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PROJECTIVE INVARIANT METRICS FOR EINSTEIN SPACES

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1. Introduction

In my recent paper [1], I associated a projectively invariant pseudo-distance d_M to every affinely connected manifold M and proved the following

THEOREM 1. *Let M be a Riemannian manifold with metric ds_M^2 and Ricci tensor* Ric_M *such that* $\mathrm{Ric}_M \leq -c^2 ds_M^2$. *Let δ_M be the Riemannian distance defined by ds_M^2. Then*

$$d_M(x, y) \geq \frac{2c}{\sqrt{n-1}}\delta_M(x, y) \qquad \text{for } x, y \in M .$$

The purpose of this paper is to show the following

THEOREM 2. *Let M be a complete Einstein manifold with*

$$\mathrm{Ric}_M = -c^2 ds_M^2 .$$

Then

$$d_M(x, y) = \frac{2c}{\sqrt{n-1}}\delta_M(x, y) \qquad \text{for } x, y \in M .$$

The following corollary has been known for some time [3], [4].

COROLLARY. *The projective transformations of a complete Einstein manifold with negative Ricci tensor are all isometries.*

Since Theorem 1 is not stated in [1] in the same manner as above, we shall first indicate how it can be derived from the results proved in [1].

We should remark that d_M vanishes identically if ds_M^2 is complete

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and $\mathrm{Ric}_M \geqq 0$, [2].

2. Proof of Theorem 1

Let ρ be the distance function on the interval $I = \{u; -1 < u < 1\}$ induced by the "real Poincaré metric" $4du^2/(1-u^2)^2$. Our Theorem 1 is an immediate consequence of the following two results proved in [1].

LEMMA 1. *Let M be a manifold with a torsionfree affine connection. Let* $\mathrm{Proj}\,(I, M)$ *denote the family of projective maps $f: I \to M$. Then*

(1) $\rho(a, b) \geqq d_M(f(a), f(b))$ *for* $a, b \in I$, $f \in \mathrm{Proj}\,(I, M)$;

(2) *If Δ_M is any pseudo-distance on M such that*

$$\rho(a, b) \geqq \Delta_M(f(a), f(b)) \qquad \text{for } a, b \in I, f \in \mathrm{Proj}\,(I, M)\ ,$$

then

$$\Delta_M(x, y) \leqq d_M(x, y) \qquad \text{for } x, y \in M\ .$$

This is Proposition 3.5 of [1] and follows directly from the construction of d_M.

LEMMA 2. *Let M be a Riemannian manifold with* $\mathrm{Ric}_M \leqq -c^2 ds_M^2$. *Let δ_M be the Riemannian distance defined by ds_M^2. Then*

$$\rho(a, b) \geqq \frac{2c}{\sqrt{n-1}}\delta_M(f(a), f(b)) \qquad \text{for } a, b \in I, f \in \mathrm{Proj}\,(I, M)\ .$$

This is Corollary 4.14 of [1] and follows from its infinitesimal version, Lemma 4.1 of [1].

3. Proof of Theorem 2

Since Theorem 1 established an inequality in one direction, we have only to prove the opposite inequality. Given two points x, y of M, we take a minimizing geodesic $x(s)$ parametrized by its arc-length s in such a way that

$$x = x(0)\ , \qquad y = x(a)\ ,$$

where a is the Riemannian distance $\delta_M(x, y)$ from x to y.

A projective parameter p for this geodesic is defined as a solution of the differential equation

$$\{p, s\} = \frac{2}{n-1} \sum R_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} ,$$

where $\{p, s\}$ is the Schwarzian derivative, (see [1]). Since $\mathrm{Ric}_M = -c^2 ds_M^2$ by assumption, the differential equation above takes the following simple form:

$$\{p, s\} = -\frac{2c^2}{n-1} .$$

This equation has a general solution of the following form ([1; § 5]):

$$p(s) = (\alpha e^{ks} + \beta e^{-ks})/(\gamma e^{ks} + \delta e^{-ks}) \quad \text{with} \quad \alpha\delta - \beta\gamma \neq 0 ,$$

where $k = c/\sqrt{n-1}$.

We use the following special solution:

$$p(s) = (e^{ks} - e^{-ks})/(e^{ks} + e^{-ks})$$

so that

$$p(-\infty) = -1 , \quad p(0) = 0 , \quad p(\infty) = 1 .$$

We use the obvious projective map $f: I \to M$ given by $p = u$. Since the points x and y correspond to $s = 0$ and $s = a$ respectively, they correspond to $p = 0$ and $p = (e^{ka} - e^{-ka})/(e^{ka} + e^{-ka}) = p(a)$. In other words, $x = f(0)$ and $y = f(p(a))$. From Lemma 1 above, we have

$$\rho(0, p(a)) \geqq d_M(x, y) .$$

Since

$$\rho(u, v) = \left| \log \frac{1+v}{1-v} \cdot \frac{1-u}{1+u} \right| ,$$

we obtain

$$\rho(0, p(a)) = \left| \log \frac{1 + p(a)}{1 - p(a)} \right| = 2ka = 2ca/\sqrt{n-1} .$$

Hence,

$$d_M(x, y) \leqq 2ca/\sqrt{n-1} = \frac{2c}{\sqrt{n-1}} \delta_M(x, y) .$$

This completes the proof of Theorem 2.

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