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ON THE LEAST DEGREE OF POLYNOMIALS BOUNDING ABOVE THE DIFFERENCES BETWEEN LENGTHS AND MULTIPLICITIES OF CERTAIN SYSTEMS OF PARAMETERS IN LOCAL RINGS

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§ 1. Introduction

Let A be a commutative local Noetherian ring with the maximal ideal $\mathfrak{m}$ and M a finitely generated A-module, $d = \dim M$. It is well-known that the difference between the length and the multiplicity of a parameter ideal $\mathfrak{q}$ of M

$$I_M(\mathfrak{q}) = l(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$$

gives a lot of informations on the structure of the module M. For instance, M is a *Cohen-Macaulay* (CM for short) module if and only if $I_M(\mathfrak{q}) = 0$ for some parameter ideal $\mathfrak{q}$ or M is *Buchsbaum* module (see [S-V]) if and only if $I_M(\mathfrak{q})$ is a constant for all parameter ideals $\mathfrak{q}$ of M. In this note we shall investigate this difference, but in a more general situation as follows: Let $x = \{x_1, \cdots, x_d\}$ be a system of parameters (s.o.p. for short) on M and $n = (n_1, \cdots, n_d)$ a d-tuple of positive integers. We consider the difference

$$I_M(n; x) = l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) - n_1 \cdots n_d e(x; M)$$

as a function in n. In general, this function is not a polynomial in n, even in the case $n_1 = n_2 = \cdots = n_d = t$ (see [G-K]). The necessary and sufficient conditions in term of x, for this function to be a polynomial, have been examined in [C_1]. Here we shall show that the least degree of all polynomials in n bounding above $I_M(n; x)$ is independent of the choice of x (Theorem 2.3). This numerical invariant of M will be called the *polynomial type* of M. The aim of this note is to study the polynomial type of a module over a local ring. In Section 2 we define the polynomial

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type of a module and give some properties of this invariant. Using the local cohomology modules of M and the notion of reducing sequence [A-B], we give in Section 3 upper and under bounds for the polynomial type of M. In Section 4 we improve these bounds in the case that A is a factor ring of a CM ring. In particular, if M is equidimensional we can show that the polynomial type of M is just the dimension of the non-Cohen-Macaulay locus of M in Supp M. In the last section we examine the behaviour of the polynomial types by a flat extension.

§ 2. The polynomial type of a module

Throughout this note we denote by A a commutative local Noetherian ring with the maximal ideal $\mathfrak{m}$ and by M a finitely generated A-module with $\dim M = d$.

We begin with the following lemma.

LEMMA 2.1. *Let $x = \{x_1, \cdots, x_d\}$ be an s.o.p. on M and $n = (n_1, \cdots, n_d)$ a d-tuple of positive integers. Then*

$$l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) \leqslant n_1 \cdots n_d \, l(M/(x_1, \cdots, x_d)M) .$$

Proof. The lemma is proved in [G] for the case $d = 1$. Put

$$E = M/(x_1^{n_1})M \quad \text{and} \quad F = M/(x_2, \cdots, x_d)M .$$

Then by induction on d we get

$$\begin{aligned} l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) &= l(E/(x_2^{n_2}, \cdots, x_d^{n_d})E) \\ &\leqslant n_2 \cdots n_d \, l(E/(x_2, \cdots, x_d)E) = n_2 \cdots n_d \, l(F/(x_1^{n_1})F) \\ &\leqslant n_1 \cdots n_d \, l(M/(x_1, \cdots, x_d)M) \quad \text{as required} . \end{aligned}$$

From now on, for an s.o.p. $x = \{x_1, \cdots, x_d\}$ of M we set

$$I_M(n; x) = l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) - n_1 \cdots n_d e(x; M) .$$

In particular, we also set $I_M(x) = I_M(n; x)$ if $n_1 = \cdots = n_d = 1$.

COROLLARY 2.2. $I_M(n; x) \leqslant n_1 \cdots n_d I_M(x)$.

The corollary 2.2 shows that if we consider $I_M(n; x)$ as a function in n then this function is bounded above by the polynomial $n_1 \cdots n_d I_M(x)$. In general we can show the following theorem.

THEOREM 2.3. *Let $x = \{x_1, \cdots, x_d\}$ be an s.o.p. on M. Then the least* *degree of all polynomials in n bounding above $I_M(n; x)$ is independent of the choice of x.*

Proof (see [C_2], Proposition 2.5). Let $\underline{t} = (t, \cdots, t)$ be a d-tuple of the same integers t. Then by [G], Theorem 6, the least degree of all polynomials in t bounding above $I_M(\underline{t}; x)$ is independent of the choice of x. Denote this invariant by $p'(M)$ and by $p(x)$ the least degree of all polynomials in n bounding above $I_M(n; x)$. It is clear that $p'(M) \leqslant p(x)$. On the other hand, we can easily verify that $I_M(\underline{t}; x) \geqslant I_M(n; x)$ if $t \geqslant n_i$, $i = 1, \cdots, d$. It follows that $p'(M) \geqslant p(x)$. Thus $p(x) = p'(M)$ is independent of the choice of x.

DEFINITION 2.4. The numerical invariant of M given in Theorem 2.3 is called the *polynomial type* of M and we denote it by $p(M)$.

Remark 2.5. (i) If we stipulate that the degree of the zero-polynomial is equal -1. Then M is a CM module if and only if $p(M) = -1$.

(ii) If $I_M(n; x)$ is a constant for some s.o.p. x on M and for $n_1, \cdots, n_d$ sufficiently large then M is called a generalized CM module (see [C-S-T]). Therefore M is a generalized CM module if and only if $p(M) \leqslant 0$.

(iii) From the limit formula of Lech

$$\lim_{\min(n_i) \to \infty} (n_1 \cdots n_d)^{-1} l(M/(x_1^{n_1}, \cdots, n_d^{n_d})M) = e(x_1, \cdots, x_d; M)\,,$$

we easily deduce that $p(M) \leqslant \dim M - 1$.

LEMMA 2.6. *Let $\hat{M}$ be the $\mathfrak{m}$-adic completion of M then $p(M) = p(\hat{M})$.*

Proof. If $x = \{x_1, \cdots, x_d\}$ is an s.o.p. on M then x is also an s.o.p. on $\hat{M}$. The lemma follows from the fact that

$$l(M/(x_1, \cdots, x_d)M) = l(\hat{M}/(x_1, \cdots, x_d)\hat{M}) \quad \text{and} \quad e(x; M) = e(x; \hat{M})\,.$$

§ 3. Bounds of $p(M)$

First of all we need some notations as follows.

We denote by $\mathfrak{a}_i$ the annihilator of the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ of M with respect to the maximal ideal $\mathfrak{m}$, and we set

$$\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M)\,.$$

A subset of an s.o.p. $x_1, \cdots, x_j$ of M is called a *reducing sequence* if the following condition holds: $x_i \notin P$ for all $P \in \operatorname{Ass}(M/(x_1, \cdots, x_{i-1})M)$ with $\dim(A/P) \geqslant d - i$, $i = 1, \cdots, j$. Note that if $x = \{x_1, \cdots, x_d\}$ is an s.o.p.

on M and $x_1, \cdots, x_{d-1}$ form a reducing sequence, then x is just a reducing s.o.p. which has been introduced in [A-B]. We set

$$r(M) = \inf\{k/\text{every subset of an s.o.p. of } M \text{ having } (d-k-1)\text{-elements is a reducing sequence of } M\}.$$

Finally, we denote by $NC(M)$ the *non-Cohen-Macaulay locus* of M, i.e. $NC(M) = \{P \in \operatorname{Supp} M/\ M_P \text{ is not a CM module}\}$. The following theorem, which will be often used in this paper, is one of the main results of [C_2].

THEOREM 3.1. *Suppose that A has a dualizing complex. Then*
(i) $p(M) = r(M) = \dim(A/\mathfrak{a}(M))$;
(ii) *if M is equidimensional then* $p(M) = \dim(NC(M))$.

Example 3.2. Ferrand and Raynaud [F-R] have constructed a two-dimensional local integral domain $(R, \mathfrak{m})$ such that the $\mathfrak{m}$-adic completion $\hat{R}$ has a one-dimensional associated prime ideal. Thus R is not a generalized CM module; it follows that $p(R) = 1$. But, it is easy to see that $NC(R) = \{\mathfrak{m}\}$, $r(R) = 0$ and $\dim(R/\mathfrak{a}(R)) = 2$. So we get inequalities

$$\dim(R/\mathfrak{a}(R)) > p(R) > \dim(NC(R)) = r(R).$$

Thus, in the general case, Theorem 3.1 does not hold without the assumption that A has a dualizing complex. However, we have the following theorem.

THEOREM 3.3. *With the previous notations it holds*
(i) $\dim(A/\mathfrak{a}(M)) \geqslant p(M) \geqslant r(M)$;
(ii) *If $NC(M)$ is closed then* $p(M) \geqslant r(M) \geqslant \dim(NC(M))$.

Proof. (i) We denote by $\hat{A}$ and $\hat{M}$ the $\mathfrak{m}$-adic completion of A and M, respectively. Then it is obvious that $\mathfrak{a}(M)\hat{A} \subseteq \mathfrak{a}(\hat{M})$. Therefore by Theorem 3.1 and Lemma 2.6.

$$\dim(A/\mathfrak{a}(M)) = \dim(\hat{A}/\mathfrak{a}(M)\hat{A}) \geqslant \dim(\hat{A}/\mathfrak{a}(\hat{M})) = p(\hat{M}) = p(M).$$

Let $p(M) = k$ and $x = \{x_1, \cdots, x_d\}$ be an arbitrary s.o.p. on M. By [A-B], Corollary 4.3, we have

$$\begin{aligned} I_M(n; x) = {} & l((x_1^{n_1}, \cdots, x_{d-1}^{n_{d-1}})M : x_d^{n_d}/(x_1^{n_1}, \cdots, x_{d-1}^{n_{d-1}})M) \\ & + \sum_{i=1}^{d-1} e(x_{i+1}^{n_{i+1}}, \cdots, x_d^{n_d}; (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i}/(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M). \end{aligned}$$

Since $I_M(n; x)$ is bounded above by a polynomial in n of degree k, therefore

$$e(x_{i+1}^{n_{i+1}}, \cdots, x_d^{n_d}; (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i}/(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M) = 0$$

for $i = 1, \cdots, d-k-1$. Thus by virtue of [A-B], Proposition 4.7 $x_1, \cdots, x_{d-k-1}$ is a reducing sequence; so $p(M) \geqslant r(M)$. For the proof of (ii) we need two auxiliary lemmas as follows. Note that these lemmas have been proved in [C_2] with the assumption that A has a dualizing complex.

LEMMA 3.4. *Let $x_1, \cdots, x_j$ ($j \geqslant 1$) be an unconditioned reducing sequence of M (i.e. from any permutation of the sequence $x_1, \cdots, x_j$ we still get a reducing sequence). Then for every prime ideal P contained in* $\mathrm{Ass}(M/(x_1, \cdots, x_j)M)$ *with* $\dim(A/P) = d - j$ *we have*

$$\dim(M_P) + \dim(A/P) = d .$$

Proof. Suppose that $\dim(M_P) = k < j$. We can choose the order of the sequence $x_1, \cdots, x_j$ such that $x_1, \cdots, x_k$ form an s.o.p. on M_P. Since $(x_1, \cdots, x_k)M_P \subseteq (x_1, \cdots, x_{j-1})M_P \subseteq PM_P$, it follows that $(x_1, \cdots, x_{j-1})M_P$ is PA_P-primary. Hence by [M], 7.C, $P \in \mathrm{Ass}(M/(x_1, \cdots, x_{j-1})M)$. This contradicts the assumption that $x_1, \cdots, x_j$ is a reducing sequence of M. Thus $\dim(M_P) = j$ as required.

LEMMA 3.5. *Let $P \in \mathrm{Supp}\, M$ with* $\dim(A/P) > r(M)$. *Then M_P is a CM module and* $\dim(M_P) + \dim(A/P) = d$.

Proof. Let $P \in \mathrm{Supp}\, M$ with $\dim(A/P) > r(M) = k$. Choose a subset of an s.o.p. $x_1, \cdots, x_j$ of M contained in P so that j is maximal. It is easy to verify that $\dim(A/P) = d - j$. Therefore $d - k - 1 \geqslant j$; so $x_1, \cdots, x_j$ is an unconditional reducing sequence of M. Hence $\dim(M_P) + \dim(A/P) = d$ by Lemma 3.4. On the other hand, $x_1, \cdots, x_j$ is also a reducing sequence on M_P. Thus by [A-B], Corollary 4.8,

$$\begin{aligned} &l(M_P/(x_1, \cdots, x_j)M_P) - e((x_1, \cdots, x_j)A_P; M_P) \\ &\quad = l((x_1, \cdots, x_{j-1})M_P : x_j/(x_1, \cdots, x_{j-1})M_P) . \end{aligned}$$

Since $x_1, \cdots, x_j$ is a reducing sequence of M we get $PA_P \notin \mathrm{Ass}(M_P/(x_1, \cdots, x_{j-1})M_P)$. Therefore $(x_1, \cdots, x_{j-1})M_P : x_j = (x_1, \cdots, x_{j-1})M_P$. It follows that $l(M_P/(x_1, \cdots, x_j)M_P) = e((x_1, \cdots, x_j)A_P; M_P)$, in other words, M_P is a CM module. The lemma is proved.

Proof of (ii). By part (i) of Theorem we only need to show that

$r(M) \geqslant \dim(NC(M))$. This inequality immediately follows from Lemma 3.5. The proof of Theorem 3.3 is now complete.

COROLLARY 3.6. *Let* $P \in \operatorname{Supp} M$ *with* $\dim(A/P) > p(M)$. *Then* M_P *is a CM module and* $\dim(M_P) + \dim(A/P) = d$.

§ 4. The case A is the homomorphic image of a CM ring

In this section we shall improve our previous inequalities in the case that A is the homomorphic image of a CM ring. We begin with the following theorem.

THEOREM 4.1. *Let* A *be the homomorphic image of a CM ring and* k *a positive integer. Then the following conditions are equivalent*:

(i) $p(M) \leqslant k$.

(ii) *Any subset of an s.o.p. of* M *having* $(d - k - 1)$*-elements is a reducing sequence.*

(iii) *For any* $P \in \operatorname{Supp} M$ *with* $\dim(A/P) > k$, M_P *is a CM modnle and* $\dim(M_P) + \dim(A/P) = d$.

(iv) *For any* $P \in \operatorname{Supp} M$ *with* $\dim(A/P) = k + 1$, M_P *is a CM module and* $\dim(M_P) + \dim(A/P) = d$.

Proof. By the proof of Theorem 3.3 we already get the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Since the CM property is stable under generalization, (iii) is equivalent to (iv). Thus we have only to show that (iii) $\Rightarrow$ (i). As above, we denote by $\hat{A}$ and $\hat{M}$ the $\mathfrak{m}$-adic completion of A and M, respectively. Put $\dim(\hat{A}/\mathfrak{a}\hat{M})) = k'$ and suppose that $k' > k$. Let $P \in \operatorname{Ass}(\hat{A}/\mathfrak{a}(\hat{M}))$ so that $\dim(\hat{A}/P\hat{A}) = k'$ and $P \cap A = \mathfrak{p}$. Then there exists a $Q \in \operatorname{Ass}(\hat{A}/\mathfrak{p}\hat{A})$ such that $Q \subseteq P$. Therefore $Q \cap A = \mathfrak{p}$. Since $\hat{A}/\mathfrak{p}\hat{A}$ is unmixed by [N], 34.9, we get

$$\dim(A/\mathfrak{p}) = \dim(\hat{A}/\mathfrak{p}\hat{A}) = \dim(\hat{A}/Q) \leqslant \dim(\hat{A}/P\hat{A}) = k' .$$

Note that $\hat{A}$ is catenery, so we deduce from 3.5 and the assumption that

$$\begin{aligned} d &= \dim(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(M_\mathfrak{p}) + \dim(\hat{A}/Q) \\ &= \dim(M_\mathfrak{p}) + \dim(\hat{A}/P) + \dim(\hat{A}_P/Q\hat{A}_P) \\ &= \dim(M_\mathfrak{p}) + \dim(\hat{A}/P) + \dim(\hat{A}_P/\mathfrak{p}\hat{A}_P) \\ &= \dim(\hat{M}_P) + \dim(\hat{A}/P) . \end{aligned}$$

On the other hand, since $\dim(\hat{A}/\mathfrak{p}) \geqslant k' > k$, $M_\mathfrak{p}$ is a CM module. So $\hat{M}_P$ is CM because any fiber of the canonical homomorphism $A \to \hat{A}$ is CM.

Therefore

$$d = \dim(\hat{M}_P) + \dim(\hat{A}/P) = \operatorname{depth}(\hat{M}_P) + \dim(\hat{A}/P)\,.$$

Applying [S], Satz 2.4.6 it follows that $\mathfrak{a}(\hat{M}) \not\subseteq P$ which is a contradiction. Thus $k \geqslant k'$. By virtue of Lemma 2.6 and Theorem 3.1 we deduce that $p(M) = p(\hat{M}) = k' \leqslant k$ as required.

It is well-known that if A is the homomorphic image of a CM ring then $NC(M)$ is closed in Spec A. Then we get the following corollary.

COROLLARY 4.2. *Suppose that A is the homomorphic image of a CM ring and M is equidimensional. Then it holds*

$$p(M) = r(M) = \dim(NC(M))\,.$$

Proof. Note that if A is catenary and M is equidimensional then for all $P \in \operatorname{Supp} M$ we have $\dim(M_P) + \dim(A/P) = d$. Therefore by Theorem 4.1 it follows that $p(M) = \dim(NC(M))$. Hence $p(M) = r(M) = \dim(NC(M))$ by Theorem 3.3.

The following corollary is one of the main results in [C-S-T].

COROLLARY 4.3. *Suppose that A is the homomorphic image of a CM ring then following conditions are equivalent*:

(i) *M is a generalized CM module.*

(ii) $p(M) \leqslant 0$.

(iii) *Every s.o.p. on M is reducing.*

(iv) $\dim(NC(M)) = 0$ *and* $\dim(M_P) + \dim(A/P) = d$ *for all* $P \in \operatorname{Supp} M$.

Proof. The corollary immediately follows from Theorem 4.1 and the definition of a generalized CM module.

§ 5. Flat extensions

Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat, local homomorphism. We denote by F the fiber $A/\mathfrak{m} \otimes_A B$. In this section we examine the relationship between polynomial types $p(A)$, $p(B)$ and $\dim F$ by such a flat homomorphism.

THEOREM 5.1. *Let $f: (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local, flat extension. Then*:

(i) *If $p(B) \geqslant \dim F$ then $p(B) \geqslant p(A) + \dim F$. Moreover, the equality holds if $\mathfrak{a}(A)B \subseteq \mathfrak{a}(B)$.*

(ii) *If $p(B) < \dim F$ then A is a CM ring.*

Proof. Denote $\hat{A}$ and $\hat{B}$ the $\mathfrak{m}$-adic (the $\mathfrak{n}$-adic) completion of $A(B)$, respectively. Let $\hat{f}: \hat{A} \to \hat{B}$ be the induced flat homomorphism. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \hat{A} & \xrightarrow{\hat{f}} & \hat{B}\,. \end{array}$$

Then by Lemma 2.6 we have $p(A) = p(\hat{A})$, $p(B) = p(\hat{B})$. It is also easily to see that $\hat{B} \otimes_{\hat{A}} \hat{A}/\hat{\mathfrak{m}} \cong \hat{F}$, $\hat{F}$ is the $\mathfrak{n}$-adic completion of F. Hence, without loss of the generality we can suppose that A and B are complete. Therefore they are homomorphic images of regular rings.

(i) Let $\mathfrak{p} \in \operatorname{Spec} A$ such that $\dim(A/\mathfrak{p}) > p(B) - \dim F$. Since the going down Theorem holds between A and B we can choose a $P \in \operatorname{Spec} B$ such that $\dim(B/P) = \dim(B/\mathfrak{p}B)$ and $P \cap A = \mathfrak{p}$. Therefore we get

$$\dim(B/P) = \dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim F > p(B)\,.$$

By virtue of Theorem 4.1, B_P is CM and $\dim(B_P) + \dim(B/P) = \dim B$; it follows that

$$\begin{aligned} \dim B = \dim(B_P) + \dim(B/P) &= \dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) + \dim F \\ &= \dim A + \dim F\,. \end{aligned}$$

Therefore $\dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim A$. Hence $p(A) \leqslant p(B) - \dim F$ by Theorem 4.1. Suppose now that $\mathfrak{a}(A)B \subseteq \mathfrak{a}(B)$. Let $P \in \operatorname{Spec} B$ such that $\mathfrak{a}(B) \subseteq P$ and $\dim(B/P) = p(B)$. Note that A and B are catenary; we have for $\mathfrak{p} = P \cap A$

$$\begin{aligned} \dim(B/P) &\leqslant \dim(B/\mathfrak{p}B) - \dim(B_P/\mathfrak{p}B_P) \\ &= \dim(A/\mathfrak{p}) + \dim F - \dim(B_P/\mathfrak{p}B_P)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \dim(A/\mathfrak{p}) &\geqslant \dim(B/P) - \dim F + \dim(B_P/\mathfrak{p}B_P) \\ &= p(B) - \dim F + \dim(B_P/\mathfrak{p}B_P) \geqslant p(B) - \dim F\,. \end{aligned}$$

Since $\mathfrak{a}(A)B \subseteq \mathfrak{a}(B)$ it follows that $\mathfrak{a}(A) \subseteq P \cap A = \mathfrak{p}$. Thus

$$p(A) = \dim(A/\mathfrak{a}(A)) \geqslant \dim(A/\mathfrak{p}) \geqslant p(B) - \dim F$$

by Theorem 3.1. So we get $p(A) = p(B) - \dim F$ as required.

(ii) Let $P \in \operatorname{Spec} B$ such that $\dim(B/P) = \dim F$ and $P \cap A = \mathfrak{m}$. As $\dim(B/P) > p(B)$, B_P is CM. Therefore A is CM by [M], 21.C. The proof of Theorem 5.1 is complete.

COROLLARY 5.2. *With the same* A, B, F *as in Theorem* 5.1, *if* $\dim F = 0$ *then* $p(A) = p(B)$.

Proof. Since $\dim F = 0$ we can easily show that $\mathfrak{a}(A)B \subseteq \mathfrak{a}(B)$. Therefore $p(A) = p(B)$ by Theorem 5.1, (i).

COROLLARY 5.3. *Let* $\mathfrak{p} \in \operatorname{Spec} A$ *with* $\dim(A/\mathfrak{p}) \leqslant p(A)$ *then*

$$p(A_\mathfrak{p}) \leqslant p(A) - \dim(A/\mathfrak{p}) .$$

Proof. Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Choose a $P \in \operatorname{Spec} \hat{A}$ such that $\dim(\hat{A}/P) = \dim(\hat{A}/\mathfrak{p}\hat{A}) = \dim(A/\mathfrak{p})$ and consider the local flat homomorphism $A_\mathfrak{p} \to \hat{A}_P$. Since $\dim(\hat{A}_P/\mathfrak{p}\hat{A}_P) = 0$ it follows by Corollary 5.2 that $p(A_\mathfrak{p}) = p(\hat{A}_P)$. Hence we can assume that A is a complete ring. Let now $\mathfrak{q} \subseteq \mathfrak{p}$ a prime ideal such that $\dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) > p(A) - \dim(A/\mathfrak{p})$. Then we have

$$\dim(A/\mathfrak{q}) \geqslant \dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) > p(A) .$$

Thus $A_\mathfrak{q}$ is CM by Corollary 3.6. Furthermore, since A is complete A is catenary; therefore $\dim(A_\mathfrak{p}) = \dim(A_\mathfrak{q}) + \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})$. Hence, applying Theorem 4.1 for $A_\mathfrak{p}$, we deduce that $p(A_\mathfrak{p}) \leqslant p(A) - \dim(A/\mathfrak{p})$ as required.

Theorem 5.1 has some interesting consequences. First, the following two corollaries say about the relationship between the polynomial types. Their proofs are trivial therefore we omit it.

COROLLARY 5.4. *With the same* A, B *as in Theorem* 5.1 *then* $p(A) \leqslant p(B)$.

COROLLARY 5.5. *With the same* A, B, F *as in Theorem* 5.1, *assume that* A *is not a CM ring then* $p(A) = p(B)$ *if and only if* $\dim F = 0$.

The following corollary is a generalization of Theorem 1, (b) in [D-E].

COROLLARY 5.6. *With the same notations as above then* A *is a generalized CM ring if* $p(B) \leqslant \dim F$.

Proof. If $p(B) < \dim F$ then A is CM by Theorem 5.1, (ii). If $p(B) = \dim F$ then by Theorem 5.1, (i)

$$\dim F = p(B) \geqslant p(A) + \dim F .$$

It follows that $p(A) \leqslant 0$; so A is generalized CM by Remark 2.5.

EXAMPLE 5.7. Let $(A, \mathfrak{m})$ be a local ring of dimension $d > 0$ and let

X a transcendental over A. We define $B = A[X]_{(\mathfrak{m},X)A[X]}$ then $A \to B$ is a local, flat homomorphism. It is easy to see that the fiber $F = B \otimes_A A/\mathfrak{m}$ is regular and of dimension 1. If A is CM, since F is regular, then B is also CM. Therefore the equality $p(B) = p(A) + \dim F$ does not hold in this case. If A is not CM then by Theorem 5.1, (i), $p(B) \geqslant p(A) + \dim F = p(A) + 1$. Especially, if A is generalized CM, i.e. $p(A) = 0$ then $p(B) \geqslant 1$; so B is not a generalized CM ring.

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