

LOCAL DEFORMATIONS OF ISOLATED SINGULARITIES ASSOCIATED WITH NEGATIVE LINE BUNDLES OVER ABELIAN VARIETIES

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Introduction

Let V be an analytic space with an isolated singularity p . In [1] M. Kuranishi approached the problem of deformations of isolated singularities (c.f. [2] and [3]) as follows; Let M be a real hypersurface in the complex manifold $V - \{p\}$. Then one has the induced CR -structure ${}^{\circ}T''(M)$ on M by the inclusion map $i: M \rightarrow V - \{p\}$ (c.f. Def. 1.6). Then deformations of the isolated singularity (V, p) give rise to ones of the induced CR -structure ${}^{\circ}T''(M)$. He established in §9 in [1] the universality theorem for deformations of the induced CR -structure ${}^{\circ}T''(M)$, when M is compact strongly pseudo-convex (Def. 1.5) of $\dim M \geq 5$. From this theorem we can know CR -structures on M which appear in deformations of ${}^{\circ}T''(M)$.

Here we assume that V is 1-convex in the sense of Andreotti-Grauert such that $\dim_{\mathbb{C}} V \geq 3$ and that M is a compact real hypersurface in $V - \{p\}$ defined by strictly plurisubharmonic function ρ on V such that $\rho \geq 0$, that is, $M = \{q \in V; \rho(q) = c\}$, here c is a constant. Then as Prof. $V \geq 2$, we find in terms of [2] that the infinitesimal deformation $H^1(V, \Theta)$ (c.f. [1]) of the isolated singularity (V, p) is regarded as a subspace of the infinitesimal deformation $H^1(M, {}^{\circ}T''(M))$ of ${}^{\circ}T''(M)$ (c.f. §3). Therefore in order to solve the problem of local deformations of (V, p) , it is enough to determine the infinitesimal deformations $H^1(M, {}^{\circ}T''(M))$ and complex structure on a neighborhood of M in $V - \{p\}$, which induce CR -structures on M appearing in deformations of ${}^{\circ}T''(M)$.

In this paper we shall prove, using the above Kuranishi's theory, the following.

Received February 28, 1978.

The first named author was a visiting member at the Research Institute for Mathematical Sciences, Kyoto University, during a part of the period of this research.

THEOREM 5.1. *Let T be an abelian variety of $\dim T \geq 2$ and B a negative line bundle over T . Let (B, T) be the isolated singularity defined by the exceptional variety $O(T)$ in B , here $O(T)$ denotes the zero section of B . Then local deformations of (B, T) are also isolated singularities (B', T') , where T' and B' are abelian varieties and negative line bundles over T' , respectively.*

Remark. (B, T) is 1-convex with $\dim_c B \geq 3$.

The above theorem has been essentially proved by M. Schlessinger [2]. However we want to publish this paper as an example of applications of Kuranishi's theory to deformations of isolated singularities.

In §1 we describe basic notions of CR -structures and in §2 determine the induced CR -structure ${}^\circ T''(B_i)$ on the unit sphere bundle B_i of the negative line bundle B over the abelian variety T using a normalized automorphic factor for B . In §3 we calculate the infinitesimal deformation $H^1(B, {}^\circ T''(B))$.

In §4 and §5 we show that $H^1(B, {}^\circ T''(B))$ has basis which are integrable CR -structures and that these integrable CR -structures are induced from some negative line bundles B' over abelian varieties T' .

§1 Basic definitions

Let M be a real oriented smooth manifold of dimension $2n + 1$, $n = 1, 2, \dots$, and let ${}^\circ T''$ be a subbundle of the complexified tangent bundle CTM . Let E be a vector bundle over M . We denote by $\Gamma(E)$ the set of C^∞ -sections of E .

DEFINITION 1.1. The subbundle ${}^\circ T''$ is called an almost CR -structure on M , when the following condition is satisfied;

$$(1.1) \quad {}^\circ T'' \cap {}^\circ \bar{T}'' = 0, \quad \dim_c {}^\circ T'' = n.$$

Moreover an almost CR -structure ${}^\circ T''$ is a CR -structure, provided that

$$(1.2) \quad \begin{array}{l} {}^\circ T'' \text{ is integrable in the sense of Frobenius, i.e. if } Z_1, Z_2 \text{ are sections} \\ \text{of } {}^\circ T'', \text{ then so is their Lie bracket } [Z_1, Z_2]. \end{array}$$

Now let ${}^\circ T''$ be a CR -structure on M . Since ${}^\circ T''$ is the complex vector subbundle of CTM of complex fiber dimension n and is invariant under complex conjugation, there is a real line bundle F of TM such that

$$(1.3) \quad CTM = {}^\circ T'' \oplus {}^\circ \bar{T}'' \oplus CF.$$

From now on we fix this decomposition of CTM . Put

$$(1.4) \quad T' = {}^\circ T'' \oplus CF,$$

and we denote by $\pi'(\pi)$ the projection of CTM onto $T'({}^\circ T'')$, respectively.

DEFINITION 1.2. An almost CR -structure E'' on M is said to be of finite distance to ${}^\circ T''$ if $\pi' \circ i_{E''}: E'' \rightarrow T'$ is an isomorphism, where $i_{E''}: E'' \rightarrow CTM$ is the inclusion map.

PROPOSITION 1.3 [1]. *Let E'' be an almost CR -structure on M of finite distance to ${}^\circ T''$. Then there is a unique element φ of $\Gamma(M, \text{Hom}({}^\circ T'', T'))$ such that*

$$(1.5) \quad E'' = \{X - \varphi(X); X \in {}^\circ T''\}.$$

Conversely let $\varphi \in \Gamma(M, \text{Hom}({}^\circ T'', T'))$ and we write

$$\varphi = \varphi_1 + \varphi_2$$

where $\varphi_1 \in \Gamma(M, \text{Hom}({}^\circ T'', {}^\circ \bar{T}''))$ and $\varphi_2 \in \Gamma(M, \text{Hom}({}^\circ T'', CF))$. For any φ satisfying $\varphi_1(\overline{\varphi_1(X)}) \neq \bar{X}$ or $\varphi_2(\overline{\varphi_1(X)}) \neq \overline{\varphi_2(X)}$ for all $X \in {}^\circ T''$, ($X \neq 0$), the formula (1.5) defines an almost CR -structure E'' of finite distance to ${}^\circ T''$.

The almost CR -structure E'' defined by (1.5) in terms of φ will be denoted by ${}^\circ T''$. We identify ${}^\circ T''$ with φ , and φ is also called an (almost) CR -structure on M , when ${}^\circ T''$ is an (almost) CR -structure.

Next we will examine when T'' is integrable, i.e., a CR -structure on M . For any $X \in \Gamma(CTM)$ we put

$$X_{T'} = \pi'(X) \quad \text{and} \quad X_{\circ T''} = \pi(X).$$

The following formulation for the integrable condition is due to T. Akahori [4].

PROPOSITION 1.4. *Let $\varphi \in \Gamma(\text{Hom}({}^\circ T'', T'))$ be an almost CR -structure on M . Let $P(\varphi)$ be a map of $\Gamma(\wedge^2 {}^\circ T'')$ into $\Gamma(T')$ defined by*

$$(1.6) \quad P(\varphi)(X, Y) = [X - \varphi(X), Y - \varphi(Y)]_{T'} + \varphi([X - \varphi(X), Y - \varphi(Y)]_{\circ T''})$$

for $X, Y \in \Gamma({}^\circ T'')$.

Then φ is integrable if and only if

$$P(\varphi) \equiv 0.$$

In order to compute the infinitesimal deformation of ${}^\circ T''$, we must define the operator $\bar{\partial}_b^{(p)}: \Gamma(T' \otimes \wedge^p ({}^\circ T'')^*) \rightarrow \Gamma(T' \otimes \wedge^{p+1} ({}^\circ T'')^*)$, $p = 0, 1, 2, \dots$. At first for $p = 0$, $\bar{\partial}_b^{(0)}: \Gamma(T'') \rightarrow \Gamma(T' \otimes {}^\circ T'')$ is defined by

$$(1.7) \quad (\bar{\partial}_b^{(0)}u)(X) = [X, u]_{T'} , \quad \text{for } u \in \Gamma(T'') \text{ and } X \in \Gamma({}^\circ T'') .$$

For convenience sake we set

$$Xu = [X, u]_{T'} .$$

For $p \geq 1$,

$$(1.8) \quad \begin{aligned} & (\bar{\partial}_b^{(p)}\psi)(X_1, \dots, X_{p+1}) \\ &= \sum_{i=1}^p (-1)^{i-1} X_i \psi(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) \\ & \quad + \sum_{i < j} (-1)^{j+k} \psi([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}) , \end{aligned}$$

where $X_1, \dots, X_{p+1} \in \Gamma({}^\circ T'')$, and $\psi \in \Gamma(T' \otimes \wedge^p ({}^\circ T'')^*)$. From definitions of $\bar{\partial}_b^{(p)}$ it is clear that $\bar{\partial}_b^{(p+1)} \cdot \bar{\partial}_b^{(p)} = 0$, i.e.,

$$0 \longrightarrow \Gamma(T'') \xrightarrow{\bar{\partial}_b^{(0)}} \Gamma(T' \otimes ({}^\circ T'')^*) \xrightarrow{\bar{\partial}_b^{(1)}} \Gamma(T' \otimes \wedge^2 ({}^\circ T'')^*) \longrightarrow \dots$$

is a complex.

We explain the convexity of the CR -structure ${}^\circ T''$ on M . Let X_1, \dots, X_n be a frame of $\Gamma({}^\circ \bar{T}''|U)$ for some open set U in M . Then $\bar{X}_1, \dots, \bar{X}_n$ become the frame of $\Gamma({}^\circ T''|U)$. Take a cross-section S of $\Gamma(F|U)$ such that $S(p) \neq 0$ for any $p \in U$. Hence we obtain smooth functions C_{jk} on U defined by

$$(1.9) \quad \begin{aligned} \sqrt{-1}[X_j, \bar{X}_k] &\equiv C_{jk} S \pmod{{}^\circ T'' \oplus {}^\circ \bar{T}''} , \\ &\text{for } 1 \leq j, k \leq n . \end{aligned}$$

It is trivial that the functional matrix

$$\|C_{jk}\| \text{ is hermitian .}$$

DEFINITION 1.5. ${}^\circ T''$ is strongly pseudo-convex, when for any $p \in M$, there is a cross-section S of $\Gamma(F|U)$ such that

$$\|C_{jk}\| > 0 , \quad \text{on some neighborhood } U \text{ of } P .$$

Finally let us consider a complex manifold V and an imbedding $i: M \rightarrow V$, where $\dim_{\mathbb{C}} V = n + 1$ and $\dim_{\mathbb{R}} M = 2n + 1$. Then a complex sub-bundle ${}^\circ T''$ on M is defined as follows: For any point $p \in M$,

$$(1.10) \quad {}^\circ T''_p = Ci_p^* TM \cap T_{i(p)} V,$$

where $T_{i(p)} V$ denotes the holomorphic tangent space of V at $i(P)$. Since $i(M)$ is a real hypersurface of V , ${}^\circ T''(M)$ becomes the subbundle of CTM of complex fiber dimension n . Clearly ${}^\circ T''$ is integrable, i.e., this subbundle defines a CR -structure on M .

DEFINITION 1.6. Let $i: M \rightarrow V$ be an imbedding as above. Then the CR -structure ${}^\circ T''$ on M defined by (1.10) is called the induced CR -structure by i , or simply the induced CR -structure.

§2. CR -structures on negative line bundles over abelian varieties

2.1. Let T be an abelian variety with an $n \times 2n$ -matrix $\omega = (\omega_a^i)_{1 \leq i \leq n, 1 \leq a \leq 2n}$ as a period matrix, that is, let C^n be the space of n complex variables (z_1, \dots, z_n) and let $Z^{2n} = \overbrace{Z \times \dots \times Z}^{2n}$. The elements of C^n and Z^{2n} are written as column vectors of length n and $2n$, respectively. For any element $d = {}^t(d^1, \dots, d^{2n})$ of Z^{2n} , we put

$$\omega \cdot d = {}^t \left(\sum_{a=1}^{2n} \omega_a^1 d^a, \dots, \sum_{a=1}^{2n} \omega_a^n d^a \right).$$

if Λ denotes the lattice $\{\omega \cdot d; d \in Z^{2n}\}$ in C^n , then

$$T = C^n / \Lambda.$$

Now let B be a negative line bundle over T and let π be the projection of C^n onto T . Then the induced bundle $\pi^{-1}(B)$ of B under π is isomorphic to the trivial bundle $C^n \times C$. From this fact there exists a holomorphic map $f: C^{2n} \times Z^{2n} \rightarrow C - \{0\}$, called an “automorphic factor” of B , satisfying the following conditions (6);

$$(C.1) \quad \text{For } d_1, d_2 \text{ in } Z^{2n}, \text{ and } z \in C^n, \\ f(z, d_1 + d_2) = f(z + \omega d_1, d_2) f(z, d_2).$$

$$(C.2) \quad \text{let } \sim \text{ be the equivalence relation in } C^n \times C \text{ defined by} \\ (z_1, \zeta_1) \sim (z_2, \zeta_2) \iff \text{there is } d \in Z^{2n} \text{ such that} \\ (z_2, \zeta_2) = (z_1 + \omega d, f(z_1, d) \zeta_1).$$

Then the line bundle B over T is isomorphic to $C^n \times C / \sim$. It is clear that automorphic factors of B depend on the choice of isomorphisms of $\pi^{-1}(B)$ onto $C^n \times C$. And we can take an automorphic factor f of B which has the following form (c.f. pp. 111, [5]);

For any $z \in C^n$ and $d \in Z^{2n}$,

$$(2.1) \quad f(z, d) = \exp \{2\pi\sqrt{-1}({}^t z Q \bar{\omega} d + \tfrac{1}{2} d A^\circ d + {}^t b \cdot d)\}$$

where

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix}$$

is an $(n \times n)$ -matrix such that ${}^t \bar{Q} = -Q$, and A° is a $(2n \times 2n)$ -matrix and b denotes a real column vector of length $2n$. We fix the above automorphic factor f with (2.1). Let $h: C^n \rightarrow R$ be the smooth positive function defined by

$$(2.2) \quad h(z) = \exp(-2\pi\sqrt{-1}{}^t z Q \bar{z}).$$

Then we have $h(z) = h(z + \omega d) |f(z, d)|^2$ for every $d \in Z^{2n}$ and $z \in C^n$, so that h induces the hermitian metric \tilde{h} on the line bundle B over T . Hence the Chern class $c(B)$ of B equals to the de Rham cohomology class of

$$\left(\frac{1}{2\pi\sqrt{-1}} \bar{\partial} \log h = \sum_{i,j=1}^n Q_{i\bar{j}} dz^i \wedge d\bar{z}^j \right).$$

However since B is negative, we have $\sqrt{-1}Q < 0$, that is, the hermitian matrix $\sqrt{-1}Q$ is negative definite.

2.2. From the negativity of B we know that if T is regarded as the zero-section of B , then there exists an analytic variety \tilde{T} and a holomorphic map g of B onto \tilde{T} such that for some point $\tilde{t}_0 \in \tilde{T}$.

g is a bi-holomorphic map of $B - T$ onto $\tilde{T} - \{\tilde{t}_0\}$, and $g(T) = \tilde{t}_0$. (c.f. [6])

Clearly \tilde{T} has the isolated singularity point \tilde{t}_0 , which is denoted by (\tilde{T}, \tilde{t}_0) . Let \tilde{S} be a real hypersurface around \tilde{t}_0 in $\tilde{T} - \{\tilde{t}_0\}$. Then local deformations of isolated singularity (\tilde{T}, \tilde{t}_0) induce ones of the induced CR -structure on \tilde{S} by the inclusion $i_{\tilde{S}}: \tilde{S} \rightarrow \tilde{T} - \{\tilde{t}_0\}$. However in terms of the biholomorphic map $g: B - T \rightarrow \tilde{T} - \{\tilde{t}_0\}$ we shall consider local deformations of the induced CR -structure on a real hypersurface around T in $B - T$.

Now let B_1 be the unit circle bundle over T defined by the hermitian metric h on B , i.e.,

$$B_1 = \{e \in B; \tilde{h}(e) = 1\}.$$

PROPOSITION 2.1. *Let ${}^\circ T''(B_1)$ be the induced CR-structure on B_1 by the inclusion $\iota_{B_1}: B_1 \rightarrow B$. Then ${}^\circ T''(B_1)$ is strongly pseudo-convex, that is, B_1 is the real $(2n + 1)$ -dimensional compact strongly pseudo-convex manifold.*

This proposition is proved by the following two lemmas. At first let $\tilde{\psi}$ be the natural projection of $C^n \times C$ onto $B = C^n \times C / \sim$ as in § 1. Here put

$$V = \tilde{\psi}^{-1}(B_1) (\subset C^n \times C).$$

Then it is trivial that

$$V = \{(z, \zeta) \in C^n \times C; h(z) |\zeta|^2 = 1\}.$$

Moreover let ψ be the diffeomorphism of $C^n \times S^1$ onto V defined by

$$\psi(z, \theta) = \left(z, \frac{e^{\sqrt{-1}\theta}}{\sqrt{h(z)}} \right), \quad \text{for } (z, \theta) \in C^n \times S^1,$$

here θ is the angular coordinate of S^1 .

LEMMA 2.2. *Let ${}^\circ T''(V) = CTV \cap T^{0,1}(C^n \times C)$ be the induced CR-structure on V . Let Z_1, \dots , and Z_n , be vector fields on $C^n \times S^1$ defined by*

$$(2.3) \quad Z_j = \frac{\partial}{\partial \bar{z}^j} - \frac{\sqrt{-1}}{2} \frac{\partial \log h}{\partial \bar{z}^j} \frac{\partial}{\partial \theta}, \quad j = 1, \dots, n.$$

Then $\{\psi_*(Z_j)\}_{j=1}^n$ become the global basis of ${}^\circ T''(V)$.

Proof. By direct calculations we obtain

$$\psi_{*(z, \theta)} \left(\frac{\partial}{\partial \bar{z}^j} \right) = \frac{\partial}{\partial \bar{z}^j} - \frac{1}{2} \frac{\partial \log h}{\partial \bar{z}^j} \frac{e^{\sqrt{-1}\theta}}{\sqrt{h}} \frac{\partial}{\partial \zeta} - \frac{1}{2} \frac{\partial \log h}{\partial \bar{z}^j} \frac{e^{-\sqrt{-1}\theta}}{\sqrt{h}} \frac{\partial}{\partial \bar{\zeta}}$$

and

$$\psi_{*(z, \theta)} \left(\frac{\partial}{\partial \theta} \right) = \sqrt{-1} \left(\frac{e^{\sqrt{-1}\theta}}{\sqrt{h}} \frac{\partial}{\partial \zeta} - \frac{e^{-\sqrt{-1}\theta}}{\sqrt{h}} \frac{\partial}{\partial \bar{\zeta}} \right).$$

Here we have

$$(2.4) \quad \psi_*(Z_j) = \frac{\partial}{\partial \bar{z}^j} - \bar{\zeta} \frac{\partial \log h}{\partial \bar{z}^j} \frac{\partial}{\partial \bar{\zeta}}, \quad j = 1, \dots, n.$$

This means $\psi_{*(z, \theta)}(Z_j) \in T_{\psi(z, \theta)}^{0,1}(C^n \times C)$, for any $(z, \theta) \in C^n \times S^1$, so that the $\psi_*(Z_j)$ are cross-sections of ${}^\circ T''(V)$. It is clear that $\{\psi_*(Z_j)\}_{j=1}^n$ is the global base of ${}^\circ T''(V)$. Q.E.D.

Next let ${}^\circ T''(C^n \times S^1)$ be the complex subspace of $CT(C^n \times S^1)$ generated by the basis $\{Z_j\}_{j=1}^n$. Then from the above lemma it is trivial that ${}^\circ T''(C^n \times S^1) = {}^\circ T''(V)$, and that ${}^\circ T''(C^n \times S^1)$ is the CR -structure on $C^n \times S^1$. $CT(C^n \times S^1)$ has the following decomposition; let $F = \{R(\partial/\partial\theta)\}$ be the real line bundle over $C^n \times S^1$, spanned by $\partial/\partial\theta$. Then we get

$$CT(C^n \times S^1) = {}^\circ T''(C^n \times S^1) \oplus {}^\circ \bar{T}''(C^n \times S^1) \oplus CF.$$

LEMMA 2.3. ${}^\circ T''(C^n \times S^1)$ is strongly convex.

Proof. Put $Z_j = \bar{Z}_j$ ($j = 1, \dots, n$). By (2.3) and (2.2) it follows that

$$(2.5) \quad [Z_j, Z_j] = \sqrt{-1} \frac{\partial^2 \log h}{\partial \bar{z}^j \partial z^j} \frac{\partial}{\partial \theta} = 2\pi Q_{jj} \frac{\partial}{\partial \theta} \quad (1 \leq i, j \leq n)$$

On the other hand since Q is negative definite, ${}^\circ T''(C^n \times S^1)$ is strongly convex. Q.E.D.

Remark. The next formulas are trivial;

$$(2.6) \quad [Z_j, Z_i] = [Z_j, Z_i] = \left[Z_j, \frac{\partial}{\partial \theta} \right] = \left[Z_j, \frac{\partial}{\partial \theta} \right] = 0$$

for $1 \leq i, j \leq n$.

We shall express the induced CR -structure ${}^\circ T''(B_1)$ by using $\{Z_j\}_{j=1}^n$. Let $\tilde{\psi}$ be the canonical projection of $C^n \times C$ onto $B = C^n \times C/\sim$ and $\hat{\psi}$ the composite map of ψ and $\tilde{\psi}$;

$$\hat{\psi} = \tilde{\psi} \circ \psi: C^n \times S^1 \xrightarrow{\psi} C^n \times C \xrightarrow{\tilde{\psi}} B.$$

It follows from (C.2) in 2.1 and the definition of ψ that for $(z, \theta), (z', \theta') \in C^n \times S^1$, $\hat{\psi}(z, \theta) = \hat{\psi}(z', \theta')$ means that there is a $d \in Z^{2n}$ such that

$$z' = z + \omega d, \quad \text{and} \quad \theta' = \theta + \arg f(z, d).$$

LEMMA 2.4. For $(z, \theta) \in C^n \times S^1$, and $d \in Z^{2n}$

$$\hat{\psi}_{*(z, \theta)}(Z_j) = \hat{\psi}_{*(z + \omega d, \theta + \arg f(z, d))}(Z_j), \quad (j = 1, \dots, n),$$

and

$$\hat{\psi}_{*(z, \theta)}\left(\frac{\partial}{\partial \theta}\right) = \hat{\psi}_{*(z + \omega d, \theta + \arg f(z, d))}\left(\frac{\partial}{\partial \theta}\right).$$

Therefore $\{\hat{\psi}_*(Z_j)\}_{j=1, \dots, n}$ and $\hat{\psi}_*(\partial/\partial\theta)$ become vector fields on B_1 .

Proof. It follows that

$$\begin{aligned} & \hat{\psi}_{*(z+\omega d, \theta + \arg f(z, d))}(Z_j) \\ &= \tilde{\psi}_{*(z+\omega d, \frac{e^{\sqrt{-1}(\theta + \arg f(z, d))}}{\sqrt{h(z+\omega d)}})} \left(\frac{\partial \bar{z}^j}{\partial} - \bar{\zeta} \frac{\log h(z + \omega d)}{\partial \bar{z}^j} \frac{\partial}{\partial \bar{\zeta}} \right). \end{aligned}$$

Here set

$$\zeta_0 = \frac{e^{\sqrt{-1}(\theta + \arg f(z, d))}}{\sqrt{h(z + \omega d)}}$$

and

$$A = \hat{\psi}_{*((z+\omega d, \theta + \arg f(z, d)))}(Z_j).$$

Using $h(z) = h(z + \omega d) |f(z, d)|^2$ as in § 1, we get

$$(2.7) \quad A = \tilde{\psi}_{*(z+\omega d, \zeta_0)} \left(\frac{\partial}{\partial \bar{z}^j} + \frac{\partial \log f(z, d)}{\partial \bar{z}^j} \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} - \frac{\partial \log h(z)}{\partial \bar{z}^j} \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right).$$

On the other hand let g_d be the bi-holomorphic map of $C^n \times C$ onto $C^n \times C$ defined by

$$g_d(z, \zeta) = (z + \omega d, f(z, d)\zeta), \quad \text{for any } d \in Z^{2n}.$$

Then we have

$$g_{d(z, f(z, d)^{-1}\zeta_0)} \left(\frac{\partial}{\partial \bar{z}^j} \right) = \left(\frac{\partial}{\partial \bar{z}^j} + \frac{\partial \log f(z, d)}{\partial \bar{z}^j} \zeta_0 \left(\frac{\partial}{\partial \bar{\zeta}} \right) \right)_{g_{d(z, f(z, d)^{-1}\zeta_0)}},$$

and

$$g_{d(z, f(z, d)^{-1}\zeta_0)} \left(\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right) = \bar{\zeta}_0 \left(\frac{\partial}{\partial \bar{\zeta}} \right)_{g_{d(z, f(z, d)^{-1}\zeta_0)}}.$$

Hence from (2.7) it follows that

$$A = (\tilde{\psi} \cdot g_d)_{(z, f(z, d)^{-1}\zeta_0)} (\psi_{*(z, \theta)} Z_j) \quad (\text{c.f. (2.4)})$$

But as $\tilde{\psi} g_d = \tilde{\psi}$, we have finally

$$A = \hat{\psi}_{*(z, \theta)} Z_j.$$

Similarly it is proved

$$\hat{\psi}_{*(z, \theta)} \left(\frac{\partial}{\partial \theta} \right) = \hat{\psi}_{*(z+\omega d, \theta + \arg f(z, d))} \left(\frac{\partial}{\partial \theta} \right). \quad \text{Q.E.D.}$$

Let us return to the proof of Proposition 2.1. By virtue of Lemmas 2.2 and 2.4. the induced CR-structure ${}^\circ T''(B_i)$ on B_i is spanned by $\hat{\psi}_*(Z_i)$,

\dots , and $\hat{\psi}_*(Z_n)$, denoted by $\{\{\hat{\psi}_*(Z_1), \dots, \hat{\psi}_*(Z_n)\}\}$. Furthermore we have the next decomposition of CTB_1 ;

$$(2.8) \quad CTB_1 = {}^\circ T''(B_1) \oplus {}^\circ \bar{T}''(B_1) \oplus CF(B_1) ,$$

where

$$(2.9) \quad \begin{cases} {}^\circ T''(B_1) = \{\{\hat{\psi}_*(Z_1), \dots, \hat{\psi}_*(Z_n)\}\} , \\ {}^\circ \bar{T}''(B_1) = \{\{\hat{\psi}_*(Z_1), \dots, \hat{\psi}_*(Z_n)\}\} , \\ CF(B_1) = \{\{\hat{\psi}_*(\partial/\partial\theta)\}\} . \end{cases}$$

Thus our proposition is completely proved.

§ 3. Infinitesimal deformations of ${}^\circ T''(B_1)$

Notations being as in § 2, let us first consider relations between almost CR -structures of finite distance to ${}^\circ T''(C^n \times S^1)$ on $C^n \times S^1$ (c.f. Lemma 2.3) and ones of finite distance to ${}^\circ T''(B_1)$ on B_1 . For this purpose we set

$${}^\circ T'(C^n \times S^1) = {}^\circ \bar{T}''(C^n \times S^1) \oplus CF$$

and

$$T'(B_1) = {}^\circ \bar{T}''(B_1) \oplus CF(B_1) .$$

From Proposition 1.3 it is enough to consider the correspondence between $\Gamma(\text{Hom}({}^\circ T''(C^n \times S^1), T'(C^n \times S^1)))$ and $\Gamma(\text{Hom}({}^\circ T''(B_1), T'(B_1)))$. Here we put for simplicity

$${}^\circ T'' = {}^\circ T''(C^n \times S^1) \quad \text{and} \quad T' = T'(C^n \times S^1) .$$

Let Z_1, \dots , and Z_n be the basis of ${}^\circ T''$ defined by Lemma 2.2. Then we have the following

PROPOSITION 3.1. *For any $\varphi \in \Gamma(\text{Hom}({}^\circ T'', T'))$ we can write*

$$(3.1) \quad \varphi(Z_j) = \sum_{k=1}^n \varphi_j^k Z_k + \varphi_j \frac{\partial}{\partial \theta} \quad (j = 1, \dots, n) ,$$

where φ_j^k and φ_j are smooth functions on $C^n \times S^1$. Then φ induces an element of $\Gamma(\text{Hom}({}^\circ T''(B_1), T'(B_1)))$ if and only if the following condition (C) is satisfied;

$$(C) \quad \begin{cases} \varphi_j^k(z, \theta) = \varphi_j^k(z + \omega d, \theta + \arg f(z, d)) \\ \varphi_j(z, \theta) = \varphi_j(z + \omega d, + \arg f(z, d)) , \end{cases}$$

for each $d \in \mathbb{Z}^{2n}$ and $(z, \theta) \in \mathbb{C}^n \times S^1$.

Proof. Let τ be the linear map from $\Gamma(\text{Hom}({}^\circ T''(B_i), T'(B_i)))$ to $\Gamma(\text{Hom}({}^\circ T'', T'))$ defined as follows; let $\tilde{\varphi}$ be any element of $\Gamma(\text{Hom}({}^\circ T''(B_i), T'(B_i)))$. Then we put, for any $(z, \theta) \in \mathbb{C}^n \times S^1$,

$$[(\tau\tilde{\varphi})(Z_j)]_{(z,\theta)} = \hat{\psi}_{*\hat{\varphi}(z,\theta)}^{-1}(\tilde{\varphi}(\hat{\psi}_{*(z,\theta)}(Z_j))) .$$

If $\tilde{\varphi}$ is an element of $\Gamma(\text{Hom}({}^\circ T''(B_i), T'(B_i)))$ with the expression

$$\tilde{\varphi}(\hat{\psi}_*(Z_j)) = \sum_{k=1}^n \tilde{\varphi}_j^k \hat{\psi}_*(Z_k) + \tilde{\varphi}_j \hat{\psi}_*\left(\frac{\partial}{\partial \theta}\right) ,$$

then it follows that

$$(\tau\tilde{\varphi})(Z_j) = \sum_{k=1}^n (\tilde{\varphi}_j^k \circ \hat{\psi})Z_k + (\tilde{\varphi}_j \circ \hat{\psi})\frac{\partial}{\partial \theta} .$$

Thus $\tau\tilde{\varphi}$ satisfies the condition (C). Conversely an arbitrary element $\varphi \in \Gamma(\text{Hom}({}^\circ T'', T))$ satisfying (C) induces an element $\tilde{\varphi}$ of $\Gamma(\text{Hom}({}^\circ T''(B_i), T'(B_i)))$, and we have

$$\varphi = \tau\tilde{\varphi} . \quad \text{Q.E.D.}$$

We denote by $\Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$ the set of all smooth-section of $\text{Hom}({}^\circ T'', T')$ satisfying the condition (C). The above Proposition 3.1 shows that

$$\tau: \Gamma(\text{Hom}({}^\circ T''(B_i), T'(B_i))) \rightarrow \Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$$

is isomorphic.

More generally let $\Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T')$ be the set of all smooth-sections φ of $\wedge^k({}^\circ T'')^* \otimes T'$, ($k = 0, 1, \dots, n$) such that, when φ is expressed as

$$(Z_{j_1} \wedge \dots \wedge Z_{j_k}) = \sum_{\ell=1}^n \varphi_{j_1, \dots, j_k}^\ell Z_\ell + \varphi_{j_1, \dots, j_k} \frac{\partial}{\partial \theta} ,$$

$$(1 \leq j_1 < \dots < j_k \leq n) .$$

Then all coefficients $\varphi_{j_1, \dots, j_k}$ and $\varphi_{j_1, \dots, j_k}$ satisfy the condition (C). Then τ induces the isomorphism of $\Gamma(\wedge^k({}^\circ T''(B_i))^* \otimes T'(B_i))$ onto $\Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T')$. Here we have the following commutative diagram;

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(T'(B_1)) & \xrightarrow{\bar{\partial}_b^{(0)}} & \Gamma(\text{Hom}({}^\circ T'' B_1), T'(B_1)) & & \\
& & \downarrow \tau & & \downarrow \tau & & \\
0 & \longrightarrow & \Gamma_{(C)}(T') & \xrightarrow{\bar{\partial}_b^{(0)}} & \Gamma_{(C)}(\text{Hom}({}^\circ T'', T')) & & \\
& & & & \xrightarrow{\bar{\partial}_b^{(1)}} & \Gamma(\wedge^2({}^\circ T''(B_1) \otimes T'(B)) & \longrightarrow \\
& & & & & \downarrow \tau & \\
& & & & \xrightarrow{\bar{\partial}_b^{(1)}} & \Gamma_{(C)}(\wedge^2({}^\circ T'')^* \otimes T') & \longrightarrow
\end{array}$$

where the $\bar{\partial}_b^{(k)}$ denote the operators defined by (1.7) and (1.8).

Let $H^k({}^\circ T''(B_1))$ and $H_{(C)}^k({}^\circ T'')$ be the k -th cohomologies of complexes $\{\Gamma(\wedge^k({}^\circ T''(B_1))^* \otimes T'(B_1)), \bar{\partial}_b^{(k)}\}_{k=0}^n$ and $\{\Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T'), \bar{\partial}_b^{(k)}\}_{k=0}^n$, respectively. Then we know that

$$H^k({}^\circ T''(B_1)) \cong H_{(C)}^k({}^\circ T'') .$$

3.1. We shall determine explicitly a basis of the first cohomology $H_{(C)}^1({}^\circ T'')$, that is, the infinitesimal deformation of ${}^\circ T''(B_1)$. First of all let φ be an element of $\Gamma_{(C)}(\text{Hom}({}^\circ T'', T')$ with

$$(3.1) \quad \varphi(Z_j) = \sum_{k=1}^n \varphi_j^k Z_k + \varphi_j \frac{\partial}{\partial \theta} \quad (j = 1, \dots, n) .$$

Then we obtain the following

PROPOSITION 3.2. *It follows that $\bar{\partial}_b^{(1)}\varphi = 0$ if and only if for all $i, j \in \{1, \dots, n\}$,*

$$(3.2) \quad Z_i \varphi_j^k - Z_j \varphi_i^k = 0, \quad (k = 1, \dots, n)$$

and

$$(3.3) \quad \sum_{k=1}^n (\varphi_j^k \Phi_{ki} - \varphi_i^k \Phi_{kj}) + Z_i \varphi_j - Z_j \varphi_i = 0 ,$$

where we put

$$(3.4) \quad \Phi_{i\bar{j}} = 2\pi Q_{i\bar{j}} \quad (\text{c.f. (2.1)}) .$$

Proof. By (1.8) and (2.6) we see that

$$\begin{aligned}
\bar{\partial}_b^{(1)}\varphi(Z_i, Z_j) &= [Z_i, \varphi(Z_j)]_{T'} - [Z_j, \varphi(Z_i)]_{T'} \\
&= \sum_{k=1}^n (Z_i \varphi_j^k - Z_j \varphi_i^k) Z_k + \sum_{k=1}^n (\varphi_j^k [Z_i, Z_k]_{T'} - \varphi_i^k [Z_j, Z_k]_{T'}) \\
&\quad + (Z_i \varphi_j - Z_j \varphi_i) \frac{\partial}{\partial \theta} .
\end{aligned}$$

However as $[Z_j, Z_i] = 2\pi Q_{ij}(\partial/\partial\theta)$ (c.f. (2.5)), it is trivial that

$$\begin{aligned} \bar{\partial}_\theta^{(1)} \varphi(Z_i, Z_j) &= \sum_{n=1}^n (Z_i \varphi_i^k - Z_j \varphi_i^k) Z_k \\ &+ \left\{ \sum_{k=1}^n (\varphi_j^k 2\pi Q_{kj} - \varphi_i^k 2\pi Q_{ki}) + Z_i \varphi_j - Z_j \varphi_i \right\} \frac{\partial}{\partial\theta}. \end{aligned}$$

This fact proves Proposition 3.2.

Q.E.D.

In order to study properties of the $\{\varphi_j^k, \varphi_j\}$ $k, j = 1, \dots, n$ on $C^n \times S^1$ satisfying the differential equations (3.2) and (3.3), we denote by $\mathfrak{S}(f)$ the Fourier expansion of any function f on $C^n \times S^1$ with respect to the angular parameter of S^1 . Let us put

$$(3.5) \quad \mathfrak{S}(\varphi_j^k)(z, \theta) = \sum_{m \in \mathbb{Z}} \varphi_{j,m}^k(z) e^{\sqrt{-1}m\theta}.$$

Then from the uniqueness of Fourier expansions and the condition (C) it follows that

$$(3.6) \quad \varphi_{j,m}^k(z + \omega d) = \varphi_{j,m}^k(z) e^{-\sqrt{-1}m \arg f(z, d)}$$

for any $d \in \mathbb{Z}^{2n}$ and $m = 0, \pm 1, \pm 2, \dots$.

At first we consider the differential systems (3.2). By (2.3), (3.2) means that

$$(3.7) \quad \frac{\partial \varphi_j^k}{\partial \bar{z}^i} - \frac{\partial \varphi_i^k}{\partial \bar{z}^j} - \frac{\sqrt{-1}}{2} \left(\frac{\partial \log h(z)}{\partial \bar{z}^i} \frac{\partial \varphi_j^k}{\partial \theta} - \frac{\partial \log h(z)}{\partial \bar{z}^j} \frac{\partial \varphi_i^k}{\partial \theta} \right) = 0, \\ (1 \leq i, j, k \leq n).$$

LEMMA 3.3. *For each $m \in \mathbb{Z}$, the $\{\varphi_{j,m}^k\}$ $k, j = 1, \dots, n$ satisfy the following equation;*

$$(3.8) \quad \frac{\partial \varphi_{j,m}^k}{\partial \bar{z}^i} - \frac{\partial \varphi_{i,m}^k}{\partial \bar{z}^j} + \frac{m}{2} \left(\frac{\partial \log h}{\partial \bar{z}^i} \varphi_{j,m}^k - \frac{\partial \log h}{\partial \bar{z}^j} \varphi_{i,m}^k \right) = 0.$$

Proof. This lemma is trivial from (3.7) and definitions of the $\varphi_{j,m}^k$.

Q.E.D.

Moreover we have

LEMMA 3.4. *Let $\hat{\varphi}_m$ be the element of $\Gamma_{(C)}(\text{Hom}({}^o T'', T'))$ defined by*

$$\hat{\varphi}_m(Z_j) = \sum_{k=1}^n e^{\sqrt{-1}m\theta} \varphi_{j,m}^k(z) Z_k \quad (m \in \mathbb{Z}).$$

Then there exists an element ζ_m of $\Gamma_{(C)}(T')$ for every non-zero integer m such that

$$\zeta_m = \sum_{k=1}^n \zeta_m^k(z) Z_k ,$$

and

$$(3.9) \quad \bar{\partial}_b^{(0)} \zeta_m(Z_j) = \phi_m(Z_j) + \sum_{k=1}^n \zeta_m^k \Phi_{kj} \frac{\partial}{\partial \theta} .$$

Proof. Put $\psi_{j,m}^k(z) = \phi_{j,m}^k(z) h^{m/2}(z)$. Then we find from (2.2) and (3.6) that

$$\psi_{j,m}^k(z + \omega \mathbf{d}) = \psi_{j,m}^k(z) e^{-\sqrt{-1}m \arg f(z, \mathbf{d})} j(z, \mathbf{d})^{-m} = \psi_{j,m}^k(z) f(z, \mathbf{d})^{-m} ,$$

$$(\mathbf{d} \in \mathbb{Z}^{2n}) .$$

Therefore if we set

$$\psi_m^k = \sum_{j=1}^n \psi_{j,m}^k d\bar{z}^j ,$$

ψ_m^k is regarded as the cross-section of the vector bundle $B^{-m} \otimes \wedge^{0,1}(T)$ over the abelian variety T , where $\wedge^{0,1}(T)$ represents the bundle consisting of $(0,1)$ -type differential forms on T . Let $\bar{\partial}$ be the usual exterior derivation of type $(0,1)$ on T . Then it is clear that

$$\bar{\partial} \psi_m^k = \sum_{i < j} \left(\frac{\partial \psi_{j,m}^k}{\partial \bar{z}^i} - \frac{\partial \psi_{i,m}^k}{\partial \bar{z}^j} \right) d\bar{z}^i \wedge \bar{z}^j .$$

But

$$\frac{\partial \psi_{j,m}^k}{\partial \bar{z}^i} = \left[\frac{\partial \phi_{j,m}^k}{\partial \bar{z}^i} + \frac{m}{2} \phi_{j,m}^k \frac{\partial \log h}{\partial \bar{z}^i} \right] h^{m/2} ,$$

so that using (3.8), we obtain, for any $m \in \mathbb{Z}$,

$$\bar{\partial} \psi_m^k = 0 , \quad (k = 1, \dots, n) .$$

Now let $m \neq 0$. Then, since B is a negative line bundle and the holomorphic tangent bundle of T is analytically trivial ($\dim_{\mathbb{C}} T \geq 2$), there is an element $\tilde{\eta}_m^k$ of $\Gamma(B^{-m})$ such that, for any $m (\neq 0)$,

$$\bar{\partial} \tilde{\eta}_m^k = \psi_m^k , \quad (k = 1, \dots, n) .$$

If we write η_m^k the pull-back of $\tilde{\eta}_m^k$ by the projection $\hat{\psi}: C^n \rightarrow T$, η_m^k is the cross-section of the trivial bundle $C^n \times C$ over C^n and satisfies the following relations;

$$(3.10) \quad \begin{cases} \eta_m^k(z + \omega \mathbf{d}) = \eta_m^k(z) f(z, \mathbf{d})^{-m} & (\mathbf{d} \in \mathbb{Z}^{2n}), \\ \bar{\partial} \eta_m^k = \psi_m^k. \end{cases}$$

Furthermore let ζ_m^k be the smooth function on $\mathbb{C}^n \times S^1$ defined by

$$\zeta_m^k(z, \theta) = e^{\sqrt{-1}m\theta} \eta_m^k(z) h^{-m/2}(z).$$

We have then from (3.10)

$$\zeta_m^k(z + \omega \mathbf{d}, \theta + \arg f(z, \mathbf{d})) = \zeta_m^k(z, \theta),$$

that is, the vector field ζ_m defined by

$$\zeta_m = \sum_{k=1}^n \zeta_m^k Z_k$$

belongs to $\Gamma_{(C)}(T')$.

This ζ_m satisfies (3.9). Indeed it follows that

$$\begin{aligned} (\bar{\partial}_b \zeta_m)(Z_j) &= \sum_{k=1}^n (Z_j \zeta_m^k) Z_k + \sum_k \zeta_m^k [Z_j, Z_k] \\ &= \sum_k e^{\sqrt{-1}m\theta} \frac{\partial \eta_m^k}{\partial \bar{z}^j} h^{-m/2} Z_k + \sum_k \zeta_m^k \Phi_{kj} \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n). \end{aligned}$$

Here using $\bar{\partial} \eta_m^k = \psi_m^k$ in (3.10), we have

$$\begin{aligned} (\bar{\partial}_b \zeta_m)(Z_j) &= \sum_k e^{\sqrt{-1}m\theta} \psi_{j,m}^k h^{-m/2} Z_k + \sum_k \zeta_m^k \Phi_{kj} \frac{\partial}{\partial \theta} \\ &= \sum_k e^{\sqrt{-1}m\theta} \phi_{j,m}^k Z_k + \sum_k \zeta_m^k \Phi_{kj}. \end{aligned} \quad \text{Q.E.D.}$$

Next for (3.3), we set

$$(\mathfrak{S}\varphi_j)(z, \theta) = \sum_{m \in \mathbb{Z}} \varphi_{j,m}(z) e^{\sqrt{-1}m\theta}, \quad (j = 1, \dots, n).$$

Let $\hat{\varphi}_m \in \Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$ be as in Lemma 3.4. If we define the element $\varphi_m \in \Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$ for each $m \in \mathbb{Z}$, by

$$\varphi_m(Z_j) = \hat{\varphi}_m(Z_j) + \varphi_{j,m} e^{\sqrt{-1}m\theta} \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n),$$

then each φ_m is the $e^{\sqrt{-1}m\theta}$ -component of the Fourier expansion $\mathfrak{S}(\varphi)$ of φ , that is,

$$\mathfrak{S}(\varphi) = \sum_{m \in \mathbb{Z}} \varphi_m.$$

Here $\mathfrak{S}(\varphi)$ is defined by

$$\mathfrak{S}(\varphi)(Z_j) = \sum_k \mathfrak{S}(\varphi_j^k) Z_k + \mathfrak{S}(\varphi_j) \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n).$$

Since $\bar{\partial}_b^{(1)}\varphi = 0$, we find

$$\bar{\partial}_b^{(1)}\varphi_m = 0, \quad \text{for every } m \in \mathbb{Z}.$$

Moreover we can write, for any non-zero integer m , using ζ_m in Lemma 3.4,

$$(3.11) \quad (\varphi_m - \bar{\partial}_b^{(0)}\zeta_m)(Z_j) = \tilde{\varphi}_{j,m}(z) e^{\sqrt{-1}m\theta} \frac{\partial}{\partial \theta}, \quad j = 1, \dots, n,$$

where the $\tilde{\varphi}_{j,m}(z)$ are smooth functions on \mathbb{C}^n such that

$$\tilde{\varphi}_{j,m}(z + \omega d) = \tilde{\varphi}_{j,m}(z), \quad \text{for all } (z, d) \in \mathbb{C}^n \times \mathbb{Z}^{2n}.$$

LEMMA 3.5. *Let m be any non-zero integer. Then $\varphi_m - \bar{\partial}_b^{(0)}\zeta_m$ in (3.11) is $\bar{\partial}_b^{(0)}$ -boundary, i.e., there exists an element $\eta_m \frac{\partial}{\partial \theta}$ of $\Gamma_{(C)}(T')$ such that*

$$\bar{\partial}_b^{(0)}\left(\eta_m \frac{\partial}{\partial \theta}\right) = \varphi_m - \bar{\partial}_b^{(0)}\zeta_m.$$

Proof. As $\bar{\partial}_b^{(1)}(\varphi_m - \bar{\partial}_b^{(0)}\zeta_m) = 0$, the family of functions $\{\tilde{\varphi}_{j,m}\}_{j=1,\dots,n}$ in the right hand side of (3.11) satisfies the following relations;

$$\frac{\partial \tilde{\varphi}_{j,m}}{\partial \bar{z}^i} - \frac{\partial \tilde{\varphi}_{i,m}}{\partial \bar{z}^j} + \frac{m}{2} \left(\frac{\partial \log h}{\partial \bar{z}^i} \tilde{\varphi}_{j,m} - \frac{\partial \log h}{\partial \bar{z}^j} \tilde{\varphi}_{i,m} \right) = 0,$$

$$1 \leq i, j \leq n.$$

Therefore this lemma is proved in the same way as Lemma 3.4. Q.E.D.

We obtain from Lemma 3.5 the following

PROPOSITION 3.6. *Let φ be an arbitrary element of $\Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$ such that $\bar{\partial}_b^{(1)}\varphi = 0$. Moreover let $\mathfrak{S}(\varphi)$ be the Fourier expansion of φ with respect to the parameter θ of S^1 ;*

$$\mathfrak{S}(\varphi) = \sum_{m \in \mathbb{Z}} \varphi_m$$

where the φ_m are elements of $\Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$ defined by

$$\varphi_m(Z_j) = e^{\sqrt{-1}m\theta} \left(\sum_k \varphi_{j,m}^k(z) Z_k + \varphi_{j,m}(z) \frac{\partial}{\partial \theta} \right).$$

Then for all $m (\neq 0) \in \mathbb{Z}$, we have,

$$\varphi_m = \bar{\partial}_b^{(0)} \zeta_m, \quad \text{for some } \zeta_m \in \Gamma_{(C)}(T').$$

Furthermore we can prove the following

PROPOSITION 3.7. *Let all notations be as in the above proposition. Let $\varphi \in \Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$, with $\bar{\partial}_b^{(1)} \varphi = 0$. Then φ is $\bar{\partial}_b$ -cohomologous to φ_0 , where $\mathfrak{S}(\varphi) = \varphi_0 - \sum_{Z-(0) \ni m} \varphi_m$.*

Proof. In general let f be any C^∞ -function on $C^n \times S^1$. Then $\mathfrak{S}(f)$ converges uniformly on every compact subset of $C^n \times S^1$. Now for any $\varphi \in \Gamma_{(C)}(\text{Hom}({}^\circ T'', T'))$, we define the norm, denoted by $|\varphi|$, as follows; Let $\varphi(Z_j) = \sum \varphi_j^k Z_k - \varphi_j \frac{\partial}{\partial \theta}$. Then we have

$$|\varphi| = \max_{1 \leq j, k \leq n} \{ \sup |\varphi_j^k|, \sup |\varphi_j| \}.$$

This is well-defined because of the condition (C). Here if we put $\mathfrak{S}_k(\varphi) = \sum_{|m| \leq k} \varphi_m$, for any non-negative integer k , we find that for any $\varepsilon > 0$, there exists an integer $k(\varepsilon) \geq 0$, such that

$$|\varphi - \mathfrak{S}_{k(\varepsilon)}(\varphi)| < \varepsilon.$$

Let τ be the isomorphism of the complex $\{ \Gamma(\wedge^k({}^\circ T''(B_i))^* \otimes T'(B_i)), \bar{\partial}_b^k \}$ onto $\{ \Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T'), \bar{\partial}_b(k) \}$ as in the proof of Proposition 3.1.

On the other hand we impose the hermitian innerproduct, \langle, \rangle on $CT(B_i)$ such that, $\psi_*(Z_1), \dots, \psi_*(Z_n), \psi_*(Z_1), \dots, \psi^*(Z_n)$ and $\psi_*(\partial/\partial \theta)$ are orthonormal basis. For every $\tilde{\varphi} \in \Gamma(\wedge^k({}^\circ T(B_i))^* \otimes T'(B_i))$, we set

$$||\tilde{\varphi}|| = \sum_{1 \leq j_1 < \dots < j_k \leq n} \langle \tilde{\varphi}(\psi_*(Z_{j_1}), \dots, \psi_*(Z_{j_k}), \tilde{\varphi}(\psi_*(Z_{j_1}), \dots, \psi_*(Z_{j_k}^*)) \rangle^{\frac{1}{2}}.$$

The L_2 -norm, denoted by $|| \cdot ||_{B_1}$, on $\Gamma(\wedge^k({}^\circ T''(B_i))^* \otimes T'(B_i))$ is defined by

$$||\tilde{\varphi}||_{B_1} = \int_{B_1} ||\tilde{\varphi}|| dv$$

where dv denotes the volume element associated with the hermitian inner product \langle, \rangle on $CT(B_i)$.

We can further form the formal adjoint

$$\bar{\partial}_2^{(k)*}: \Gamma(\wedge^k({}^\circ T''(B_i))^* \otimes T'(B_i)) \rightarrow \Gamma(\wedge^{k-1}({}^\circ T''(B_i))^* \otimes T'(B_i))$$

of $\bar{\partial}_b^{(k-1)}$ with respect to the above norm $|| \cdot ||_{B_1}$, ($k = 1, \dots, n$).

Now take an element φ of $\Gamma_{(C)}({}^\circ T'' T')$ with $\bar{\partial}_b^{(1)} \varphi = 0$. For any $\varepsilon > 0$, there exists an integer $k(\varepsilon)$ such that

$$(3.12) \quad \|\tau^{-1}\varphi - \tau^{-1}(\mathfrak{S}_{k(\varepsilon)}(\varphi))\| < \varepsilon .$$

Moreover it is clear that $\bar{\partial}_b^{(1)}\tau^{-1}\varphi = 0$. Since ${}^\circ T''(B_1)$ is strongly pseudoconvex and $\dim_{\mathbb{R}} B_1 \geq 5$, we know from § 6 [1] that there exists $\tilde{\zeta} \in \Gamma(T''(B_1))$ and $\tilde{\eta} \in \Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ such that

$$\tau^{-1}\varphi - \tau^{-1}\varphi_0 = \tilde{\eta} + \bar{\partial}_b^{(0)}\tilde{\zeta} ,$$

where $(\bar{\partial}_b^{(2)*} \cdot \bar{\partial}_b^{(1)} + \bar{\partial}_b^{(0)}\bar{\partial}_b^{(1)*})\tilde{\eta} = 0$.

We shall show $\tilde{\eta} = 0$. Indeed suppose $\tilde{\eta} \neq 0$. Then it follows that

$$\|\tau^{-1}\varphi - \tau^{-1}\varphi_0\|_{B_1} \geq \|\tilde{\eta}\|_{B_1} > \varepsilon_1 , \quad \text{for some } \varepsilon_1 > 0 .$$

Here for any $\varepsilon > 0$ with $\varepsilon < \varepsilon_1$, we choose an integer $k(\varepsilon)$ satisfying (3.12), and put

$$\mathfrak{S}'_{k(\varepsilon)}(\varphi) = \mathfrak{S}_{k(\varepsilon)}(\varphi) - \varphi_0 .$$

It follows from Proposition 3.6 that there is an element $\varepsilon \in \Gamma(T')$ such that

$$\mathfrak{S}'_{k(\varepsilon)}(\varphi) = \bar{\partial}_b^{(0)}\zeta ,$$

so that we have

$$\varepsilon > \|\tau^{-1}\varphi - \tau^{-1}\varphi_0 - \tau^{-1}\mathfrak{S}'_{k(\varepsilon)}(\varphi)\|_{B_1} = \|\tau^{-1}\varphi - \tau^{-1}\varphi_0\bar{\partial}_b^{(0)}\tau^{-1}\zeta\|_{B_1} \geq \|\tilde{\eta}\|_{B_1} .$$

This is a contradiction and so our proposition is proved. Q.E.D.

By virtue of the above arguments, in order to determine the infinitesimal deformation $H^1_{(C)}({}^\circ T'') (\cong H^1({}^\circ T''(B_1)))$, it is enough to consider the following subcomplex (3.13) of $\{\Gamma_{(C)}(\wedge^p({}^\circ T'')^* \otimes T'), \bar{\partial}_b^{(p)}\}$; Let

$$\varphi \in \Gamma(\wedge^k({}^\circ T'')^* \otimes T')$$

and set

$$\varphi(Z_{j_1}, \dots, Z_{j_k}) = \sum_{i=1}^n \varphi_{j_1, \dots, j_k}^i Z_i + \varphi_{j_1, \dots, j_k} \frac{\partial}{\partial \theta} \quad (1 \leq j_1, \dots, j_k \leq n) .$$

We denote by $\Gamma_T(\wedge^k({}^\circ T'')^* \otimes T')$ the set of all $\varphi \in \Gamma(\wedge^k({}^\circ T'')^* \otimes T')$ such that $\varphi_{j_1, \dots, j_k}^i$ and $\varphi_{j_1, \dots, j_k}$ are smooth functions on the abelian variety T . Then the complex

$$(3.13) \quad \begin{aligned} 0 \longrightarrow \Gamma_T(T'') &\xrightarrow{\bar{\partial}_b^{(0)}} \Gamma_T(({}^\circ T'')^* \otimes T') \\ &\xrightarrow{\bar{\partial}_b^{(1)}} \Gamma_T(\wedge^2({}^\circ T'')^* \otimes T') \xrightarrow{\bar{\partial}_b^{(2)}} \dots , \end{aligned}$$

is the required one.

THEOREM 3.8. *Let ${}^\circ T''$ be the strongly pseudo-convex CR-structure on $C^n \times S^1$ ($n \geq 2$) induced from the negative line bundle B over the abelian variety T ($\dim_C T = n$), as before. Let $\varphi \in \Gamma_{(C)}({}^\circ T'')^* \otimes T'$ with*

$$(3.14) \quad \varphi(Z_{\bar{j}}) = \sum_{k=1}^n C_j^k Z_k + a_j \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n),$$

where the C_j^k and the a_j are arbitrary constants such that

$$(3.15) \quad \sum_{k=1}^n C_j^k Q_{k\bar{i}} = \sum_{k=1}^n C_i^k Q_{k\bar{j}}, \quad (1 \leq i, j \leq n).$$

Then any element of $H_{(C)}^1({}^\circ T'')$ can be represented by some element ρ as above.

Proof. Take $\varphi \in \Gamma_T({}^\circ T'')^* \otimes T'$ such that $\bar{\partial}_0^{(1)}\varphi = 0$ and

$$(3.16) \quad \varphi(Z_{\bar{j}}) = \sum_{k=1}^n \varphi_j^k Z_k + \varphi_j \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n).$$

Recall that φ_j^k and φ_j are C^∞ -functions on T . By Proposition 3.2, $\bar{\partial}_0^{(1)}\varphi = 0$ means that φ satisfies equations (3.2) and (3.3). Here using φ_j^k and φ_j in (3.16), we put

$$\psi = \sum_{j,k=1}^n \varphi_j^k \frac{\partial}{\partial z^k} \otimes d\bar{z}^j, \quad \omega = \sum_{j=1}^n \varphi_j d\bar{z}^j,$$

and

$$\Phi = \sum_{i,j=1}^n \Phi_{i\bar{j}} dz^i \wedge d\bar{z}^j (= \sqrt{-1} \partial \bar{\partial} \log h), \quad (\text{c.f. (1.2)}),$$

where $\Phi_{i\bar{j}} = 2\pi Q_{i\bar{j}}$.

Let $\bigwedge^{(p,q)}(T)$ be the set of all differential forms of type (p, q) on T . Moreover let \frown be the generalized interior product, that is, for any $\tilde{\psi} \in \bigwedge^{(p,q)}(T) \otimes \Gamma(T(T))$ with

$$\begin{aligned} \tilde{\psi} = \sum_{j_1, \dots, j_q} \sum_{i_1, \dots, i_p} \sum_k A_{i_1, \dots, i_p, j_1, \dots, j_q}^k dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \\ \dots \wedge d\bar{z}^{j_q} \otimes \frac{\partial}{\partial z^k} \end{aligned}$$

the linear map $\tilde{\psi} \frown : \bigwedge^{(\ell, m)}(T) \rightarrow \bigwedge^{(\ell+p-1, m+q)}(T)$ is defined by

$$\begin{aligned}
\tilde{\psi} \frown & \left(\sum_{\substack{\alpha_1, \dots, \alpha_\ell \\ \beta_1, \dots, \beta_m}} B_{\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_\ell} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_m} \right) \\
&= \sum_{i_1, \dots, i_p} \sum_{j_1, \dots, j_q} \sum_{\alpha_1, \dots, \alpha_\ell} \sum_{\beta_1, \dots, \beta_m} \sum_k \left\{ \sum_{\ell'=1}^{\ell} (-1)^{\ell'+1} A_{i_1, \dots, i_p, j_q}^k dz^{i_1} \wedge \right. \\
&\quad \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} \wedge B_{\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m} dz^{\alpha_1} \wedge \\
&\quad \left. \dots \wedge dz^{\alpha_{\ell'-1}} \wedge \left\langle \frac{\partial}{\partial z^k} dz^{\alpha_{\ell'}} \right\rangle \wedge \dots \wedge dz^{\alpha_\ell} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_m} \right\}.
\end{aligned}$$

Then (3.2) and (3.3) are rewritten as follows;

$$(3.17) \quad \bar{\partial}\psi = 0,$$

and

$$(3.18) \quad \psi \frown \Phi - \bar{\partial}\omega = 0.$$

Next we examine conditions for $\bar{\partial}_b$ -boundary in the complex (3.13). For this aim let

$$\psi = \sum_{j=1}^n \psi^j Z_j + \eta \frac{\partial}{\partial \theta}$$

be an element of $\Gamma_T(T')$ such that $\varphi = \bar{\partial}_b^{(0)}\psi$. Then we obtain

$$(3.19) \quad \begin{cases} \varphi_j^k = \frac{\partial \psi^k}{\partial \bar{z}^j} & (1 \leq j, k \leq n), \\ \varphi_j = \sum_{k=1}^n \psi^k \Phi_{kj} + \frac{\partial \eta}{\partial \bar{z}^j}, & (j = 1, \dots, n). \end{cases}$$

Set here

$$\xi = \sum_{k=1}^n \psi^k \frac{\partial}{\partial z^k}.$$

Then (3.19) means that

$$(3.20) \quad \psi = \bar{\partial}\xi \quad \text{and} \quad \omega = \xi \frown \Phi + \bar{\partial}\eta.$$

Therefore in terms of (3.17), (3.18) and (3.20) the complex (3.13) reduces to the following one;

$$\begin{aligned}
0 \longrightarrow \Gamma(T(T)) \oplus C^\infty(T) &\xrightarrow{\bar{\partial}_\Phi^{(0)}} (\wedge^{(0,1)}(T) \otimes \Gamma(T(T))) \oplus \wedge^{(0,1)}(T) \\
&\xrightarrow{\bar{\partial}_\Phi^{(1)}} (\wedge^{(0,2)}(T) \otimes \Gamma(T(T))) \oplus \wedge^{(0,2)}(T),
\end{aligned}$$

where

$$\bar{\partial}_\Phi^{(0)}(\xi, \eta) = (\bar{\partial}\xi, \xi \lrcorner \Phi + \bar{\partial}\eta)$$

and

$$\bar{\partial}_\Phi^{(1)}(\psi', \omega') = (\bar{\partial}\psi', \psi' \lrcorner \Phi - \bar{\partial}\omega'),$$

for $(\xi, \eta) \in \Gamma(T(T)) \oplus C^\infty(T)$ and $(\psi', \omega') \in (\wedge^{(0,1)}(T) \otimes T(T)) \oplus \wedge^{(0,1)}(T)$.

Now let $(\psi', \omega') \in \text{Ker } \bar{\partial}_\Phi^{(1)}$. As $\bar{\partial}\psi' = 0$, it follows that there exist constants $\{C_j^k\}$ $1 \leq j, k \leq n$ and $\xi \in \Gamma(T(T))$ such that

$$\psi' = \sum_{j,k=1}^n C_j^k d\bar{z}^j \otimes \frac{\bar{\partial}}{\partial z^k} + \bar{\partial}\xi.$$

Moreover we have

$$(3.21) \quad \psi' \lrcorner \Phi - \bar{\partial}\omega' = \sum_{i,j,k} C_j^k \Phi_{ki} d\bar{z}^j \wedge d\bar{z}^i + \bar{\partial}(\xi \lrcorner \Phi - \omega') = 0$$

Therefore the $\bar{\partial}$ -cohomology class of $\sum_{i,j,k} C_j^k \Phi_{ki} d\bar{z}^j \wedge d\bar{z}^i$ is zero. However since C_j^k and Φ_{ki} are constants, we get

$$(3.22) \quad \sum_{i,j,k} C_j^k \Phi_{ki} d\bar{z}^j \wedge d\bar{z}^i = 0$$

This fact shows (3.15). Moreover it follows from (3.21) and (3.22) that $\bar{\partial}(\xi \lrcorner \Phi - \omega') = 0$, so that we are able to write

$$(3.23) \quad \xi \lrcorner \Phi - \omega' = \sum_{j=1}^n a_j d\bar{z}^j + \bar{\partial}\eta_0,$$

where the a_j are constants and η_0 is a smooth function on T . Clearly

$$\left(\sum_{j,k} C_j^k \frac{\bar{\partial}}{\partial z^k} \otimes d\bar{z}^j, \sum_j a_j d\bar{z}^j \right)$$

belongs to $\text{Ker } \bar{\partial}_\Phi^{(1)}$. It follows that (ψ', ω') and

$$\left(\sum_{j,k} C_j^k \frac{\bar{\partial}}{\partial z^k} \otimes d\bar{z}^j, \sum_j a_j d\bar{z}^j \right)$$

are $\bar{\partial}_\Phi^{(0)}$ -cohomologous. Indeed we have

$$\begin{aligned} (\psi', \omega') - \left(\sum_{j,k} C_j^k \frac{\bar{\partial}}{\partial z^k} \otimes d\bar{z}^j, \sum_j a_j d\bar{z}^j \right) \\ = (\bar{\partial}\xi, \xi \lrcorner \Phi - \bar{\partial}\eta_0) = \bar{\partial}_\Phi^{(0)}(\xi, -\eta_0). \end{aligned}$$

Thus our theorem is completely proved.

Q.E.D.

§4. Integrable CR-structures on $\mathbf{C}^n \times S^1$.

Let all notations be as before. Recall that ${}^\circ T''$ is the strongly pseudoconvex CR-structure on $\mathbf{C}^n \times S^1$ induced by $\psi: \mathbf{C}^n \times S^1 \rightarrow \mathbf{C}^n \times \mathbf{C}$ (see the bottom of Proposition 2.1). Let now $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ be at finite distance from ${}^\circ T''$. Then ${}^\circ T'' = \{X - \varphi(X); X \in {}^\circ T''\}$ becomes an almost CR-structure on $\mathbf{C}^n \times S^1$. It follows from Proposition 1.4 that φ is integrable if and only if $P(\varphi) = 0$, where the linear map $P: \Gamma(({}^\circ T'')^* \otimes T') \rightarrow \Gamma(\wedge^2 ({}^\circ T'')^* \otimes T')$ is defined by (1.6).

PROPOSITION 4.1. *Let $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ be as in Theorem 3.8. Then φ is integrable.*

Proof. This is trivial from the definition of P and (3.14). Q.E.D.

For convenience sake we write \mathcal{H}^1 the set of all $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ satisfying (3.14) and (3.15) in Theorem 3.8, so that \mathcal{H}^1 is isomorphic to $H_{(C)}^1({}^\circ T'')$. Let φ be any element of \mathcal{H}^1 with

$$(4.1) \quad \varphi(Z_j) = \sum_{k=1}^n C_j^k Z_k - \frac{\sqrt{-1}}{2} a_j \frac{\partial}{\partial \theta}, \quad (j = 1, \dots, n),$$

where the norm $|\varphi|$ of φ is sufficiently small. Then φ is at finite distance to ${}^\circ T''$ and we get

$$(4.2) \quad \begin{aligned} Z_j - \varphi(Z_j) &= \left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^n C_j^k \frac{\partial}{\partial z^k} \right) \\ &\quad - \frac{\sqrt{-1}}{2} \left(\frac{\partial \log h}{\partial \bar{z}^j} + \sum_{k=1}^n C_j^k \frac{\partial \log h}{\partial z^k} - a_j \right) \frac{\partial}{\partial \theta}, \\ &\quad j = 1, \dots, n. \end{aligned}$$

And the CR-structure $\varphi T''$ is generated by $\{Z_j - \varphi(Z_j)\} \ j = 1, \dots, n$.

We shall next determine complex structures on $\mathbf{C}^n \times \mathbf{C}$ which induce CR-structures $\varphi \in \mathcal{H}^1$ on $\mathbf{C}^n \times S^1$ by the map $\psi: \mathbf{C}^n \times S^1 \rightarrow \mathbf{C}^n \times \mathbf{C}$. Remember that ψ is defined by

$$\psi(z, \theta) = \left(z, \frac{e^{\sqrt{-1}\theta}}{\sqrt{h(z)}} \right) \quad \text{for } (z, \theta) \in \mathbf{C}^n \times S^1.$$

PROPOSITION 4.2. *Let $\varphi \in \mathcal{H}^1$ with (4.1). Let z^1, \dots, z^n and ζ be the canonical coordinates of $\mathbf{C}^n \times \mathbf{C}$, and put*

$$(4.3) \quad \begin{cases} \check{Z}_j = \left(\frac{\partial}{\partial z^j} - \sum_{k=1}^n C_j^k \frac{\partial}{\partial z^k} \right) + \left(\sum_{k=1}^n C_j^k \frac{\partial \log h}{\partial z^k} - a_j \right) \zeta \frac{\partial}{\partial \zeta}, \\ \frac{\partial}{\partial \bar{\zeta}} \end{cases} \quad (j = 1, \dots, n).$$

Then the CR-structure ${}^{\circ}T''$ on $C^n \times S^1$ is the one which is induced from the complex structure $\{\{\check{Z}_1^{\circ}, \dots, \check{Z}_n^{\circ}, \partial/\partial \bar{\zeta}\}\}$ on $C^n \times C$ by ψ , that is,

$$(4.4) \quad \psi_*(\psi T'') = \psi_*(CT(C^n \times S^1)) \cap \left\{ \left\{ \check{Z}_1^{\circ}, \dots, \check{Z}_n^{\circ}, \frac{\partial}{\partial \bar{\zeta}} \right\} \right\}.$$

Proof. First of all we see the system (4.3) is integrable. In fact it follows that

$$\begin{aligned} [\check{Z}_i^{\circ}, \check{Z}_j^{\circ}] &= - \sum_{k=1}^n \left(C_i^k \frac{\partial^2 \log h}{\partial \bar{z}^j \partial z^k} - C_j^k \frac{\partial^2 \log h}{\partial \bar{z}^i \partial z^k} \right) \zeta \frac{\partial}{\partial \zeta} \\ &= -\sqrt{-1} \sum_{k=1}^n (C_i^k Q_{kj} - C_j^k Q_{ki}) \zeta \frac{\partial}{\partial \zeta} = 0, \end{aligned}$$

and

$$\left[\check{Z}_j^{\circ}, \frac{\partial}{\partial \bar{\zeta}} \right] = 0, \quad (1 \geq i, j \leq n).$$

Thus (4.3) is integrable. We next show the relation (4.4). For this aim it is enough to prove that

$$\psi_*({}^{\circ}T'') \subset \left\{ \left\{ \check{Z}_1^{\circ}, \dots, \check{Z}_n^{\circ}, \frac{\partial}{\partial \bar{\zeta}} \right\} \right\}.$$

However it follows that for all $(z, \theta) \in C^n \times S^1$,

$$\begin{aligned} \psi_{*(z, \theta)}(Z_j - \varphi(Z_j)) &= \left(\frac{\partial}{\partial \bar{z}^j} - \sum_k C_j^k \frac{\partial}{\partial z^k} \right)_{\psi(z, \theta)} + \left\{ \left(\sum_k C_j^k \frac{\partial \log h}{\partial z^k} - a_j \right) \zeta \frac{\partial}{\partial \zeta} \right\}_{\psi(z, \theta)} \\ &\quad - \left\{ \left(\frac{1}{2} \frac{\partial \log h}{\partial z^j} - \sum_k C_j^k \frac{\partial \log h}{\partial z^k} - a_j \right) \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right\}_{\psi(z, \theta)}, \\ &\quad (j = 1, \dots, n). \quad \text{Q.E.D.} \end{aligned}$$

Now for any $\varphi \in \mathcal{H}^1$ with (4.1), we denote by $T''_{\varphi}(C^n \times C)$ the subbundle of $CT(C^n \times C)$ generated by the system (4.3) in Proposition 4.2, or the complex structure on $C^n \times C$ defined by (4.3). Moreover let T^{φ} be the diffeomorphism of $C^n \times C$ onto $C^n \times C$ defined by

$$(4.5) \quad \begin{cases} z^j \circ T^\varphi(w, t) = w^j - \sum_{k=1}^n C_k^j \bar{w}^k, & (j = 1, \dots, n), \\ \zeta \circ T^\varphi(w, t) = t \exp \left\{ 2\pi\sqrt{-1} \sum_{i,j,k,\ell=1}^n Q_{kj} C_i^k C_i^\ell w^\ell \bar{w}^j \right. \\ \quad \left. - \sqrt{-1}\pi \sum_{k,i,j=1}^n Q_{kj} C_i^k \bar{w}^i \bar{w}^j + \sum_{j=1}^n a_j \bar{z}^j \right\} \end{cases}$$

where (w, t) and (z, ζ) are usual coordinates on $C^n \times C$.

Then we have the following

PROPOSITION 4.3. *Let φ be an element of \mathcal{H}^1 with (4.1). Then the complex structure $T''_\varphi(C^n \times C)$ is induced from the standard complex structure on $C^n \times C$ by the diffeomorphism $T^\varphi: C^n \times C \rightarrow C^n \times C$ defined by (4.5), that is,*

$$T''_\varphi(C^n \times C) = \left\{ \left\{ T_*^\varphi \left(\frac{\partial}{\partial \bar{w}^1} \right), \dots, T_*^\varphi \left(\frac{\partial}{\partial \bar{w}^n} \right), T_*^\varphi \left(\frac{\partial}{\partial \bar{t}} \right) \right\} \right\}.$$

Proof. By direct calculations we have

$$\begin{aligned} T_*^\varphi \left(\frac{\partial}{\partial \bar{w}^j} \right) &= \left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^n C_j^k \frac{\partial}{\partial z^k} \right) \\ &\quad - \left\{ -2\pi\sqrt{-1} \sum_{i,k=1}^n Q_{kj} C_i^k (\bar{z}^i T\varphi) - a_j \right\} \zeta \circ T^\varphi \frac{\partial}{\partial \zeta} \\ &\quad + \frac{\partial \zeta \circ T^\varphi}{\partial \bar{w}^j} \frac{\partial}{\partial \bar{\zeta}}, \quad (j = 1, \dots, n). \end{aligned}$$

However from $h(z) = \exp(2\pi\sqrt{-1} \sum_{j=1}^n Q_{ij} z^i \bar{z}^j)$, it follows that

$$\sum_{k=1}^n C_j^k \frac{\partial \log h}{\partial z^k} = -2\pi\sqrt{-1} \sum_{i,k=1}^n Q_{kj} C_i^k \bar{z}^i,$$

so that we find, for every $(w, t) \in C \times C$,

$$\begin{aligned} T_{*(u,t)}^\varphi \left(\frac{\partial}{\partial \bar{w}^j} \right) &= \left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^n C_j^k \frac{\partial}{\partial z^k} \right)_{T^\varphi(w,t)} \\ &\quad + \left\{ \left(\sum_{k=1}^n C_j^k \frac{\partial \log h}{\partial z^k} + a_j \right) \zeta \frac{\partial}{\partial \zeta} \right\}_{T^\varphi(w,t)} + \left(\frac{\partial \bar{\zeta}}{\partial \bar{w}^j} \frac{\partial}{\partial \bar{\zeta}} \right)_{T^\varphi(w,t)}, \\ &\quad (j = 1, \dots, n) \end{aligned}$$

On the other hand it is clear that

$$T_*^\varphi \left(\frac{\partial}{\partial \bar{t}} \right) = \frac{\partial \bar{\zeta} \circ T^\varphi}{\partial \bar{t}} \frac{\partial}{\partial \bar{\zeta}}.$$

Then the above equations show

$$T''_\varphi(C^n \times C) = \left\{ \left\{ T_*^\varphi \left(\frac{\partial}{\partial \bar{w}^1} \right), \dots, T_*^\varphi \left(\frac{\partial}{\partial \bar{w}^n} \right), T_*^\varphi \left(\frac{\partial}{\partial \bar{t}} \right) \right\} \right\}. \quad \text{Q.E.D.}$$

§5. The main theorem

We shall complete in this section the proof of the following.

THEOREM 5.1. *Let T be an abelian variety of complex dimension n ($n \geq 2$), and let B be a negative line bundle over T . We denote by (B, T) the isolated singularity defined from B and T , as stated at 2.2 in §2. Then any local deformation of (B, T) is also (B', T') , where T' is an abelian variety of $\dim_{\mathbb{C}} T' = n$, and B' represents a negative line bundle over T' .*

First of all recall that the infinitesimal deformation $H^1({}^\circ T''(B))$ of (B, T) is isomorphic to \mathcal{H}^1 as in the previous section, and that any element of \mathcal{H}^1 is integrable (c.f. Proposition 4.1.). Therefore we shall at first prove that small CR -structures in \mathcal{H}^1 are induced ones from negative line bundles over abelian varieties.

Now let us take any element φ of \mathcal{H}^1 with $\varphi(Z_j) = \sum_{k=1}^n C_j^k Z_k - \sqrt{-1}a_j(\partial/\partial\theta)$, ($j = 1, \dots, n$), whose norm $|\varphi|$ is sufficiently small, and fix φ . Then the CR -structure ${}^p T''(C^n \times S^1)$ on $C^n \times S^1$ defined by (4.2) is induced from the complex structure $T''_\varphi(C^n \times C)$ on $C^n \times C$ determined in terms of (4.3) in Proposition 4.2. Moreover from Proposition 4.3, $T''_\varphi(C^n \times C)$ arises out of the standard complex structure on $C^n \times C$ by the map $T^\varphi: C^n \times C \rightarrow C^n \times C$ which is defined by (4.5). We regard this map T^φ as the bundle map between two trivial bundles $C^n \times C$ over C^n . Thus in order to prove the above statement it is enough to show that there exist an abelian variety T' and a negative line bundle B' over T' such that the bundle map $T^\varphi: C^n \times C \rightarrow C^n \times C$ induces canonically a bundle map $\tilde{T}^\varphi: B \times B$.

From now on we shall show the above statements. For this aim let us put

$$C = \begin{pmatrix} C_1^1, & \dots, & C_n^1 \\ & \dots & \\ C_1^n, & \dots, & C_n^n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

and

$$Q = \begin{pmatrix} Q_{1\bar{1}}, & \dots, & Q_{1\bar{n}} \\ & \dots & \\ Q_{n\bar{1}}, & \dots, & Q_{n\bar{n}} \end{pmatrix}$$

Then the condition (3.15) becomes

$$(3.15)' \quad {}^t C Q = {}^t Q C$$

where t denotes the transpose of matrices. Furthermore the map T^φ is also represented as follows;

$$(4.5)' \quad T^\varphi(w, t) = (w - C - \bar{w}, t \cdot \exp \{2\pi\sqrt{-1} {}^t \bar{w} {}^t Q C \bar{C} w - \sqrt{-1} \pi {}^t \bar{w} {}^t Q C \bar{w} + {}^t a \bar{w}\}) ,$$

where

$$w = \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix} \in C^n .$$

Here we set

$$(5.1) \quad g(w) = \exp \{2\pi\sqrt{-1} {}^t \bar{w} {}^t Q C \bar{C} w - \sqrt{-1} \pi {}^t \bar{w} {}^t Q C \bar{w} + {}^t a \bar{w}\} .$$

so that it follows that

$$(5.2) \quad T^\varphi(w, t) = (w - C \bar{w}, g(w)t) .$$

Now we write ω' and $f'(w, d)$ a periodic matrix and an automorphic factor respectively, corresponding to an abelian variety T' and a negative line bundle B' over T' to be required for the given CR -structure $\varphi \in \mathcal{H}^1$. Then since T^φ in (5.2) induces a bundle map of B' onto B , ω' and f' must satisfy the following conditions;

$$(5.3) \quad \omega' - c\bar{\omega}' = \omega$$

and

$$(5.4) \quad g(w + \omega' d) f'(w, d) = g(w) f(w - C \bar{w}, d) ,$$

for all $(w, d) \in C^n \times Z^{2n}$,

where ω is the periodic matrix of T and f denotes the automorphic factor of B in § 2. Since $|\varphi|$ is sufficiently small, there exists a unique ω' satisfying (5.3), and we fix ω' . Furthermore we obtain the following.

PROPOSITION 5.2. *Let f' be the map of $C^n \times Z^{2n}$ into C defined by (5.4). Then f' becomes an automorphic factor for the periodic matrix, ω' , that is,*

$$(5.5) \quad f' \text{ is the holomorphic map of } C^n \times Z^{2n} \text{ into } C - \{0\} ,$$

and

$$(5.6) \quad \begin{aligned} f'(w, d_1 + d_2) &= f'(w + \omega' d_1, d_2) f'(w, d_1), \\ &\text{for } w \in C^n \text{ and } d_i \in Z^{2n} \ (i = 1, 2). \end{aligned}$$

The proof of this proposition is due to the following two lemmas.

LEMMA 5.3. *For an arbitrary $(w, d) \in C^n \times Z^{2n}$, we have*

$$(5.7) \quad f(w, d) = f(w, d) g(\omega' d)^{-1} \exp \{-2\pi\sqrt{-1} {}^t(\bar{\omega}' d)' Q C \bar{C} w\}.$$

Proof. At the beginning we get, using (5.1), (5.3) and ${}^t Q C = {}^t C Q$,

$$(5.8) \quad \begin{aligned} g(w + \omega' d) &= g(w) g(\omega' d) \exp \{2\pi\sqrt{-1} {}^t \bar{w}' Q C (\bar{C} \omega' - \bar{\omega}') d \\ &\quad + 2\pi\sqrt{-1} {}^t(\bar{\omega}' d)' Q C \bar{C} w\}. \end{aligned}$$

Therefore it follows from (5.4) that

$$\begin{aligned} f'(w, d) &= f(w - C \bar{w}, d) g(\omega' d)^{-1} \exp \{-2\pi\sqrt{-1} {}^t \bar{w}' Q C (\bar{C} \omega' - \bar{\omega}') d \\ &\quad - 2\pi\sqrt{-1} {}^t(\bar{\omega}' d)' Q C \bar{C} s\}. \end{aligned}$$

On the other hand we find from (2.1)

$$f(w - C \bar{w}, d) = \exp \{2\pi\sqrt{-1} {}^t(w - C \bar{w}) Q \bar{\omega} d + A(d)\},$$

where we put $A(d) = \frac{1}{2} {}^t d A^\circ d + {}^t b d$, so that we obtain

$$\begin{aligned} f'(w, d) &= g(\omega' d)^{-1} \exp \{-2\pi\sqrt{-1} {}^t(\bar{\omega}' d)' Q C \bar{C} w\} \\ &\quad \times \exp \{-2\pi\sqrt{-1} {}^t \bar{w}' Q C (\bar{C} \omega' - \bar{\omega}') d\} \\ &\quad \times \exp \{2\pi\sqrt{-1} {}^t(w - C \bar{w}) Q (\bar{\omega}' - \bar{C} \omega') d + A(d)\} \\ &= g(\omega' d) \exp \{-2\pi\sqrt{-1} {}^t(\bar{\omega}' d)' Q C \bar{C} w\} f(w, d). \end{aligned} \quad \text{Q.E.D.}$$

Moreover we have

LEMMA 5.4. *f' satisfies (5.6).*

Proof. From (5.7) it is clear that

$$\begin{aligned} f'(w, d_1 + d_2) &= f(w, d_1 + d_2) g(\omega' (d_1 + d_2))^{-1} \\ &\quad \times \exp \{-2\pi\sqrt{-1} {}^t(\bar{\omega}' (d_1 + d_2))' Q C \bar{C} w\}. \end{aligned}$$

Using the fact that f is the automorphic factor, we get from (5.8)

$$\begin{aligned} g(\omega' d_1 + \omega' d_2) &= g(\omega' d_1) g(\omega' d_2) \exp \{2\pi\sqrt{-1} {}^t(\bar{\omega}' d_1)' Q C (\bar{C} \omega' - \bar{\omega}') d_2\} \\ &\quad + 2\pi\sqrt{-1} {}^t(\bar{\omega}' d_2)' Q C \bar{C} \omega' d_1\}. \end{aligned}$$

Hence it follows that

$$\begin{aligned}
f'(w, d_1 + d_2) &= f(w + \omega d_1, d_2) f(w, d_1) g(\omega' d_1)^{-1} g(\omega' d_2)^{-1} \\
(5.9) \quad &\times \exp(-2\pi\sqrt{-1}'(\bar{\omega}' d_1)' Q C(\bar{C}\omega' - \bar{\omega}') d_2 - 2\pi\sqrt{-1}'(\bar{\omega} d_1)' Q C \bar{C} \omega) \\
&\times \exp\{-2\pi\sqrt{-1}'(\bar{\omega}' d_2)' Q C \bar{C}(w + \omega' d_1)\}.
\end{aligned}$$

But we get

$$f(w + \omega' d_1, d_2) = f(w + \omega' d_1, d_2) \exp\{2\pi\sqrt{-1}'(-C\bar{\omega}' d_1) Q(\bar{\omega}' - \bar{C}\omega') d_2\}.$$

Finally noting $'(C\bar{\omega}' d_1) Q = '(\bar{\omega}' d_1)' Q C$, we find in terms of (5.9) and the above relation,

$$\begin{aligned}
f'(w, d_1 + d_2) &= f(w + \omega' d_1, d_2) g(\omega' d_2)^{-1} \\
&\times \exp\{-2\pi\sqrt{-1}'(\bar{\omega}' d_2)' Q C \bar{C}(w + \omega' d_1)\} \\
&\times f(w, d_1) g(\omega' d_1)^{-1} \exp\{-2\pi\sqrt{-1}'(\bar{\omega}' d_1)' Q C \bar{C} \omega\}.
\end{aligned}$$

This equation shows (5.6).

Q.E.D.

From the above lemmas, Proposition 5.2 is proved. Further let T' and B' the abelian variety and the line bundle over T' , respectively, defined from ω' and f' in Proposition 5.2. Then we have that following.

PROPOSITION 5.5. *Let all notations be as above. If C is sufficiently small, then B' is negative. Moreover the map $T^\circ: C^n \times C \rightarrow C^n \times C$ defined by (4.5) (or (4.5)') induces canonically the bundle map $\tilde{T}^\circ: B' \rightarrow B$.*

Proof. This is trivial from constructions of T° and B' . Q.E.D.

Thus all small CR -structures in \mathcal{H}^1 are induced ones from negative line bundles over abelian varieties.

Now from the universality theorem in § 9 [1], we know the following facts; Let $\iota_{B_1}: B_1 \rightarrow B$ be the inclusion map as before and let N be any neighborhood of B_1 in B . When N_ω is a deformation of the complex structure on N which is the open submanifold of B , where $\omega \in \Gamma(N, T''(N)^* \otimes T'(N))$ and an embedding $i: B_1 \rightarrow N$ is given, we denote by $\omega \circ i$ the induced CR -structure on V by $i: B_1 \rightarrow N$. Finally let \mathcal{H}_K^1 be the harmonic space in $\Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ with respect to the operator $\bar{\partial}_b^{(2)*} \bar{\partial}_b^{(1)} + \bar{\partial}_b^{(0)} \bar{\partial}_b^{(1)*}$ ($\dim_{\mathbb{R}} \mathcal{H}_K^1 < \infty$). Then there exists a differential map

$\psi_K: \mathcal{H}_K^1 \rightarrow \Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ satisfying the following conditions;

(α) $\psi'_K(0)t = t$ for any $t \in \mathcal{H}_K^1$.

(β) If N_ω is a deformation of N such that n_1 -Sobolev-norm $\|\omega\|_{n_1}$ of ω is sufficiently small (n_1 is a sufficiently large integer), then there are a point $t_\omega \in \mathcal{H}_K^1$ and an embedding $i_\omega: B_1 \rightarrow N_\omega$ such that

$$\omega \circ i_\omega = \psi_K(i_\omega) .$$

Moreover i_ω and t_ω are infinitely differentiable on ω , and when ω is zero, it follows that $t_\omega = 0$ and $\omega \circ t_\omega = \iota_V$.

Here we can prove the following

PROPOSITION 5.6. *For any sufficiently small point $t \in \mathcal{H}_K^1$, $\psi_K(t)$ is an induced CR-structure on V form a negative line bundle over an abelian variety.*

Proof. At first, for any small $\varphi \in \mathcal{H}^1$ with the expression (4.1) we have shown that the complex structure T''_φ on B defined by (4.3) becomes one of a negative line bundle B' over an abelian variety T' (c.f. Proposition 5.5) and that the CR-structure is induced from B' . Here let \hat{T} be the smooth map of \mathcal{H}^1 into $\Gamma(T''(B)^* \otimes T'(B))$ defined by

$$\begin{aligned} \hat{T}(\varphi) \left(\frac{\partial}{\partial \bar{z}^j} \right) &= \sum_{k=1}^n C_j^k \frac{\partial}{\partial z^k} + \left(\sum_{k=1}^n \frac{\partial \log h}{\partial z^k} - a_j \right) \zeta \frac{\zeta}{\partial \bar{\zeta}} , \quad (j = 1, \dots, n) \\ \hat{T}(\varphi) \left(\frac{\partial}{\partial \bar{\zeta}} \right) &= 0 \end{aligned}$$

for any $\varphi \in \mathcal{H}_*^1$ with (4.1), (see (4.3)). Moreover let \mathcal{H}^1 be a sufficiently small neighborhood of 0 in \mathcal{H}^1 such that Proposition 5.5 and the above statement (β) hold good. Then we have from (β) a smooth map

$$\tau: \mathcal{H}_*^1 \longrightarrow \mathcal{H}_K^1 \quad \text{such that } \hat{T}(\varphi) \circ i_{T(\varphi)} = \psi_K(\tau(\varphi)) \text{ and } \tau(0) = 0 .$$

In order to prove this proposition it is enough to show that the derivation $\tau_{(0)}^1$ of τ at 0 is injective, because of $\dim \mathcal{H}^1 = \dim \mathcal{H}_K^1$. Here let s be a small real number. Then it follows from Theorem 7.1 and (5.16) in [1] that

$$(5.10) \quad \lim_{s \rightarrow 0} \frac{\hat{T}(s\varphi) \circ i_{T(s\varphi)} - \hat{T}(s\varphi) \circ \iota_{B_1}}{s} = \bar{\partial}_b^{(0)} \xi$$

where ξ is an element of $\Gamma(T'(B_1))$.

On the other hand we get from the definition of \hat{T}

$$(5.11) \quad \lim_{s \rightarrow 0} \frac{\hat{T}(s\varphi) \circ \iota_{B_1} - \hat{T}(0) \circ \iota_{B_1}}{s} = \varphi .$$

Therefore we have from (5.10) and (5.11)

$$\left. \frac{d\hat{T}(s\varphi) \circ i_{T(s\varphi)}}{ds} \right|_{s=0} = \varphi + \bar{\partial}_b^{(0)} \xi .$$

Furthermore it follows, using (β) , that

$$\left. \frac{d\psi_K(\tau(s\varphi))}{ds} \right|_{s=0} = \tau'(0) \cdot \varphi ,$$

so that

$$(5.12) \quad \tau'(0) \cdot \varphi = \bar{\partial}_b^{(0)} \xi + \varphi .$$

Now suppose that $\tau'(0) \cdot \varphi = 0$. Then it is clear from (5.12) that the $\bar{\partial}_b^{(1)}$ -cohomology class of φ is zero, but as φ belongs to \mathcal{H}^1 we have $\varphi = 0$. Thus $\tau'(0)$ is injective. Q.E.D.

Our main Theorem 5.1 is obtained from the above Proposition 5.6.

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