightarrow C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi)/_{\widetilde{U_p}}\,,$$

where $\nu = v_p(N)$, $\mu = v_p(\mathfrak{f}_\chi)$ *and*

$$\tilde{\Omega}_p = \begin{cases} \{\xi \in \tilde{\Omega}_p(\nu, \Phi, \Lambda_p) \mid \Phi(\xi) \not\equiv 0 \pmod{p^{3\nu+2\mu+1}}\} & \text{if } \nu \neq 2\mu \\ \{\xi \in \tilde{\Omega}_p(\nu, \Phi, \Lambda_p) \mid \Phi(\xi) \not\equiv 0 \pmod{p^{3\nu+2\mu+1}}\,, \\ \qquad s \not\equiv \xi \pmod{p^{\nu+2\mu+1}}\} & \text{if } \nu = 2\mu\,. \end{cases}$$

Proof. The assertions (1) and (2) are contained in Hijikata [8]. We prove (3). The theorem of Hijikata quoted in [11] as Th. 2.4 says that for Λ_p containing $Z_p[\tilde{X}]$, φ gives a bijective map

$$\varphi\colon \Omega_p(\nu, \Phi, \Lambda_p) \cap \Omega'_p(\nu, \Phi, \Lambda_p) \longrightarrow C_p(\nu, \Phi, \Lambda_p)/\tilde{U}_p\,.$$

By the definition of $\Xi_p(U_\chi)$, we see $s \equiv 0 \pmod{p^{\nu+2\mu}}$ if $C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi)$ is not empty. If $\varphi'_\eta(\tilde{X}) \in \Xi_p(U_\chi)$ for $\eta \in \Omega'_p(\nu, \Phi, \Lambda_p)$, it must hold $\nu + \rho = 2\nu + \mu$ and $\nu + \mu = v_p(\Phi(\eta)) - \nu - \rho$, hence $\rho = \nu + \mu$ and $v_p(\Phi(\eta)) = 3\nu + 2\mu$. But if $\rho = \nu + \mu$, then η saitsfies $\Phi(\eta) \equiv 0 \pmod{p^{3\nu+2\mu+1}}$ hence $\varphi'_\eta(X) \notin \Xi_p(U_\chi)$. Assume $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi)$ for $\xi \in \Omega_p(\nu, \Phi, \Lambda_p)$. Then as above, we have $\rho = \nu + \mu$ and $v_p(\Phi(\xi)) = 3\nu + 2\mu$. When these conditions are satisfied, $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi)$ if and only if $\xi \not\equiv s \pmod{p^{\nu+2\mu+1}}$. We note the last condition is always satisfied if $\nu \neq 2\mu$. For otherwise, put $s = p^{\nu+2\mu}s'$ and $\xi = p^{\nu+2\mu}(s' + p\xi')$, then we have

$$p^{2\nu+4\mu}(s'p\xi' + p^2\xi'^2 + n) \equiv 0 \pmod{p^{3\nu+2\mu}}\,.$$

Since n is prime to p, this condition is satisfied only if $\nu = 2\mu$. This proves the assertion (3).

By means of this Lemma, in the same way as in § 2 of [11], we obtain the following formula for tr $U_\chi T_n$.

THEOREM 2.5. *The notation being as above, let n be a positive integer prime to N, $\kappa \geq 2$, and C the constant in Cor. 2.2. Then it holds*

$$\operatorname{tr} U_\chi T_n \mid S_\kappa(N, \psi) = C(t_e + t_h + t_p)\,,$$

where t_e, t_h and t_p are given as follows.

(1) $$t_e = -\frac{1}{2}\sum_s \frac{\alpha^{\kappa-1} - \beta^{\kappa-1}}{\alpha - \beta} \sum_f \prod_{p\mid N} c_p(s, f) h(\mathfrak{f}_\chi^2(s^2 - 4n)/f^2).$$

Here s runs through all integers such that $s^2 - 4n < 0$, and f runs through all positive integers which satisfy the condition $f^2 \mid (s^2 - 4n)$, $(f, \mathfrak{f}_\chi) = 1$, and $\mathfrak{f}_\chi^2(s^2 - 4n)/f^2 \equiv 0$ *or* $1 \pmod 4$. *For a negative integer D such that $D \equiv 0$*

or 1 (mod 4), $h(D)$ *denotes the class number of the order of* $Q(\sqrt{D})$ *with the discriminant* D. α *and* β *are the two roots of* $F_s(X)=X^2-sX+n=0$. *The number* $c_p(s,f)$ *is given by*

$$(2.6)\qquad c_p(s,f)=\begin{cases} C_p \sum\limits_{\substack{\xi \bmod p^{\nu-\mu} \\ F_s(\xi)\equiv 0 \bmod p^{\nu-2\mu} \\ (resp.\ \xi\not\equiv s \bmod p)}} \bar{\chi}_p(F_s(\xi)/p^{\nu-2\mu})\bar{\psi}_p(\xi-s) & \text{if } p\,|\,\mathfrak{f}_\chi \text{ and } \nu\neq 2\mu \\ & (resp.\ \nu=2\mu) \\ \psi_p(N_1f^2)\Big(\sum\limits_{\xi\in\tilde{\Omega}_p(\nu,F_s,\Lambda_p)}\bar{\psi}_p(s-\xi)+\sum\limits_{\eta\in\tilde{\Omega}'_p(\mu,F_s,\Lambda_p)}\bar{\psi}_p(\eta)\Big) & \text{if } p\nmid\mathfrak{f}_\chi \end{cases}$$

where Λ_p *is the order of* $K(F_s)$ *such that* $[\Lambda_p\colon Z_p[\tilde{X}]]=p^\rho$ *for* $\rho=v_p(f)$, *and* $C_p=\bar{\chi}_p(N_1^2\mathfrak{f}_\chi^4/p^{2\nu+4\mu})\bar{\psi}_p(N_1\mathfrak{f}_\chi^2/p^{\nu+2\mu})$ *for* $\nu=v_p(N)$ *and* $\mu=v_p(\mathfrak{f}_\chi)$.

(2) $$t_h=-\sum_d \frac{d^{\kappa-1}}{n/d-d}\sum_f\prod_{p|N}c_p(d+n/d,f)\varphi(\mathfrak{f}_\chi(n/d-d)/f)\,.$$

Here d *runs through all positive integers such that* $0<d<\sqrt{n}$, $d\,|\,n$, *and* f *runs through all positive integers satisfying* $f\,|\,(n/d-d)$ *and* $(f,\mathfrak{f}_\chi)=1$. $c_p(d+n/d,f)$ *is given by* (2.6) *for* $s=d+n/d$, *and* φ *is the Euler function.*

(3) *If there exists a prime divisor* p *of* $\mathfrak{f}_\chi$ *such that* $v_p(N)$ *is odd, then* $t_p=0$. *Otherwise we have*

$$t_p=-\frac{n^{(\kappa-1)/2}}{2}\frac{\mathfrak{f}_\chi}{N}\delta(n)\sum_{\substack{m \bmod N \\ (m,f_\chi)=1}}\prod_{p|N}c_p(m)\,,$$

where $c_p(m)=c_p(2\sqrt{n},m)$ *for* p *which divides* N, *and* $\delta(n)=1$ *or* 0 *according as* n *is a square or not.*

In the rest of this section, we assume ψ is the trivial character. Then for a divisor L of N such that $(L,N/L)=1$, $U_\chi W_L$ acts on $S_\kappa(N)$, and we can give a formula for tr $U_\chi W_L T_n$. We write $N=M_1M_2M_3M_4$ in such a way $N_1=M_1M_2$ and $L=M_2M_3$. For a prime p which divides M_2, we define a subset $\Xi_p(U_\chi W_L)$ of $R(N)\otimes Z_p$ by

$$\Xi_p(U_\chi W_L)=\left\{g\in\begin{pmatrix}p^{2\nu+\mu}Z_p^\times & p^{\nu+2\mu}Z_p\\ p^{2\nu+2\mu}Z_p^\times & p^{2\nu+\mu}Z_p^\times\end{pmatrix}\middle|\, v_p(\det g)=3\nu+4\mu\right\},$$

and for a prime divisor p of M_3, put

$$\Xi_p(W_L)=\begin{pmatrix}p^\nu Z_p & Z_p^\times\\ p^\nu Z_p^\times & p^\nu Z_p\end{pmatrix}.$$

For $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Xi_p(U_\chi W_L)$, we put

$$(2.7)\qquad \tilde{\chi}'_p(g)=\bar{\chi}_p(ad/p^{4\nu+2\mu})\,,$$

where $\nu = v_p(N)$, $\mu = v_p(\mathfrak{f}_\chi)$. Then for $\gamma, \gamma' \in U_p$, we see

$$\chi'_p(\gamma g \gamma') = \chi_p(\det(\gamma\gamma'))\chi'_p(g) . \tag{2.8}$$

We define a union of U-double cosets $\Xi(U_\chi W_L T_n)$ by

$$\Xi(U_\chi W_L T_n) = \prod_{p|M_1} \Xi_p(U_\chi) \prod_{p|M_2} \Xi_p(U_\chi W_L) \prod_{p|M_3} \Xi_p(W_L) \prod_{p\nmid M_1M_2M_3} \Xi_p(T_n) \times U_\infty ,$$

and for $g \in \Xi(U_\chi W_L T_n)$, put

$$\tilde{\chi}'(g) = \prod_{p|M_1} \tilde{\chi}_p(g_p) \prod_{p|M_2} \tilde{\chi}'_p(g_p) ,$$

where g_p is the p-th component of g. Corresponding to Lemma 2.2, we have

LEMMA 2.6. *The notation being as in Lemma* 2.2, *for a divisor L of N with* $(L, N/L) = 1$, *the union*

$$\Xi(U_\chi W_L T_n) \cap GL_2(\boldsymbol{Q}) = \bigcup_{ij} \bigcup_{k=1}^{d} \Gamma_0(N)\alpha_{ij}\eta_L\beta_k$$

is disjoint, where i and j runs through a complete system of representatives of $(\boldsymbol{Z}/\mathfrak{f}_\chi\boldsymbol{Z})^\times$.

Proof. As in the proof of Lemma 2.2, it is enough to prove the union $\prod_{p|M_1} \Xi_p(U_\chi) \prod_{p|M_2} \Xi_p(U_\chi W_L) \prod_{p|M_3} \Xi_p(W_L) = \bigcup_{ij} \prod_{p|M_1M_2M_3} U_p\alpha_{ij}\eta_L$ is disjoint. But this follows easily from the proof of Lemma 2.2 and the fact that $\Xi_p(U_\chi W_L) = \Xi_p(U_\chi)\eta_L$ and $\Xi_p(W_L) = U_p\eta_L$.

COROLLARY 2.7. *The notation being as above, then for $f \in S_\kappa(N)$, it holds*

$$f|U_\chi W_L T_n = C' \sum_{g \in \Gamma_0(N)\backslash \Xi(U_\chi W_L T_n)\cap GL_2(\boldsymbol{Q})} \chi'(g) f|[g]_\kappa$$
$$C' = \chi(n) \prod_{p|M_1} \chi_p(A'_p) \prod_{p|M_2} \chi_p(B'_p)/\mathfrak{g}(\bar{\chi})^2$$

where $A'_p = LN_1^3\mathfrak{f}_\chi^3/p^{3\nu+2\mu}$ and $B'_p = LN_1^3\mathfrak{f}_\chi^2/p^{4\nu+2\mu}$ for $\nu = v_p(N)$ and $\mu = v_p(\mathfrak{f}_\chi)$.

Proof. The right hand side of the above equality is independent of the choice of the representatives because of (2.2) and (2.8). If β_k is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then we see

$$\alpha_{ij}\beta_k = \begin{cases} \begin{pmatrix} a(ij\tilde{M}^2 - \tilde{M}M^2)L & b(ij\tilde{M}^2 - \tilde{M}M^2) - id\tilde{M}M \\ aj\tilde{M}^2ML & b\tilde{M}^2M - d\tilde{M}M^2 \end{pmatrix} & (\bmod M_1^4) \\ \begin{pmatrix} L(b(ij\tilde{M}^2 - \tilde{M}M^2) - id\tilde{M}M & a(ij\tilde{M}^2 - \tilde{M}M^2) \\ L(b\tilde{M}^2M - d\tilde{M}M^2) & aj\tilde{M}^2M \end{pmatrix} & (\bmod M_2^4) , \end{cases}$$

where $\tilde{M} = N_1$ and $M = \mathfrak{f}_\chi$. Hence we have

$$\tilde{\chi}'(\alpha_{ij}\beta_k) = \bar{\chi}(ij)C'^{-1} .$$

Our assertion follows from this and Lemma 2.6.

To express $\mathrm{tr}\, U_\chi W_L T_n$ in an explicit way, we prove

LEMMA 2.8. *Let $\Phi(X) = X^2 - sX + M_1^2 M_2^2 L f_\chi^4 n$, and for a prime divisor p of N, let $\nu = v_p(N)$ and $\mu = v_p(\mathfrak{f}_\chi)$. Then for an order Λ_p of $K(\Phi)_p$ containint $Z_p[\tilde{X}]$, the followings hold.*

(1) *For p dividing M_3, $C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(W_L) \neq \phi$ only if $s \equiv 0 \pmod{p^\nu}$ and $\Lambda_p = Z_p[\tilde{X}]$. When this condition is satisfied, one has*

$$|C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(W_L)/_{\widetilde{U_p}}| = 1 .$$

(2) *For p dividing M_2, $C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi W_L) \neq \phi$ only if $s \equiv 0 \pmod{p^{2\nu+\mu}}$ and $[\Lambda_p : Z_p[\tilde{X}]] = p^\rho$, where $\rho = \nu + 2\mu$. When this condition is satisfied, φ induces a bijective map*

$$\varphi : \tilde{\Omega}'_p \longrightarrow C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi W_L)/_{\widetilde{U_p}} ,$$

where $\tilde{\Omega}_p$ is a complete system of representatives modulo $p^{2\nu+2\mu}$ of the set $\{p^{2\nu+\mu}\xi \mid \xi \in Z_p^\times,\ \xi \not\equiv s/p^{2\nu+\mu} \pmod p\}$ $(\subset \Omega_p(\nu, \Phi, \Lambda_p))$ (resp. $\{p^{2\nu+\mu}\xi \mid \xi \in Z_p^\times,\ \xi \not\equiv s/p^{2\nu+\mu} \pmod p),\ \Phi(p^{2\nu+\mu}\xi) \not\equiv 0 \pmod{p^{3\nu+4\mu+1}}\}$ $(\subset \Omega_p(\nu, \Phi, \Lambda_p)) \cup \{p^{2\nu+\mu}\eta \mid \eta \in Z_p^\times,$ $\eta \not\equiv s/p^{2\nu+\mu} \pmod p),\ \Phi(p^{2\nu+\mu}\eta) \equiv 0 \pmod{p^{3\nu+4\mu+1}}\}$ $(\subset \Omega'_p(\nu, \Phi, \Lambda_p)))$ if $\nu > 2\mu$ (resp. if $\nu = 2\mu$).

Proof. The assertion (1) is contained in Yamauchi [18]. If $C_p(\nu, \Phi, \Lambda_p) \cap \Xi_p(U_\chi W_L) \neq \phi$, then we see that $s \equiv 0 \pmod{p^{2\nu+\mu}}$ and $[\Lambda_p : Z_p[\tilde{X}]] = p^\rho$, where $\rho = \nu + 2\mu$. Assume this condition is satisfied. First we treat the case where $\nu > 2\mu$. In this case, we note $v_p(b) = \nu + 2\mu$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Xi_p(U_\chi W_L)$, hence $\varphi'_\eta(\tilde{X}) \notin \Xi_p(U_\chi W_L)$. If $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi W_L)$ for $\xi \in \Omega_p(\nu, \Phi, \Lambda_p)$, then ξ is of the form $p^{2\nu+\mu}\xi'$ with $\xi' \in Z_p$. We note $v_p(\Phi(p^{2\nu+\mu}\xi')) = 3\nu + 4\mu$ for $\xi' \in Z_p$. Hence $\xi = p^{2\nu+\mu}\xi' \in \Omega_p(\nu, \Phi, \Lambda_p)$ for $\xi' \in Z_p$, and $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi W_L)$ if and only if $\xi \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$ and $s - \xi \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$. This prove the case $\nu > 2\mu$. Next assume $\nu = 2\mu$. Also in this case, if $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi W_L)$ (resp. $\varphi'_\eta(\tilde{X}) \in \Xi_p(U_\chi W_L)$), then $\xi = p^{2\nu+\mu}\xi'$ with $\xi' \in Z_p$ (resp. $\eta = p^{2\nu+\mu}\eta'$ with $\eta' \in Z_p$). For $\xi' \in Z_p$, put $\xi = p^{2\nu+\mu}\xi'$, then $v_p(\Phi(\xi)) \geq 3\nu + 4\mu$. Hence $\xi \in \Omega_p(\nu, \Phi, \Lambda_p)$, and $\varphi_\xi(\tilde{X}) \in \Xi_p(U_\chi W_L)$ if and only if $\xi \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$, $s - \xi \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$ and $\Phi(\xi) \not\equiv 0 \pmod{p^{3\nu+4\mu+1}}$. For $\eta = p^{2\nu+\mu}\eta'$ with $\eta' \in Z_p$,

$\eta \in \Omega'_p(\nu, \Phi, \Lambda_p)$ if and only if $\Phi(\eta) \equiv 0 \pmod{p^{3\nu+4\mu+1}}$, and for such $\eta' \in Z_p$ $\varphi'_\eta(\tilde{X}) \in \Xi_p(U_\chi W_L)$ if and only if $\eta \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$ and $s - \eta \not\equiv 0 \pmod{p^{2\nu+\mu+1}}$. Our assertion follows from this.

By means of this Lemma, in the similar way as in § 3 of [11], we obtain the following.

THEOREM 2.9. *The notation being as above, let L be a divisor of N such that $(L, N/L) = 1$. We write $f_\chi = F_1F_2$, where $F_1 = (\mathfrak{f}_\chi, M_1)$ and $F_2 = (\mathfrak{f}_\chi, M_2)$. Then we have*

$$\mathrm{tr}\, U_\chi W_L T_n \,|\, S_\kappa(N) = C'(t_e + t_h + t_p)\,,$$

where C' is the constant in Cor. 2.7, and t_e, t_h and t_p are given as follows.

(1) $$t_e = -\frac{1}{2}\sum_s \frac{\alpha^{\kappa-1} - \beta^{\kappa-1}}{\alpha - \beta}(LF_2^4)^{1-\kappa/2}\sum_f \prod_{p|M_1M_2M_4} c'_p(s, f) \times h(F_1^2(L^2F_2^{-2}s^2 - 4Ln)/f^2)\,.$$

Here s runs through all integers such that $L^2F_2^{-2}s^2 - 4Ln < 0$, and f runs through all positive integers which satisfy the condition $f^2 | (L^2F_2^{-2}s^2 - 4Ln)$, $(f, \mathfrak{f}_\chi L) = 1$ and $F_1^2(L^2F_2^{-2}s^2 - 4Ln)/f^2 \equiv 0$ or $1 \pmod 4$. For s, put $G_s(X) = X^2 - LF_2sX + LF_2^4n$, then α and β are the two roots of $G_s(X) = 0$. The number $c'_p(s, f)$ is given by

$$c'_p(s,f) = \begin{cases} \bar{\chi}_p(M_1^2F_1^4M_2^2/p^{2\nu+4\mu}) \displaystyle\sum_{\substack{\xi \bmod p^{\nu-\mu} \\ G_s(\xi)\equiv 0 \bmod p^{\nu-2\mu} \\ (resp.\ \xi \not\equiv LF_2 s \bmod p)}} \bar{\chi}_p(G_s(\xi)/p^{\nu-2\mu}) & \text{if } p|M_1 \text{ and } \nu > 2\mu \ (resp.\ \nu = 2\mu) \\ \bar{\chi}_p(M_1^2F_1^4M_2^2/p^{2\nu}) \displaystyle\sum_{\xi \bmod p^\mu} \bar{\chi}(\xi(LF_2s/p^{\nu+\mu} - \xi)) & \text{if } p|M_2 \\ |\tilde{\Omega}_p(\nu, G_s, \Lambda_p)| + |\tilde{\Omega}'_p(\nu, G_s, \Lambda_p)| & \text{if } p|M_4\,, \end{cases}$$

where $\nu = v_p(N)$, $\mu = v_p(\mathfrak{f}_\chi)$, and Λ_p is the order of $K(G_s)_p$ such that $[\Lambda_p : Z_p[\tilde{X}]] = p^\rho$ for $\rho = v_p(f)$.

(2) *If L is not a square, then $t_h = 0$. If L is a square, then one has*

$$t_h = -\sum_d \frac{d^{\kappa-1}}{n/d - d}(LF_2^4)^{1-\kappa/2}\sum_f \prod_{p|M_1M_2M_4} c'_p(\sqrt{L}\,F_2^2(d + n/d), f) \times \varphi(\sqrt{L}\,F_1(n/d - d)/f)\,,$$

where d runs through all positive integers such that $0 < d < \sqrt{n}$, $d|n$, and $d + n/d \equiv 0 \pmod{\sqrt{L}\,F_2^{-1}}$, and f runs through all positive integers which satisfy $f|(n/d - d)$ and $(f, \mathfrak{f}_\chi L) = 1$. $c'_p(\sqrt{L}\,F_2^2(d + n/d), f)$ is the same as in (1) *for $s = \sqrt{L}\,F_2^2(d + n/d)$.*

(3) t_p *does not vanish only if* $M_2 = F_2^2$, $M_3 = 1$ *or* 4, *and* M_1 *and* n *are squares. When this condition is satisfied,*

$$t_p = -\frac{n^{(\kappa-1)/2}}{2}\mathfrak{f}_\chi \prod_{p|M_1M_2}\left(1-\frac{1}{p}\right)\prod_{p|M_1M_2M_4} c'_p ,$$

where $c'_p = c'_p(2\sqrt{L}F_2^2\sqrt{n}, 1)$.

§ 3. A decomposition of $S_\kappa(N, \psi)$

Let χ be a character modulo N, and χ_0 the primitive character associated with χ. For χ, we define

$$U_\chi = U_{\chi_0},\ \mathfrak{g}(\chi) = \mathfrak{g}(\chi_0) .$$

For characters χ and χ' with prime power conductors, we have

THEOREM 3.1. *For positive integers N and κ, let ψ be a character modulo N such that $\psi(-1) = (-1)^\kappa$. Let p be a prime divisor of N, and χ, χ' characters with $\mathfrak{f}_\chi = p^\mu$, $\mathfrak{f}_{\chi'} = p^{\mu'}$ which satisfy the condition* (1.1). *Suppose $\mu \leq [v_p(N)/3]$, $\mu' \leq [v_p(N)/3]$, and $v_p(\mathfrak{f}_\psi) \leq [v_p(N)/3]$. Then for $f \in S_\kappa^0(N, \psi)$, it holds*

$$f|U_\chi U_{\chi'} = \psi_P(-1)\bar{\psi}_{N/P}(P)f|U_{\chi\chi'} \qquad \text{if } \chi \neq \bar{\chi}'$$
$$f|U_\chi U_{\chi'} = \psi_P(-1)^2\bar{\psi}_{N/P}(P)^2 f \qquad \text{if } \chi = \bar{\chi}' ,$$

where $P = p^\nu$ for $\nu = v_p(N)$.

Proof. We may assume χ and χ' are primitive. For integers i, j, i', and j', put

$$\alpha_{ij} = \begin{pmatrix}\mathfrak{f}_\chi & i\\ 0 & \mathfrak{f}_\chi\end{pmatrix}\eta_P\begin{pmatrix}\mathfrak{f}_\chi & j\\ 0 & \mathfrak{f}_\chi\end{pmatrix}\eta_P , \qquad \alpha'_{i'j'} = \begin{pmatrix}\mathfrak{f}_{\chi'} & i'\\ 0 & \mathfrak{f}_{\chi'}\end{pmatrix}\eta_P\begin{pmatrix}\mathfrak{f}_{\chi'} & j'\\ 0 & \mathfrak{f}_{\chi'}\end{pmatrix}\eta_P .$$

Then by the definition of U_χ and $U_{\chi'}$, we have

$$f|U_\chi U_{\chi'} = \frac{1}{\mathfrak{g}(\bar{\chi})^2\mathfrak{g}(\bar{\chi}')^2}\sum_{\substack{i,j \in (Z/p^\mu Z)^\times\\ i',j' \in (Z/p^{\mu'}Z)^\times}} \bar{\chi}(ij)\bar{\chi}'(i'j')f|[\alpha_{ij}\alpha'_{i'j'}]_\kappa .$$

Since $f|U_\chi U_{\chi'} = f|U_{\chi'}U_\chi$ for $f \in S_\kappa^0(N, \psi)$ by (1) of Prop. 1.4, we may assume $\mu \geq \mu'$.

Case I. First we assume $\mu > \mu'$. Let $\alpha_{ij}\alpha'_{i'j'} = -p^{\nu+2\mu'}\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, then by the assumption on $\mathfrak{f}_\chi$, $\mathfrak{f}_{\chi'}$ and $\mathfrak{f}_\psi$, we have

$$A \equiv -p^{\nu+2\mu} + i_0 j_0 p^{2\nu} \pmod{p^{\nu+3\mu}}$$

$$B \equiv -i_0 p^{\nu+\mu} \pmod{p^{\nu+2\mu}}$$
$$C \equiv j_0 p^{2\nu+\mu} \pmod{p^{2\nu+2\mu}}$$
$$D \equiv -p^{\nu+2\mu} \pmod{p^{\nu+3\mu}},$$

where $i_0 = i + p^{\mu-\mu'}i'$ and $j_0 = j + p^{\mu-\mu'}j'$. Since $\det \alpha_{ij}\alpha'_{i'j'}$ and $\det \alpha_{i_0j_0}$ are powers of p, by Lemma 2.1 we see $\beta = -p^{-\nu-2\mu'}\ \alpha_{ij}\alpha'_{i'j'}\alpha_{i_0j_0}^{-1} \in \Gamma_0(N)$ and $\psi_P(\beta) = 1$, where $\alpha_{i_0j_0} = \begin{pmatrix} p^\mu & i_0 \\ 0 & p^\mu \end{pmatrix}\eta_P\begin{pmatrix} p^\mu & j_0 \\ 0 & p^\mu \end{pmatrix}\eta_P$. For the other prime divisors of N, we have

$$\beta \equiv \begin{pmatrix} -P & * \\ 0 & -P^{-1} \end{pmatrix} \qquad \bmod (N/P)^4 .$$

Hence we obtain

$$f|[\alpha_{ij}\alpha'_{i'j'}]_\kappa = (-1)^\kappa \bar{\psi}_{N/P}(-P) f|[\alpha_{i_0j_0}]_\kappa .$$

Since $\psi(-1) = (-1)^\kappa$, we see

$$\begin{aligned} f|U_\chi U_{\chi'} &= \frac{\psi_P(-1)\bar{\psi}_{N/P}(P)}{\mathfrak{g}(\bar{\chi})^2\mathfrak{g}(\bar{\chi}')^2} \sum_{\substack{i_0, j_0 \bmod p^\mu \\ i', j' \bmod p^{\mu'}}} \bar{\chi}((i_0 - p^{\mu-\mu'}i')(j_0 - p^{\mu-\mu'}j')) \\ &\quad \times \bar{\chi}'(i'j') f|[\alpha_{i_0j_0}]_\kappa \\ &= \frac{\psi_P(-1)\bar{\psi}_{N/P}(P)}{\mathfrak{g}(\bar{\chi})^2\mathfrak{g}(\bar{\chi}')^2} \sum_{i', j' \bmod p^{\mu'}} \bar{\chi}((1 - p^{\mu-\mu'}i')(1 - p^{\mu-\mu'}j'))\bar{\chi}'(i'j') \\ &\quad \times \sum_{i_0, j_0 \bmod p^\mu} \bar{\chi}\bar{\chi}'(i_0j_0) f|[\alpha_{i_0j_0}]_\kappa . \end{aligned}$$

We note (c.f. Shimura [16, Lemma 8])

$$\frac{1}{\mathfrak{g}(\bar{\chi})\mathfrak{g}(\bar{\chi}')} \sum_{i' \bmod p^{\mu'}} \bar{\chi}(1 - p^{\mu-\mu'}i')\bar{\chi}'(i') = \frac{1}{\mathfrak{g}(\bar{\chi}\bar{\chi}')} .$$

Thus we obtain

$$f|U_\chi U_{\chi'} = \psi_P(-1)\bar{\psi}_{N/P}(P) f|U_{\chi\chi'} .$$

Case II. Next we assume $\mathfrak{f}_\chi = \mathfrak{f}_{\chi'} = \mathfrak{f}_{\chi\chi'}$. In the same way as above, we obtain

$$f|[\alpha_{ij}\alpha_{i'j'}]_\kappa = \psi_P(-1)\bar{\psi}_{N/P}(P) f|[\alpha_{i_0j_0}]_\kappa ,$$

where $i_0 = i + i'$ and $j_0 = j + j'$. We note $\alpha_{i_0j_0} \in \Xi(U_\chi T_1) \cap GL_2(\boldsymbol{Q})$ if and only if i_0 and j_0 are prime to p. Taking notice of (c.f. ibid.)

$$\frac{1}{\mathfrak{g}(\bar{\chi})\mathfrak{g}(\bar{\chi}')} \sum_{i' \bmod p^\mu} \bar{\chi}(1 - i')\bar{\chi}'(i') = \frac{1}{\mathfrak{g}(\bar{\chi}\bar{\chi}')} ;$$

we have

$$f|U_\chi U_{\chi'} = \psi_P(-1)\bar{\psi}_{N/P}(P)f|U_{\chi\chi'} + S_1 + S_2 + S_3\ ,$$

where

$$S_k = \frac{\psi_P(-1)\bar{\psi}_{N/P}(P)}{\mathfrak{g}(\bar{\chi})^2\mathfrak{g}(\bar{\chi}')^2}\sum \bar{\chi}((i_0 - i')(j_0 - j'))\bar{\chi}'(i')\bar{\chi}'(j')f|[\alpha_{i_0j_0}]_\kappa\ .$$

Here the summation is extended over i_0, j_0, i', j' modulo p^μ which satisfy the condition (1) $i_0 \not\equiv 0 \pmod p$, $j_0 \equiv 0 \pmod p$, (2) $i_0 \equiv 0 \pmod p$, $j_0 \not\equiv 0 \pmod p$ or (3) $i_0 \equiv 0 \pmod p$, $j_0 \equiv \pmod p$ according as $k = 1, 2,$ or 3. We show $S_1 = S_2 = S_3 = 0$. Let $f = \sum_{m\geq 1} a_m e^{2\pi imz}$ be the Fourier expansion of f. In the case of S_1, put $i_0 = pu$. Then we see

$$\begin{aligned}
&\sum_{\substack{i' \bmod p^\mu \\ u \bmod p^{\mu-1}}} \bar{\chi}(pu - i')\bar{\chi}'(i')f\left|\left[\begin{pmatrix} P & pu \\ 0 & P\end{pmatrix}\right]_\kappa\right. \\
&= \sum_m a_m \sum_{u,i'} \bar{\chi}(pu - i')\bar{\chi}'(i')e^{2\pi ipum/p^\mu}e^{2\pi imz} \\
&= \sum_m a_m \sum_u \bar{\chi}(pu - 1) \sum_{(i',p)=1} \bar{\chi}\bar{\chi}'(i')e^{2\pi ipum/p^\mu}e^{2\pi imz} \\
&= 0\ ,
\end{aligned}$$

since the conductor of $\chi\chi'$ is p^μ. This shows $S_1 = 0$. We can treat the cases of S_2 and S_3 in the same way, and we omit the proof.

Case III. Finally we assume $\mathfrak{f}_\chi = \mathfrak{f}_{\chi'} > \mathfrak{f}_{\chi\chi'}$. Put $\chi'' = \chi\chi'$, then $\chi' = \bar{\chi}\chi''$. By Case I, we have $U_{\chi'} = \psi_P(-1)\psi_{N/P}(P)U_{\bar{\chi}}U_{\chi''}$. If we prove $U_\chi U_{\bar{\chi}} = (\psi_P(-1)\bar{\psi}_{N/P}(P))^2$, we obtain $U_\chi U_{\chi'} = \psi_P(-1)\psi_{N/P}(P)U_\chi U_{\bar{\chi}}U_{\chi''} = \psi_P(-1)\bar{\psi}_{N/P}(P)U_{\chi''}$. Hence it is enough to show $U_\chi U_{\bar{\chi}} = (\psi_P(-1)\bar{\psi}_{N/P}(P))^2$, and we may assume $\chi' = \bar{\chi}$. As in the case II, we have

$$f|U_\chi U_{\bar{\chi}} = \frac{\psi_P(-1)\bar{\psi}_{N/P}(P)}{(\mathfrak{g}(\bar{\chi})\mathfrak{g}(\chi))^2}(T_1 + T_2 + T_3 + T_4)\ ,$$

where

$$T_k = \sum \chi((i_0 - i')(j_0 - j'))\chi(i'j')f|[\alpha_{i_0j_0}]_\kappa\ .$$

Here the summation is extended over i_0, j_0, i', j' modulo p^μ which satisfy the condition (1) $i_0 \not\equiv 0 \pmod p$, $j_0 \not\equiv 0 \pmod p$ (2) $i_0 \not\equiv 0 \pmod p$, $j_0 \equiv 0 \pmod p$, (3) $i_0 \equiv 0 \pmod p$, $j_0 \not\equiv 0 \pmod p$, or (4) $i_0 \equiv 0 \pmod p$, $j_0 \equiv 0 \pmod p$ according as $k = 1, 2, 3,$ or 4. Let $f = \sum_{m\geq 1} a_m e^{2\pi imz}$ be the Fourier expansion of f, then $a_m = 0$ if p divides m. We see

$$T_1 = (\sum_{i'} \chi(1 - i')\chi(i'))^2 \sum_{\substack{(i_0,p)=1 \\ (j_0,p)=1}} f|[\alpha_{i_0j_0}]_\kappa$$

and

$$\sum_{i_0 \in (\boldsymbol{Z}/p^\mu \boldsymbol{Z})^\times} f\left|\left[\begin{pmatrix} p^\mu & i_0 \\ 0 & p^\mu \end{pmatrix}\right]_\kappa = \begin{cases} -f & \text{if } \mu = 1 \\ 0 & \text{otherwise}, \end{cases}\right.$$

$$\sum_{i' \bmod p^\mu} \chi(1-i')\bar{\chi}(i') = -\chi(-1) \qquad \text{if } \mu = 1 .$$

From this we obtain

$$T_1 = \begin{cases} f|[\eta_P^2]_\kappa & \text{if } \mu = 1 \\ 0 & \text{otherwise}. \end{cases}$$

In the similar way, we can verify

$$T_2 = T_3 = \begin{cases} (p-1)f|[\eta_P^2]_\kappa & \text{if } \mu = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$T_4 = \begin{cases} (p-1)^2 f|[\eta_P^2]_\kappa & \text{if } \mu = 1 \\ p^{2\mu} f|[\eta_P^2]_\kappa & \text{otherwise}. \end{cases}$$

Our assertion follows from this and Lemma 1.1. This completes the proof.

By the above theorem and Cor. 1.3, we obtain

COROLLARY 3.2. *Let χ and χ' be the characters which satisfy* (1.1). *Suppose $v_p(\mathfrak{f}_\chi) \leq v_p(N)/3$, $v_p(\mathfrak{f}_{\chi'}) \leq v_p(N)/3$, and $v_p(\mathfrak{f}_\chi) \leq v_p(N)/3$ for each prime divisor p of $\mathfrak{f}_\chi \mathfrak{f}_{\chi'}$. Then for $f \in S_\kappa^0(N, \psi)$, it holds*

$$f|\tilde{U}_\chi \tilde{U}_{\chi'} = f|\tilde{U}_{\chi\chi'} .$$

Let M be a divisor of N such that $M^3|N$, and assume $3v_p(\mathfrak{f}_\psi) \leq v_p(N)$ for any pirme divisor p of M. Let $X(M)$ be the group of all characters defined modulo M, and $\tilde{U}$ the group consisting of operators $\tilde{U}_\chi$ acting on $S_\kappa^0(N, \psi)$ for $X(M)$. Then Cor. 3.2 says that the map $\mathfrak{U}: \chi \to \tilde{U}_\chi$ gives a homomorphism from $X(M)$ to $\tilde{U}$. By means of this homomorphism, we can decompose $S_\kappa^0(N, \psi)$ as follows;

$$S_\kappa^0(N, \psi) = \bigoplus_{a \in (\boldsymbol{Z}/M\boldsymbol{Z})^\times} S_\kappa^0(N, \psi, a) ,$$

where

$$S_\kappa^0(N, \psi, a) = \{f \in S_\kappa^0(N, \psi) | f|\tilde{U}_\chi = \chi(a)f \qquad \text{for } \chi \in X(M)\} .$$

On these subspace, the Hecke operator T_n acts and the trace of T_n on them are given by

$$\operatorname{tr} T_n | S_\kappa^0(N, \psi, a) = \frac{1}{|(\boldsymbol{Z}/M\boldsymbol{Z})^\times|} \sum_{\chi \in X(M)} \bar{\chi}(a) \operatorname{tr} \tilde{U}_\chi T_n | S_\kappa^0(N, \psi) .$$

the trace $\operatorname{tr}\tilde{U}_\chi T_n | S_\kappa^0(N, \psi)$ are given by Hijikata [8] for the trivial χ and by Th. 2.5 in this paper for general χ. In the case where ψ is the trivial character, we can consider also the action of W_L to decompose $S_\kappa(N)$. Let $\tilde{W}$ denote the group of all W_L for $L|N$, and $E(W)$ the character group of $\tilde{W}$. We define $S_\kappa^0(N, a, e)$ for $a \in (Z/MZ)^\times$ and $e \in E(W)$ by

$$S_\kappa^0(N, a, e) = \{f \in S_\kappa^0(N) | f | \tilde{U}_\chi = \chi(a) f \quad \text{for } \chi \in X(M)\,, \quad f | W_L = e(W_L) f \quad \text{for } W_L \in E(W)\}\,.$$

Then we have

$$S_\kappa^0(N) = \bigoplus_{a \in (Z/MZ)^\times} \bigoplus_{e \in E(W)} S_\kappa^0(N, a, e)\,,$$

and the trace of T_n on $S_\kappa^0(N, a, e)$ is expressed as follows;

$$\operatorname{tr} T_n | S_\kappa^0(N, a, e) = \frac{1}{|(Z/MZ)^\times||E(W)|} \sum_{\substack{\chi \in X(M) \\ W \in \tilde{W}}} \bar{\chi}(a) \bar{e}(W) \operatorname{tr} \tilde{U}_\chi W_L T_n | S_\kappa^0(N)\,.$$

a formula for $\operatorname{tr} U_\chi W T_n$ is given by Yamauchi [18] for the trivial χ and by Th. 2.9 for the general χ.

Now we take $N = p^\nu$ with a prime p and a positive integer $\nu \geq 3$ and ψ the trivial character. Under such a condition, we have given in [9] a decomposition of $S_\kappa^0(p^\nu)$ into four subspaces S_{I}, S_{II}, $S_{\mathrm{II}_\varepsilon}$, S_{III}. We compare this decomposition with that given above. Put $M = p^{[\nu/3]}$. Then for example, the subspace S_{I} is defined by

$$S_{\mathrm{I}} = \{f \in S_\kappa^0(N) | f | U_\varepsilon = f,\ f | W_N = f\}\,,$$

where ε is the quadratic residue symbol modulo p. This space is expressed by our spaces $S_\kappa^0(N, a, e)$ as follows;

$$S_{\mathrm{I}} = \bigoplus_{\substack{a \in (Z/MZ)^\times \\ \varepsilon(a) = 1}} S^0(N, a, 1)\,,$$

where 1 denotes the trivial character of $\tilde{W}$. This shows that even in the case where $\nu = 3$ our decomposition of $S_\kappa^0(N)$ gives a finer one that in [11]. In the next section, we give a numerical example in the case where $p = 11$, $\kappa = 2$, and $\nu = 3$.

We prove two more properties of U_χ.

PROPOSITION 3.3. *The notation being as above, let f be a primitive form in $S_\kappa^0(N, \psi)$. For a character χ with $\mathfrak{f}_\chi = p^\mu$ which satisfies* (1.1), *let $f | \tilde{U}_\chi = c_\chi f$. For $\sigma \in \operatorname{Gal}(\bar{Q}/Q)$ and $\zeta = e^{2\pi i/p^\mu}$, let $\zeta^\sigma = \zeta^n$ with $n \in Z$, and for $f = \sum_{m \geq 1} a_m e^{2\pi i m z}$, put $f^\sigma = \sum_{m \geq 1} a_m^\sigma e^{2\pi i m z}$. Then it holds*

$$f^{\sigma}|\tilde{U}_{\chi}\sigma = \chi(n^2)^{\sigma}(\sqrt{p}^{\,\sigma}/\sqrt{p})^{\kappa}c_{\chi}^{\sigma}f^{\sigma} .$$

Proof. Let $G_+ = \{x \in GL_2(\boldsymbol{Q}_A) \mid \det x_\infty > 0\}$, and $\boldsymbol{Q}_{ab}$ the maximal abelian extension of $\boldsymbol{Q}$. Let ρ be a homomorphism of G_+ onto $\mathrm{Gal}(\boldsymbol{Q}_{ab}/\boldsymbol{Q})$ obtained by defining $\rho(x)$ to be the action of $(\det x)^{-1}$ on $\boldsymbol{Q}_{ab}$. Let G be a subgroup of $G_+ \times \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ given by

$$G = \{(x, \sigma) \in G_+ \times \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \mid \rho(x) = \sigma \text{ on } \boldsymbol{Q}_{ab}\} .$$

Then Shimura [17, Th. 1.5] defined an action of G on modular forms. We denote the action of (x, σ) by $f^{(x,\sigma)}$. Let t be an element of $\prod_p \boldsymbol{Z}_p^{\times}$ such that $\rho(x) = \sigma$ on $\boldsymbol{Q}_{ab}$ for $x = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. Let α_{ij} and $\tilde{\chi}$ be the same as in the proof of Th. 3.1, and consider the action of (x, σ) on the both sides of

$$\frac{\psi_P(-1)\bar{\psi}_{N/P}(P)\chi(N/P)}{\mathfrak{g}(\bar{\chi})^2} \sum_{i,j} \tilde{\chi}(\alpha_{ij}) f|[\alpha_{ij}]_\kappa = c_\chi f ,$$

where $P = p^\nu$. Then the right hand side becomes $c_\chi^\sigma f^\sigma$. Observe that $(\mathfrak{g}(\bar{\chi})^2)^\sigma = \chi(n^2)^\sigma \mathfrak{g}(\bar{\chi}^\sigma)^2$ and $f^{(\alpha_{ij},1)(x,\sigma)} = (f^\sigma)^{(x^{-1}\alpha_{ij}x,1)}$. Choose $t_0 \in \boldsymbol{Z}$ such that $t_0 \equiv t_q \pmod{q^4}$ for each prime $q|N$. Let i' and j' be integers such that $i' \equiv t_0 i \pmod{P^4}$ and $t_0 j' \equiv j \pmod{P^4}$, and let A be an integer such that $A \equiv p^{\nu-\mu}(-t_0 j + j') \pmod{(N/P)^4}$ and $A \equiv 0 \pmod{P^4}$. Then we see

$$x^{-1}\alpha_{ij}x \equiv \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \alpha_{i'j'} \qquad \pmod{N^4} .$$

Hence $f^{(\alpha_{ij},1)(x,\sigma)} = (f^\sigma)^{(\alpha_{i'j'},1)}$, and we obtain

$$(f|[\alpha_{ij}]_\kappa)^{(x,\sigma)} = (\sqrt{p}^{\,\sigma}/\sqrt{p})^\kappa f^\sigma |[\alpha_{i'j'}]_\kappa .$$

Noting $\chi(\alpha_{ij}) = \chi(\alpha_{i'j'})$, we obtain

$$\frac{\psi_P^\sigma(-1)\bar{\psi}_{N/P}^\sigma(P)\chi^\sigma(N/P)}{\mathfrak{g}(\bar{\chi}^\sigma)^2} \sum_{i,j} \tilde{\chi}^\sigma(\alpha_{ij}) f^\sigma |[\alpha_{ij}]_\kappa = \chi(n^2)^\sigma(\sqrt{p}^{\,\sigma}/\sqrt{p})^\kappa c_\chi^\sigma f^\sigma .$$

Since $f \in S_\kappa^0(N, \psi^\sigma)$, this prove our proposition.

COROLLARY 3.4. *Let f be a primitive form in $S_\kappa^0(N, \psi)$, and K_f the field generated by all the Fourier coefficients a_m of f over $\boldsymbol{Q}$. Suppose $v_p(\mathfrak{f}_\psi) \leq v_p(N)/3$ and $\mu = [v_p(N)/3] \geq 1$ for a prime divisor p of N. Then K_f contains $F_{p^\mu} = \boldsymbol{Q}(e^{2\pi i/p^\mu} + e^{-2\pi i/p^\mu})$ (resp. $F_{p^{\mu-1}}$) if κ is even and p is odd (resp. $p = 2$), and $K_f(\sqrt{p})$ contains F_{p^μ} (resp. $F_{p^{\mu-1}}$) if κ is odd and p is odd (resp. $p = 2$).*

Proof. We prove only the case where κ is even and p is odd. The other case can be treated in a similar way. In this case, it is enough to prove that for $\sigma \in \mathrm{Gal}(\bar{Q}/Q)$ $\sigma | F_{p^\mu}$ = the identity if $\sigma | K_f$ = the identity. Assume $\sigma | K_f$ is the identity, then $f^\sigma = f$ and $\psi^\sigma = \psi$. In the above notation, we may assume $f \in S_\kappa^0(N, \psi, a)$ for some a. Then $c_\chi = \chi(a)$ for $\chi \in X(p^\mu)$. From this and the above proposition, it follows

$$\chi(a)^\sigma = \chi(n^2)^\sigma \chi(a)^\sigma (\sqrt{p}^{\,\sigma}/\sqrt{p})^\kappa ,$$

for all $\chi \in X(p^\mu)$, where n is an integer such that $(e^{2\pi i/p^\mu})^\sigma = e^{2\pi i n/p^\mu}$. Since κ is even, $\chi(n^2) = 1$ for all $\chi \in X(p^\mu)$, and $n^2 \equiv 1 \pmod{p^\mu}$. If p is odd, this implies $n = \pm 1 \pmod{p^\mu}$ hence $\sigma | F_{p^\mu}$ = the identity. This proves our corollary.

PROPOSITION 3.5. *The notation being as in Prop.* 3.3, *assume* $\nu - 2\mu > 0$ *and* $v_p(\mathfrak{f}_\psi) < \nu - 2\mu$ *for* $\nu = v_p(N)$ *and* $\mu = v_p(\mathfrak{f}_\chi)$. *Then it holds*

$$f | U_\chi W_P = f | W_P U_\chi ,$$

where $P = p^\nu$.

Proof. First we note η_P normalizes the set $\Xi(U_\chi T_1) \cap GL_2(Q)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Xi(U_\chi T_1) \cap GL_2(Q)$, we note

$$\eta_P^{-1} g \eta_P = \begin{cases} \begin{pmatrix} d & -c/p^\nu \\ -bp^\nu & a \end{pmatrix} & \pmod{P^3} \\ \begin{pmatrix} a & b/p^\nu \\ cp^\nu & d \end{pmatrix} & \pmod{(N/P)^4} , \end{cases}$$

and $\bar{\psi}_p(-d/p^{\nu+2\mu}) = \psi_p(-a/p^{\nu+2\mu})$ by the assumption on ψ. Our assertion follows from this and Cor. 2.3.

§ 4. Numerical examples and a congruence between cusp forms

We shall gives examples of characteristic polynomials of Hecke operators taking $N = 11^3$, $\kappa = 2$ and ψ = the trivial character and discuss a congruence property between cusp forms. We use the notation in § 3. Let S_{III} be the subspace of $S_\kappa^0(p^\nu)$ given by

$$S_{\mathrm{III}} = \{ f \in S_\kappa^0(N) \,|\, f | U_\varepsilon = f,\ f | W_P = -f \} ,$$

where ε is the quadratic residue symbol modulo p and $P = p^\nu$. In our case, we find $\dim S_{\mathrm{I}} = 15$ and $\dim S_{\mathrm{III}} = 35$. By means of the decomposition introduced in § 3, these subspaces can be written as follows;

$$S_{\mathrm{I}} = \bigoplus_{\substack{a \bmod 11 \\ \varepsilon(a)=1}} S_2(11^3, a, 1) , \quad S_{\mathrm{III}} = \bigoplus_{\substack{a \bmod 11 \\ \varepsilon(a)=1}} S_2(11^3, a, -1) ,$$

where -1 denotes the non-trivial homomorphism from $\{W_P, 1\}$ to $\{\pm 1\}$. For a such that $\varepsilon(a)=1$, we find $\dim S_2(11^3, a, 1) = 3$ and $\dim S_2(11^3, a, -1) = 7$. Taking $a=4$, we give characteristic polynomial of Hecke operator T_n acting on these subspace for some n.

n	$\varepsilon(n)$	a_n	$f_{T_n}(X)$	$N(f_{T_n}(a_n))$
2	-1	0	$X^2+\alpha^3-3\alpha-3$	199
3	1	$-\alpha^4-2\alpha^3+3\alpha^2+5\alpha-2$	$(X-\alpha^3+3\alpha)^2$	199^2
5	1	$-\alpha^4+5\alpha^2-\alpha-5$	$(X-\alpha+1)^2$	199^2
199	1	$-6\alpha^4-13\alpha^3+30\alpha^2+39\alpha-18$	$(X-4\alpha^4+8\alpha^3+13\alpha^2-16\alpha+11)^2$	$(11\cdot 23\cdot 43\cdot 199)^2$

Here $\alpha = e^{2\pi i/11} + e^{-2\pi i/11}$ and N denotes the norm from $F_{11} = \boldsymbol{Q}(\alpha)$ to $\boldsymbol{Q}$. For an explanation of the table, we remark that $S_2(11^3, 4, 1)$ contains a primitive form θ_{I} associated with a Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$. a_n denotes the n-th Fourier coefficient of θ_{I}, that is, the eigenvalue for T_n. $f_{T_n}(X)$ denotes the characteristic polynomial for T_n on the orthogonal complement S_{I}^0 of the one dimensional subspace spanned by θ_{I}. We note $N(f_{T_n}(a_n))$ is divided by the prime 199 in our table and this suggest a congruence between θ_{I} and a primitive form $f \in S_{\mathrm{I}}^0$ modulo a prime ideal $\mathfrak{p}$ in K_f which divides 199. In fact, Prop. 4.2 in [11] implies such a congruence, and this proposition has been proved as an application of the Shimura's theory on the construction of class fields over real quadratic fields [15].

Now we take $S_2(11^3, 4, -1)$. This space also contains a primitive form θ_{III} associated with a Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$. Let b_n be the n-th Fourier coefficients of θ_{III}, and S_{III}^0 the orthogonal complement of the one dimensional subspace spanned by θ_{III}. We denote by $g_{T_n}(X)$ the characteristic polynomial of T_n on S_{III}^0.

n	$\varepsilon(n)$	$g_{T_n}(X)$
2	-1	$X^6-(\alpha^3-3\alpha+12)X^4+(-2\alpha^4+7\alpha^3+8\alpha^2-21\alpha+35)X^2-(-14\alpha^4+4\alpha^3+56\alpha^2-18\alpha-4)$
3	1	$(X^3-(-\alpha^4-\alpha^3+3\alpha^2+3\alpha)X^2+(-\alpha^4-2\alpha^3+\alpha^2+5\alpha-2)X-(2\alpha^4-7\alpha^2-2\alpha+3))^2$
5	1	$(X^3-(2\alpha^4-7\alpha^2+4)X^2+(\alpha^4-\alpha^3-3\alpha^2-5)X-(-8\alpha^4-5\alpha^3+28\alpha^2+11\alpha-15))^2$

n	b_n	$N(g_{T_n}(b_n))$
2	0	$2^2\cdot 99527$
3	$\alpha^4+2\alpha^3-3\alpha^2-6\alpha+2$	$(11\cdot 99527)^2$
5	$-2\alpha^4+7\alpha^2+\alpha-1$	$(1429\cdot 99527)^2$

Here α and N are as above. This table also suggests a congruence between θ_{III} and a primitive form g in S^0_{III} modulo a prime ideal $\mathfrak{p}$ in K_g which divides 99527. By virtue of the theory of Shimura, we may prove this congruence if we can compute $g_{T_{99527}}$. However, it is difficult. So we proceed in quite another way.

For positive integers N and λ, let ψ be a character modulo N such that $\psi(-1)=(-1)^\kappa$. For a prime divisor p of N, put $\nu=v_p(N)$, $\nu_0=[(\nu-1)/2]$, and $M=N/p^\nu$. Let κ' and be κ'' positive integers such that $\kappa=\kappa'+\kappa''$ and ω be a character modulo p such that $\omega(-1)=(-1)^{\kappa''}$. First we prove

LEMMA 4.1. *The notation being as above, for a primitive form $f\in S^0_{\kappa'}(N,\psi\omega)$ and $g\in G_{\kappa''}(pM,\bar{\omega})$, put $F(z)=g(p^{\nu_0}z)f(z)$. Let χ be a character with $\mathfrak{f}_\chi=p^\mu$, and assume $1\le\mu\le\nu_0$, and $v_p(\mathfrak{f}_\psi)\le v_p(N)/3$. Then $F(z)$ belongs to $S_\kappa(N,\psi)$, and it holds*

$$F(z)\,|\,\tilde{U}_\chi=g(p^{\nu_0}z)(f(z)\,|\,\tilde{U}_\chi)\,.$$

Proof. The first assertion is obvious. We prove the above equality. By the assumption $1\le\mu\le\nu_0$, we have

$$F(z)\,|\,R_\chi=g(p^{\nu_0}z)(f(z)\,|\,R_\chi)\,.$$

Let $P=p^\nu$, then we see $g(p^{\nu_0}z)|W_P=h(p^{\nu_0}z)$ for $h\in M_{\kappa''}(pM,\omega)$, since we have

$$\begin{pmatrix}p^{\nu_0}&0\\0&1\end{pmatrix}\eta_P\equiv\begin{cases}p^{\nu_0}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}p&0\\0&1\end{pmatrix}\begin{pmatrix}p^{\nu-\nu_0-1}&0\\0&1\end{pmatrix} & (\mathrm{mod}\ P^4)\\ p^{\nu_0}\begin{pmatrix}p&0\\0&1\end{pmatrix}\begin{pmatrix}p^{\nu-\nu_0-1}&0\\0&1\end{pmatrix} & (\mathrm{mod}\ (N/P)^4)\,,\end{cases}$$

and $\nu-\nu_0-1\ge\nu_0$. Hence we obtain

$$\begin{aligned}F(z)|\,U_\chi&=(h(p^{\nu_0}z)(f(z)\,|\,R_\chi W_P))\,|\,R_\chi W_P\\&=g(p^{\nu_0}z)|W_P^2(f(z)|U_\chi)\\&=\omega(-1)g(p^{\nu_0}z)(f(z)|U_\chi)\,.\end{aligned}$$

This proves our lemma.

COROLLARY 4.2. *The notation being as above, let* $N = p^{\nu}$ *with an odd prime* p *and* $\nu \geq 3$. *Then* $F(z)$ *is contained in* $S_{\kappa}^{0}(N, \psi)$.

Proof. This follows from (2), (3) of Prop. 1.4, and the above Lemma 4.1 by taking, for example, $\chi = \varepsilon$.

We apply this Lemma taking as f a primitive form associated with a Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$ and as g an Eisenstein series. First of all, we study the eigenvalues for $\tilde{U}_{\chi}$ of primitive forms associated with Grössencharacters. Let p be a prime congruent to 3 modulo 4, and a Grössencharacter of $\boldsymbol{Q}(\sqrt{-p})$ which satisfies

$$\lambda((a)) = \left(\frac{a}{|a|}\right)^{u} \tag{4.1}$$

for $a \in \boldsymbol{Q}(\sqrt{-p})$ with $a \equiv 1 \pmod{(\sqrt{-p})^{\alpha}}$, where α is a positive integer. For λ with $u = \kappa - 1$ put

$$\theta_{\kappa}(z) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N\mathfrak{a}^{(\kappa-1)/2} e^{2\pi i N\mathfrak{a} z} ,$$

where the summation is extended over all integral ideal of $\boldsymbol{Q}(\sqrt{-p})$ prime to $(\sqrt{-p})$. Then it is known [14] that θ_{λ} belongs to $S_{\kappa}(P, \psi)$ for $P = p^{\alpha+1}$ and a character ψ modulo P defined by

$$\psi(a) = \lambda((a))\left(\frac{-p}{a}\right) \qquad \text{for } 0 \neq a \in \boldsymbol{Z},$$

and θ_{λ} is a primitive form in $S_{\kappa}^{0}(P, \psi)$ if λ is of conductor $(\sqrt{-p}^{\alpha})$.

PROPOSITION 4.3. *Let* λ *be a Grössencharacter of* $\boldsymbol{Q}(\sqrt{-p})$ *of conductor* $(\sqrt{-p}^{\alpha})$ *for a positive integer* α, *and* χ *a character with* $\mathfrak{f}_{\chi} = p^{\mu}$. *Assume* $\mu \leq \alpha/2$. *Then it holds*

$$\theta_{\kappa} | \tilde{U}_{\chi} = (\mathfrak{g}(\lambda\chi \circ N)/\mathfrak{g}(\lambda))\theta_{\lambda} ,$$

where N *is the norm from* $\boldsymbol{Q}(\sqrt{-p})$ *to* $\boldsymbol{Q}$, *and* $\mathfrak{g}(\lambda\chi \circ N)$ *and* $\mathfrak{g}(\lambda)$ *are the Gauss sum of* $\lambda\chi \circ N$ *and* λ *respectively.*

Proof. For a Grössencharacter λ' of $\boldsymbol{Q}(\sqrt{-p})$ with the conductor $(\sqrt{-p}^{\alpha})$, by means of the functional equation of the L-function of λ', we obtain

$$\theta_{\lambda'} | W_P = (\sqrt{-1})^{2\kappa+1} \frac{\mathfrak{g}(\lambda')}{p^{\alpha/2}} \theta_{\bar{\lambda}'} ,$$

where $P = p^{\alpha+1}$. Observe $\theta_{\lambda'} | R_\chi = \theta_{\lambda'\chi\circ N}$. From this, it follows $\theta_\lambda | U_\chi = -(\mathfrak{g}(\lambda\chi\circ N)\mathfrak{g}(\bar{\lambda})/p^\alpha)\theta_\lambda$. Since $\mathfrak{g}(\lambda)\mathfrak{g}(\bar{\lambda}) = (-1)^{\kappa-1}p^\alpha$, we obtain

$$\theta_\kappa | U_\chi = (-1)^\kappa(\mathfrak{g}(\lambda\chi\circ N)/\mathfrak{g}(\lambda))\theta_\lambda .$$

Since $\psi(-1) = (-1)^\kappa$, this proves the proposition.

PROPOSITION 4.4. *The notation being as in Prop.* 4.3, *put* $c_\lambda(\chi) = \mathfrak{g}(\lambda\chi\circ N)/\mathfrak{g}(\lambda)$. *If* η *is a Grössencharacter of* $\boldsymbol{Q}(\sqrt{-p})$ *of conductor* $(\sqrt{-p})$ *which satisfies* (4.1) *for* $u = k' - 1$, *then it holds*

$$c_{\lambda\eta}(\chi) = c_\lambda(\chi) ,$$

for any character χ *which satisfies* $\mu \leq \alpha/2$.

Proof. To prove this proposition, it is enough to show $\mathfrak{g}(\lambda\eta\chi\circ N)/\mathfrak{g}(\lambda\chi\circ N) = \mathfrak{g}(\lambda\eta)/\mathfrak{g}(\lambda)$. Let $\mathfrak{o}$ be the ring of integers of $\boldsymbol{Q}(\sqrt{-p})$, and for $a \in \mathfrak{o}$, put

$$\lambda_0(a) = \lambda((a))\left(\frac{a}{|a|}\right)^{-(\kappa-1)}, \quad \eta_0(a) = \eta((a))\left(\frac{a}{|a|}\right)^{-(\kappa'-1)} .$$

Then we have

$$\mathfrak{g}(\lambda\eta) = (b/|b|)^{\kappa+\kappa'-2} \sum_{a\in\mathfrak{o} \bmod (\sqrt{-p}^\alpha)} \lambda_0\eta_0(a)e^{2\pi i \operatorname{tr}(a/b)} ,$$

where $b = \sqrt{-p}^{\alpha+1}$ and tr denotes the trace from $\boldsymbol{Q}(\sqrt{-p})$ to $\boldsymbol{Q}$. Since the function $\lambda_0\eta_0(1 + \sqrt{-p}^{\alpha-1}x) = \lambda_0(1 + \sqrt{-p}^{\alpha-1}x)$ is additive in $x \in \mathfrak{o}$, we can find an element y in $\mathfrak{o}$ such that

$$\lambda_0(1 + \sqrt{-p}^{\alpha-1}x) = e^{2\pi i \operatorname{tr}(xy/\sqrt{-p}^2)} ,$$

for $x \in \mathfrak{o}$. Then we see

$$\begin{aligned}\sum_{a\in\mathfrak{o} \bmod (\sqrt{-p}^\alpha)} \lambda_0\eta_0(a)e^{2\pi i \operatorname{tr}(a/b)} &= \sum_{a\in\mathfrak{o} \bmod (\sqrt{-p}^{\alpha-1})} \lambda_0\eta_0(a)e^{2\pi i \operatorname{tr}(a/b)} \\ &\quad\times \sum_{x\in\mathfrak{o} \bmod (\sqrt{-p})} \lambda_0(1 + \sqrt{-p}^{\alpha-1}x)e^{2\pi i \operatorname{tr}(ax/\sqrt{-p}^2)} \\ &= p \sum_{\substack{a\in\mathfrak{o} \bmod (\sqrt{-p}^{\alpha-1}) \\ a+y\equiv 0 \bmod (\sqrt{-p})}} \lambda_0\eta_0(a)e^{2\pi i \operatorname{tr}(a/b)} \\ &= \eta_0(-y) \sum_{a\in\mathfrak{o} \bmod (\sqrt{-p}^\alpha)} \lambda_0(a)e^{2\pi i \operatorname{tr}(a/b)} .\end{aligned}$$

Hence we obtain

$$\mathfrak{g}(\lambda\eta) = (b/|b|)^{\kappa'-1}\eta_0(-y)\mathfrak{g}(\lambda) . \tag{4.2}$$

If we note

$$N(1 + \sqrt{-p}^{\alpha-1}x) \equiv 1 \pmod{p^{[\alpha/2]}} ,$$

we see the above argument also gives

$$\mathfrak{g}(\lambda\eta\chi \circ N) = (b/|b|)^{\kappa'-1}\eta_0(-y)\mathfrak{g}(\lambda\chi \circ N) . \tag{4.3}$$

From (4.2) and (4.3), we obtain $\mathfrak{g}(\lambda\eta\chi \circ N)/\mathfrak{g}(\lambda\chi \circ N) = \mathfrak{g}(\lambda\eta)/\mathfrak{g}(\eta)$. This completes the proof.

Let $P = p^\nu$, and ψ a character modulo P such that $v_p(f_\psi) \leq [\nu/2]$. For a primitive form θ_λ in $S_\kappa^0(P, \psi)$ associated with Grössencharacter λ of $\boldsymbol{Q}(\sqrt{-p})$, put

$$S(\theta_\lambda) = \{f \in S_\kappa^0(P, \psi) \,|\, f|\tilde{U}_\chi = c_\lambda(\chi)f \qquad \text{for } \chi \in X(p^{[(\nu-1)/2]})\}$$

where $\theta_\lambda|\tilde{U}_\chi = c_\lambda(\chi)\theta_\lambda$. Then the above proposition shows that if $\kappa \geq 2$, we can find a Grössencharacter η and a modular form g such that $F(z) = g(p^{[(\nu-1/2]}z)\theta_\eta(z)$ belongs to $S(\theta_\lambda)$.

Now we return to our example. In the above notation we have

$$S(\theta_{\rm III}) = S_2^0(11, 4, 1) \oplus S_2^0(11, 4, -1) .$$

We can choose primitive forms $f \in S_2^0(11, 4, 1)$ and $g^i \in S^0(11, 4, -1)$, $1 \leq i \leq 3$, so that $\theta_{\rm I}$, $\theta_{\rm III}$, f, $f|R_\varepsilon$, g^i, and $g^i|R_\varepsilon (1 \leq i \leq 3)$ form a basis of $S(\theta_{\rm III})$, where ε is the quadratic residue symbol as before. Let ω be a character modulo 11 such that $\omega(-1) = -1$, and $E_{\bar{\omega}}(z)$ the Eisenstein series in $M_1(11, \bar{\omega})$, that is,

$$E_{\bar{\omega}}(z) = -\frac{L(0, \bar{\omega})}{2} + \sum_{n=1}^{\infty} \sum_{d|n} \bar{\omega}(d)e^{2\pi i n z} .$$

Then we can find a uniquely determined Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$ modulo $(\sqrt{-11}^2)$ which satisfies $\theta_\eta \in S_1(11^3, \omega)$ and $F(z) = E_{\bar{\omega}}(pz)\theta_\eta(z) \in S(\theta_{\rm III})$. By noting $F(z)|R_\varepsilon = F(z)$, we see $F(z)$ can be expressed as follows;

$$F(z) = a\theta_{\rm I} + b\theta_{\rm III} + c(f + f|R_\varepsilon) + \sum_{i=1}^{3} d_i(g^i + g^i|R_\varepsilon) . \tag{4.4}$$

Let K be the field generated by all the Fourier coefficients of $F(z)$, $\theta_{\rm I}$, $\theta_{\rm III}$, f, and g^i, then a, b, c, and d_i are contained in K. Assume $a \neq 0$, and let $\mathfrak{p}$ be a prime ideal of K which divides the denominator of a. If we can verify that b/a, c/a, and d_i/a are $\mathfrak{p}$-integral and $b/a \equiv 0$, $c/a \equiv 0 \pmod{\mathfrak{p}}$, then by Deligne and Serre [2, Lemma 6.11], we can find a primitive form g in $\{g^i, g^i|R_\varepsilon\}$ such that

$$\theta_{\mathrm{III}} \equiv g \pmod{\mathfrak{p}}.$$

Let us check this. First we must calculate a. In order to do this, the following Lemma is useful.

LEMMA 4.5. *Let f, and g_i $(1 \leq i \leq n)$ be primitive forms, and $F(z)$ a cusp forms such that*

$$F(z) = \alpha f + \sum_{i=1}^{n} \beta_i g_i \, .$$

Let a_n, b_n^i, and c_n denote the n-th Fourier coefficients of f, g_i, and F respectively. For a polynomial $T(X) = \sum_{j=1}^{\ell} A_j X^j$ and a prime q, assume $T(b_q^i) = 0$ for i, $1 \leq i \leq n$. Then one has

$$T(a_q)\alpha = \sum_{m=0}^{\ell} \sum_{r=0}^{[m/2]} \left(\binom{m}{r} - \binom{m}{r-1} \right) (p^{\kappa-1})^r c_{p^{m-2r}} A_{\ell-m}$$

where $\binom{m}{r} = m!/r!(m-r)!$.

This is an easy consequenceə of Exercise 3.27′ in [13], and we omit the proof. As $T(X)$, we can take the characteristic polynomial of T_q acting on the space spanned by g_i.

Applying the above Lemma taking $\omega = \varepsilon$, we find $a = 0$, and we cannot proceed anymore. In stead of $F(z)$ for $\omega = \varepsilon$, we take the following as F;

$$F'(z) = \sum_{\omega} E_{\bar{\omega}}(pz)\theta_{\eta}(z) \, ,$$

where ω runs through all characters modulo 11 such that $\omega(-1) = -1$ and η is the Grössencharacter of $\boldsymbol{Q}(\sqrt{-11})$ such that $\theta_\eta \in S_1^0(11^3, \omega)$. Put

$$F'(z) = a'\theta_{\mathrm{I}} + b'\theta_{\mathrm{III}} + c'(f + f|R_\varepsilon) + \sum_{i=1}^{3} d_i'(g^i + g^i|R_\varepsilon) \tag{4.5}$$

as before. Then we find

$$a' = (5/22)(200\alpha^4 + 314\alpha^3 - 612\alpha^2 - 856\alpha + 54)/(262\alpha^4 + 368\alpha^3 - 895\alpha^2 - 1003\alpha + 353)$$

$$N(200\alpha^4 + 314\alpha^3 - 612\alpha^2 - 856\alpha + 54) = 2^5 \cdot 11^4 \cdot 23 \cdot 197$$

$$N(262\alpha^4 + 368\alpha^3 - 895\alpha^2 - 1003\alpha + 353) = 11^4 \cdot 23 \cdot 99527 \, .$$

Let $\mathfrak{p}$ be a prime ideal of K which divides $(262\alpha^4 + 368\alpha^3 - 895\alpha^2 - 1003\alpha + 353)$ and 99527. We note the Fourier coefficients of $22F'(z)$ are integral. By means of Lemma 4.5 and some calculation, we can check the condition on a', b', c', and d'_i mentioned before. For example, the assertion that d'_i/a' is $\mathfrak{p}$-integral can be verified in the following way. Let a_n, b_n, $f_{T_n}(X)$, and $g_{T_n}(X)$ be as in the table. Let q be a prime such that $\varepsilon(q)=1$, then $g_{T_q}(X)$ (resp. $f_{T_q}(X)$) is of the form $g_q(X)^2$ (resp. $(X-c_q)^2$), where $g_q(X)$ is a polynomial of degree 3. To prove d'_i/a' is $\mathfrak{p}$-integral, it is enough to show $g_q(a_q)$ and $g_q(c_q)$ are prime to $\mathfrak{p}$ and $g_q(X)\equiv 0 \bmod \mathfrak{p}$ does not have mutiple roots for a prime q with $\varepsilon(q)=1$. We take $q=3$. Then we have

$$g_3(a_3) = -6\alpha^4 - 2\alpha^3 + 24\alpha^2 + 6\alpha - 18\,, \quad N(g_3(a_3)) = 2^5\cdot 11$$
$$g_3(c_3) = 4\alpha^4 + 6\alpha^3 - 8\alpha^2 - 14\alpha - 8\,, \quad N(g_3(c_3)) = 2^5\cdot 11^2\,.$$

Hence $g_3(a_3)$ and $g_3(c_3)$ are prime to $\mathfrak{p}$. The second condition can be checked easily, since we know one root b_3 of $g_3(X)\equiv 0 \pmod{p}$. We omit the details. Thus we obtain

PROPOSITION 4.6. *Let $\theta_{\rm III}\in S_2(11, 4, -1)$ and $S^0_{\rm III}$ ($\subset S_2(11, 4, -1)$) be as before. Let K be the field generated by the Fourier coefficients of $\theta_{\rm III}$ and the primitive forms in $S^0_{\rm III}$, and $\mathfrak{p}$ be a prime ideal of K which divides $262\alpha^4+368\alpha^3-895\alpha^2-1003\alpha+353$ and 99527. Then there exists a primitive form g in $S^0_{\rm III}$ which satisfies*

$$\theta_{\rm III}\equiv g \pmod{\mathfrak{p}}\,.$$

Now the coefficient a in (4.4) can be written as follows;

$$a = \frac{\langle \theta_{\rm III}, F(z)\rangle}{\langle \theta_{\rm III}, \theta_{\rm III}\rangle}\,,$$

where $\langle\ ,\ \rangle$ denotes the Petersson inner product, and the coefficient a' in (4.5) can be expressed as a sum of such numbers. By means of a result of Shimura [16], we can relate the number a to the special values of zeta functions. We introduce some notations. For positive integer N, κ and a Dirichlet character ω modulo N such that $\omega(-1)=(-1)^\kappa$, put

$$E^*_{\kappa,N}(z,s,\omega) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \omega(d)(cz+d)^{-\kappa}|cz+d|^{-2s}\,, \qquad \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix},$$

where $\Gamma_\infty = \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\middle|\, b\in \boldsymbol{Z}\right\}$, and

$$E_{\kappa,N}(z,s,\omega) = \sum_{m,n}\omega(n)(mNz+n)^{-\kappa}|mNz+n|^{-2s}\,,$$

where the summation is taken over all $(m,n)\in\boldsymbol{Z}^2$, $\neq 0$. These are absolutely convergent for $\mathrm{Re}\,(2s) > 2 - \kappa$, and we have

$$E_{\kappa,N}(z, s, \omega) = 2L_N(2s + \kappa, \omega)E^*_{\kappa,N}(z, s, \omega)\ ,$$

where $L_N(s, \omega) = \sum_{(N,n)=1} \omega(n)n^{-s}$. For $\kappa > 0$, we put

$$E_{\kappa,N}(z, \omega) = E_{\kappa,N}(z, 0, \omega), \quad E^*_{\kappa,N}(z, \omega) = E^*_{\kappa,N}(z, 0, \omega)\ .$$

If $\kappa \neq 2$, or ω is not trivial, $E_{\kappa,N}(z, \omega)$ and $E_{\kappa,N}(z, \omega)$ belongs to $G_\kappa(N, \bar{\omega})$.

PROPOSITION 4.7. *For a prime $p \equiv 3 \pmod 4$, let ω be a character modulo p and θ_λ (resp. θ_η) a primitive form associated with a Grössen-character λ(resp. η) of $\boldsymbol{Q}(\sqrt{-p})$ belonging to $S^0_\kappa(P, \psi)$ (resp. $S^0_{\kappa'}(P, \psi\omega)$) for $P = p^\nu$ and a character ψ which satisfy $v_p(\mathfrak{f}_\psi) \leq \nu/3$. Assume that $\kappa > \kappa'$ and that $\kappa - \kappa' \neq 2$ or ω is not trivial. Put $F(z) = E_{\kappa-\kappa'}(p^{[(\nu-1)/2]}z, \omega)\theta_\eta(z)$. If $F(z)$ belongs to $S(\theta_\lambda)$, then*

$$\frac{\langle \theta_\lambda, F\rangle}{\langle \theta_\lambda, \theta_\lambda\rangle} = \frac{4(\kappa - 1)\pi^2}{p^{\nu-[(\nu-1)/2]}L(1, \varepsilon)} \frac{L((\kappa - \kappa')/2, \lambda'\eta)L((\kappa - \kappa')/2, \lambda'\eta'^{-1})}{L(1, \lambda'\lambda)}\ ,$$

where $\lambda'(\mathfrak{a}) = \overline{\lambda(\bar{\mathfrak{a}})}$, $\eta'(\mathfrak{a}) = \overline{\eta(\bar{\mathfrak{a}})}$ for an ideal $\mathfrak{a}$ in $\boldsymbol{Q}(\sqrt{-p})$.

Proof. Let Φ denote a fundamental domain of $\mathfrak{H}$ with respect to $\Gamma_0(P)$. Put $\mu = [(\nu - 1)/2]$. Let Γ be a subgroup of $\Gamma_0(P)$ given by

$$\Gamma = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(P) \,|\, a \equiv d \equiv 1 \pmod{p^\mu}\right\}\ ,$$

and Φ' a fundamental domain for Γ. We note Γ is a normal subgroup of $\Gamma_0(p^{\mu+1})$. Let $\{a_j\}$ be a complete system of representatives of $\boldsymbol{Z}$ modulo p^μ, then $\Gamma_0(p^{\mu+1}) = \bigcup_j \Gamma_0(P)\alpha_j$ is a disjoint union, where $\alpha_j = \begin{pmatrix} 1 & 0 \\ p^{\mu+1}a_j & 1\end{pmatrix}$. For the sake of simplicity, we put

$$E(z, s) = E_{\kappa-\kappa',P}(z, s, \omega),\ E(z, s)^* = E^*_{\kappa-\kappa',P}(z, s, \omega)\ .$$

We note $E_{\kappa-\kappa',p^{\mu+1}}(z, s, \omega) = E_{\kappa-\kappa',p}(p^\mu z, s, \omega)$, and

$$E^*_{\kappa-\kappa',p^{\mu+1}}(z, s, \omega) = \sum_j E(z, s)^* \,|\, [\alpha_j]\ ,$$

where $E(z, s)^* \,|\, [\gamma] = \omega(d)(cz + d)^{-(\kappa-\kappa')}|cz + d|^{-2s}E(\gamma(z), s)^*$ for $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\boldsymbol{Z})$. We have

$$\text{(4.6)} \qquad I = \int_\Phi \bar{\theta}_\lambda\theta_\eta E_{\kappa-\kappa',p}(p^\mu z, s, \omega)y^{s+\kappa-2}dxdy$$
$$= c(s) \sum_j \int_{\Phi'} \bar{\theta}_\lambda\theta_\eta(E(z, s)^* \,|\, [\alpha_j])y^{s+\kappa-2}dxdy\ ,$$

where $c(s) = 2L_P(2s + \kappa - \kappa', \omega)/[\Gamma_0(P): \Gamma]$. If $a_j \equiv 0 \pmod{p^\mu}$, then for $\mathrm{Re}\,(2s) > 2 - (\kappa - \kappa')$ as in § 2 of [16]

$$(4.7) \qquad \int_{\Phi'} \bar\theta_\lambda \theta_\eta (E(z, s)^* | [\alpha_j]) y^{s+\kappa-2} dxdy$$

$$= [\Gamma_0(P): \Gamma] \int_{\Phi} \bar\theta_\lambda \theta_\eta E(z, s)^* y^{s+\kappa-2} dxdy$$

$$= [\Gamma_0(P): \Gamma](4\pi)^{-(s+\kappa-1)} \Gamma(s + \kappa - 1) D(s + \kappa - 1, \theta_{\lambda'}, \theta_\eta) ,$$

where $D(s, f, g) = \sum_{n=1}^{\infty} a_n b_n n^{-s}$ for $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$. λ' is the Grössencharacter given by $\lambda'(\mathfrak{a}) = \overline{\lambda(\bar{\mathfrak{a}})}$. If $a_j \not\equiv 0 \pmod{p^\mu}$, then put $a_j = vp^{\nu-\mu-1-\tau}$ with a positive integer τ and v prime to p. If we define β_v by

$$\beta_v = \begin{pmatrix} 1 & v/p^\tau \\ 0 & 1 \end{pmatrix},$$

then $\alpha_j^{-1} = \eta_P^{-1} \beta_v \eta_P$. Since $\alpha_j \in \Gamma_o(p^{\mu+1})$ and Γ is a normal subgroup of $\Gamma_0(p^{\mu+1})$, we see

$$\int_{\Phi'} \bar\theta_\lambda \theta_\eta (E(z, s)^* | [\alpha_j]) y^{s+\kappa-2} dxdy$$

$$= \int_{\Phi'} \overline{(\theta_\lambda | [\alpha_j^{-1}]_\kappa)} (\theta_\eta | [\alpha_j^{-1}]_{\kappa'}) E(z, s)^* y^{s+\kappa-2} dxdy$$

$$= \int_{\Phi'} \overline{(\theta_\lambda | W_P^{-1}[\beta_v]_\kappa)} (\theta_\eta | W_P^{-1}[\beta_v]_{\kappa'}) E(z, s)^* | W_P^{-1} y^{s+\kappa-2} dxdy,$$

where $E(z, s)^* | W_P^{-1} = E(\eta_P^{-1}(z), s)^* (-p^{\nu/2} z)^{-(\kappa-\kappa')} |p^{\nu/2} z|^{-2s}$. Now we have

LEMMA 4.8. *For a character ψ modulo $p^{\nu-1}$, let f be a primitive form in $S_\kappa^0(P, \psi)$ for $P = p^\nu$. For a character χ, put $f_\chi = f | R_\chi$. If $\nu \geq 2$, for $\beta_v = \begin{pmatrix} 1 & v/p^\sigma \\ 0 & 1 \end{pmatrix}$ with $\tau \geq 1$ and $(v, p) = 1$, it holds*

$$f | [\beta_v]_\kappa = \begin{cases} \dfrac{1}{p - 1} \sum_\chi \chi(v) \mathfrak{g}(\bar\chi) f_\chi & \text{if } \tau = 1 \\ \dfrac{1}{p^\tau(1 - 1/p)} \sum_\chi \chi(v) \mathfrak{g}(\bar\chi) f_\chi & \text{otherwise} \end{cases}$$

where χ runs through all characters modulo p if $\tau = 1$ and all characters with the conductor p^τ if $\tau \geq 2$. For the trivial character χ_1, we put $\mathfrak{g}(\chi_1) = -1$.

Proof. By the definition of the twisting operator, we have

$$\mathfrak{g}(\bar{\chi})f_\chi = \sum_{\substack{u \bmod p^\sigma \\ (u,p)=1}} \bar{\chi}(u)f|[\alpha_u]_\kappa ,$$

where $\mathfrak{f}_\chi = p^\sigma$ and $\alpha_u = \begin{pmatrix} 1 & u/p^\sigma \\ 0 & 1 \end{pmatrix}$. If $\tau = 1$, we see

$$\begin{aligned}\sum_{\mathfrak{f}_\chi \leq p} \chi(v)\mathfrak{g}(\bar{\chi})f_\chi &= \sum_{\mathfrak{f}_\chi = p} \chi(v) \sum_{(u,p)=1} \bar{\chi}(u)f|[\alpha_u]_\kappa - f \\ &= \sum_{\substack{\mathfrak{f}_\chi \leq p \\ (u,p)=1}} \chi(v)\bar{\chi}(u)f|[\alpha_u]_\kappa \\ &= (p-1)f|[\alpha_v]_\kappa .\end{aligned}$$

This prove the case where $\tau = 1$. We can treat the case where $\tau \geq 2$ in the same way, because for χ' with, $\mathfrak{f}_{\chi'} \leq p^{\sigma-1}$, we have

$$\sum_{\substack{v \bmod p^\sigma \\ (v,p)=1}} \chi'(v)f|[\alpha_v]_\kappa = 0$$

and we omit the details.

Put $f = \theta_\lambda | W_P^{-1}$, $g = \theta_\eta | W_P^{-1}$, and $E'(z, s) = E(z, s)^* | W_P^{-1}$.

For β_v with $\tau = 1$, we have

$$\begin{aligned}I_1 &= \sum_{\substack{v \bmod p \\ (v,p)=1}} \int_{\Phi'} \overline{(f|[\beta_v]_\kappa)}(g|[\beta_v]_{\kappa'})E'(z, s)y^{s+\kappa-2}dxdy \\ &= \frac{1}{(p-1)^2} \int_{\Phi'} \sum_v (\sum_\chi \overline{\chi(v)\mathfrak{g}(\bar{\chi})f_\chi})(\sum_{\chi'} \chi'(v)\mathfrak{g}(\bar{\chi}')g_{\chi'}) \\ &\qquad\qquad \times E'(z, s)y^{s+\kappa-2}dxdy \\ &= \frac{1}{(p-1)} \int_{\Phi'} \sum_\chi \overline{\mathfrak{g}(\bar{\chi})}\mathfrak{g}(\bar{\chi})\bar{f}_\chi g_\chi E'(z, s)y^{s+\kappa-2}dxdy .\end{aligned}$$

We have by Prop. 3.5

$$\begin{aligned}\overline{(f_\chi | W_P)}(g_\chi | W_P) &= \overline{(\theta_\lambda | W_P^{-1}R_\chi W_P)}(\theta_\mu | W_P^{-1}R_\chi W_P) \\ &= \overline{(\theta_\lambda | \tilde{U}_\chi R_{\bar{\chi}})}(\theta_\mu | \tilde{U}_\chi R_{\bar{\chi}}) \\ &= \overline{(\theta_\lambda | R_{\bar{\chi}})}(\theta_\mu | R_{\bar{\chi}}) ,\end{aligned}$$

since $F(z) \in S(\theta_\lambda)$. Hence we obtain

$$\begin{aligned}(4.8) \qquad I_1 &= \frac{1}{(p-1)} \sum_\chi \overline{\mathfrak{g}(\bar{\chi})}\mathfrak{g}(\bar{\chi}) \int_{\Phi'} \overline{(f_\chi | W_P)}(g_\chi | W_P)E(z, s)^*y^{s+\kappa-2}dxdy \\ &= \frac{1}{(p-1)} \sum \overline{\mathfrak{g}(\bar{\chi})}\mathfrak{g}(\bar{\chi}) \int_{\Phi'} \overline{(\theta_\lambda | R_{\bar{\chi}})}(\theta_\eta | R_{\bar{\chi}})E(z, s)^*y^{s+\kappa-2}dxdy \\ &= (p-1)[\Gamma_0(P): \Gamma](4\pi)^{-(s+\kappa-1)}\Gamma(s+\kappa-1)D(s+\kappa-1; \theta_{\lambda'}, \theta_\eta) .\end{aligned}$$

For $\beta_v = \begin{pmatrix} 1 & v/p^\tau \\ 0 & 1 \end{pmatrix}$ with $\tau \geq 2$, we can show in the same way

$$(4.9) \qquad \sum_{\substack{v \bmod p \\ (v,p)=1}} \int_{\Phi'} \overline{(f|[\beta_v]_\kappa)}(g|[\beta_v]_{\kappa'})E'(z, s)y^{s+\kappa-2}dxdy$$

$$= \frac{1}{(p-1)}(p^\tau - 2p^{\tau-1} + p^{\tau-2})(4\pi)^{-(s+\kappa-1)}\Gamma(s+\kappa-1) \, D(s+\kappa-1, \theta_{\lambda'}, \theta_\eta) .$$

By (4.6), (4.7), (4.8), and (4.9), we obtain

$$I = 2L_P(2s+\kappa-\kappa', \omega)p^\mu(4\pi)^{-(s+\kappa-1)}\Gamma(s+\kappa-1)D(s+\kappa-1, \theta_{\lambda'}, \theta_\eta) .$$

By Lemma 1 of [16], this is equal to

$$2p^\mu(4\pi)^{-(s+\kappa-1)}\Gamma(s+\kappa-1)L\left(s+\frac{\kappa-\kappa'}{2}, \lambda'\eta\right)L\left(s+\frac{\kappa-\kappa'}{2}, \lambda'\eta'^{-1}\right),$$

where $\eta'(\mathfrak{a}) = \overline{\eta(\bar{\mathfrak{a}})}$ for ideals $\mathfrak{a}$ in $\boldsymbol{Q}(\sqrt{-p})$. Putting $s = 0$, we obtain

$$\langle \theta_\lambda, F(z) \rangle = 2p^\mu(4\pi)^{-(\kappa-1)}\Gamma(\kappa-1)L\left(\frac{\kappa-\kappa'}{2}, \lambda'\eta\right)L\left(\frac{\kappa-\kappa'}{2}, \lambda'\eta'^{-1}\right).$$

On the other hand, by (2.5) in [14], we have

$$\langle \theta_\lambda, \theta_\lambda \rangle = (4\pi)^{-\kappa}\Gamma(\kappa)\frac{\pi}{3}P(1+1/p) \operatorname{Res}_{s=\kappa} D(s, \theta_{\lambda'}, \theta_\lambda) .$$

As above, we have

$$D(s, \theta_{\lambda'}, \theta_\lambda) = \frac{L(s-\kappa+1, \lambda'\lambda)L(s-\kappa+1, \lambda_1)}{L_P(2s-2\kappa+2, \chi_1)},$$

where χ_1 is the trivial character and $\lambda_1(\mathfrak{a}) = 1$ if $\mathfrak{a}$ is prime to p and $\lambda_1(\mathfrak{a}) = 0$ otherwise. Hence we obtain

$$\langle \theta_\lambda, \theta_\lambda \rangle = (4\pi)^{-(\kappa-1)}\Gamma(\kappa)(2\pi^2)^{-1}PL(1, \lambda'\lambda)L(1, \varepsilon) ,$$

and thus

$$\frac{\langle \theta_\lambda, F \rangle}{\langle \theta_\lambda, \theta_\lambda \rangle} = \frac{4(\kappa-1)\pi^2}{p^{\nu-\mu}L(1, \varepsilon)} \frac{L((\kappa-\kappa')/2, \lambda'\eta)L((\kappa-\kappa')/2, \lambda'\eta'^{-1})}{L(1, \lambda'\lambda)} .$$

This completes the proof.

Appendix

I. Let $N = 13^3$, $\kappa = 2$, and $\psi =$ the trivial character. Then we find dim $S_2(13^3, 4, 1) = 6$, and dim $S_2(13^3, 4, -1) = 8$. Let $f_{T_n}(X)$ and $g_{T_n}(X)$ denote the characteristic polynomial of T_n on the spaces $S_2(13^3, 4, 1)$ and $S_2(13^3, 4, -1)$ respectively. Then for $n = 2$ and 3, $f_{T_n}(X)$ and $g_{T_n}(X)$ are given by

n	$f_{T_n}(X)$
2	$X^6 - (-\alpha^3 + 3\alpha + 8)X^4 + (\alpha^5 - \alpha^4 - 9\alpha^3 + 3\alpha^2 + 17\alpha + 15)X^2 - (-\alpha^5 - 5\alpha^4 + \alpha^3 + 17\alpha^2 + 6\alpha - 1)$
3	$(X^3 - (-2)X^2 + (\alpha^2 - 5)X - (\alpha^3 - \alpha^2 - 4\alpha + 5))^2$

n	$g_{T_n}(X)$
2	$X^8 - (\alpha^3 - 3\alpha + 13)X^6 + (-3\alpha^5 - \alpha^4 + 24\alpha^3 + 3\alpha^2 - 42\alpha + 51)X^4 - (-18\alpha^5 + 108\alpha^3 - 8\alpha^2 - 145\alpha + 80)X^2 + (-17\alpha^5 - \alpha^4 + 91\alpha^3 - 9\alpha^2 - 108\alpha + 41)$
3	$(X^4 - 2X^3 + (-\alpha^2 - 5)X^2 - (-2\alpha^5 - 2\alpha^4 + 9\alpha^3 + 5\alpha^2 - 8\alpha - 9)X + (-4\alpha^5 - 2\alpha^4 + 16\alpha^3 + 8\alpha^2 - 10\alpha - 2))^2$,

where $\alpha = e^{2\pi i/13} + e^{-\pi i/13}$. We remark the following. Let N denote the norm from $\boldsymbol{Q}(\alpha)$ to $\boldsymbol{Q}$, then

$$N(f_{T_2}(0)) = 443, \quad N(g_{T_2}(0)) = 53\cdot 79\,.$$

On the other hand, let $\varepsilon_0 = (3+\sqrt{13})/2$ be a fundamental unit of $\boldsymbol{Q}(\sqrt{13})$, then

$$N_{\boldsymbol{Q}(\sqrt{13})/\boldsymbol{Q}}(\varepsilon_0^{13} - 1) = -3\cdot 53\cdot 79\cdot 443\,.$$

Such a relation has been noticed in [3, Remark 2.1.] for the case $N = 5^3$.

II. Let $N = 19^3$, $\kappa = 2$, and ψ the trivial character. Then we find dim $S_2(19^3, 4, 1) = 12$ and dim $S_2(19^3, 4, -1) = 16$. Let $\theta_{\mathrm{I}}(z) = \sum a_n e^{2\pi i n z} \in S_2(19^3, 4,1)$ (resp. $\theta_{\mathrm{III}}(z) = \sum b_n e^{2\pi i n z} \in S_2(19, 4, -1)$) be a primitive form associated with a Grössencharacter of $\boldsymbol{Q}(\sqrt{-19})$ and S_{I}^0(resp. S_{III}^0) the orthogonal complement of the space spanned by θ_{I} (resp. θ_{III}). We denote by $f_{T_n}(X)$ (resp. $g_{T_n}(X)$) the characteristic polynomial of T_n acting on S_{I}^0(resp. S_{III}^0). Let $\alpha = e^{2\pi i/19} + e^{-2\pi i/19}$ and let $(x_1, x_2, \cdots, x_9)$ denote the number $\sum_{i=1}^{9} x_i\alpha^{9-i}$ in $\boldsymbol{Q}(\alpha)$. Then we have

In the preparation of the tables in the Appendix, we used FACOM M190 at Data Processing center of Kyoto University.

$$f_{T_2}(X) = X^{12} - A_{10}X^{10} + A_8X^8 - A_6X^6 + A_4X^4 - A_2X^2 + A_0$$

$$A_{10} = (0, 0, 0, 0, 0, 0, 0, 0, 18)$$
$$A_8 = (0, 3, 0, -21, 0, 42, 0, -21, 120)$$
$$A_6 = (0, 30, -3, -210, 17, 419, -24, -209, 373)$$
$$A_4 = (-2, 94, -4, -655, 76, 1298, -136, -651, 558)$$
$$A_2 = (-18, 99, 103, -687, -124, 1356, -50, -711, 351)$$
$$A_0 = (-21, 26, 145, -176, -291, 336, 163, -187, 44)$$

$a_2 = 0,\ N(f_{T_2}(a_2)) = 37^2 \cdot 56536856647$

$$f_{T_5}(X) = (X^6 - A'_5X^5 + A'_4X^4 - A'_3X^3 + A'_2X^2 - A'_1X + A'_0)^2$$

$$A'_5 = (0, 0, 0, 0, 1, 1, -4, -3, -1)$$
$$A'_4 = (0, 1, 0, -7, -2, 11, 9, 3, -15)$$
$$A'_3 = (-4, -4, 32, 27, -91, -61, 105, 50, -2)$$
$$A'_2 = (4, -5, -26, 32, 59, -31, -73, -60, 38)$$
$$A'_1 = (13, 2, -119, -10, 354, 18, -356, -22, 47)$$
$$A'_0 = (16, 18, -113, -105, 233, 141, -125, 19, 10)$$

$a_5 = (0, 1, 0, -7, -1, 13, 5, -4, -5)$

$N(f_{T_5}(a_5)) = -37 \cdot 227 \cdot 150707 \cdot 56536856647$

$$g_{T_2}(X) = X^{16} - B_{14}X^{14} + B_{12}X^{12} - B_{10}X^{10} + B_8X^8 - B_6X^6 + B_4X^4 - B_2X^2 + B_0$$

$$B_{14} = (0, 0, 0, 0, 0, 0, 0, 0, 27)$$
$$B_{12} = (0, -3, 0, 21, 0, -42, 0, 21, 294)$$
$$B_{10} = (0, -57, 1, 399, -7, -799, 12, 404, 1657)$$
$$B_8 = (4, -398, -13, 2795, -51, -5639, 164, 2928, 5157)$$
$$B_6 = (32, -1263, -149, 8940, -108, -18340, 844, 9980, 8723)$$
$$B_4 = (53, -1847, -254, 13227, -255, -27848, 1845, 16178, 7321)$$
$$B_2 = (15, -1076, -67, 7756, -325, -16788, 1453, 10589, 2464)$$
$$B_0 = (-24, -110, 168, 708, -418, -1458, 450, 1112, 194)$$

$a_2 = 0,\ N(g_{T_2}(a_2)) = 2^9 \cdot 19^2 \cdot 5736557 \cdot 6463381$

$$g_{T_5}(X) = (X^8 - B'_7X^7 + B'_6X^6 - B'_5X^5 + B'_4X^4 - B'_3X^3 + B'_2X^2 - B'_1X + B'_0)^2$$

$$B'_7 = (0, 1, 0, -7, 0, 14, 1, -6, 2)$$
$$B'_6 = (0, 3, 0, -21, -2, 42, 8, -17, -19)$$
$$B'_5 = (4, -10, -29, 67, 54, -127, -29, 49, -30)$$

$$B'_4 = (7, -27, -51, 175, 126, -335, -143, 145, 112)$$
$$B'_3 = (-35, 21, 254, -143, -492, 246, 243, -35, 121)$$
$$B'_2 = (-56, 43, 395, -236, -857, 383, 664, -133, -196)$$
$$B'_1 = (44, -13, -313, 109, 574, -189, -264, 0, -98)$$
$$B'_0 = (43, -4, -281, 6, 505, 13, -248, -32, 13)$$

$$b_5 = (0, -1, 0, 7, 1, -13, -5, 3, 5)$$

$$N(g_{T_5}(a_5)) = 571 \cdot 3457 \cdot 51679 \cdot 28579723 \cdot 5736557 \cdot 6463381.$$

Here N denotes the norm from $\boldsymbol{Q}(\alpha)$ to $\boldsymbol{Q}$. We remark $N(f_{T_2}(a_2))$ and $N(f_{T_5}(a_5))$ (resp. $N(g_{T_2}(a_2))$ and $N(g_{T_5}(a_5))$) have a common factor 56536856647 (resp. $5736557 \cdot 6463381$).

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