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# THE FUNDAMENTAL UNIT AND BOUNDS FOR CLASS NUMBERS OF REAL QUADRATIC FIELDS

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## Introduction

Although class number one problem for imaginary quadratic fields was solved in 1966 by A. Baker [3] and by H. M. Stark [25] independently, the problem for real quadratic fields remains still unsettled. However, since papers by Ankeny-Chowla-Hasse [2] and H. Hasse [9], many papers concerning this problem or giving estimate for class numbers of real quadratic fields from below have appeared. There are three methods used there, namely the first is related with quadratic diophantine equations ([2], [9], [27, 28, 29, 31], [17]), and the second is related with continued fraction expantions ([8], [4], [16], [14], [18]). The third is related with Dirichlet's classical class number formula

$$h_D = (2 \log \varepsilon_D)^{-1} \sqrt{D} L(1, \chi_D),$$

where $L(1, \chi_D)$ is the value at $s = 1$ of the $L$-function

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s}$$

with Kronecker character $\chi_D$ belonging to the real quadraric field $\mathbf{Q}(\sqrt{D})$ ([12], [30], [20]). There, T. Tatuzawa's lower bound for $L(1, \chi_D)$:

$$L(1, \chi_D) > 0.655 (c/D^c) \quad \text{(with one possible exception of } D)$$

plays very important role (cf. [26], [10]).

On the other hand, regarding estimate for the class number of real quadratic fields from above, there are two methods. One of them uses L. K. Hua's upper bound (cf. [11]) for $L(1, \chi_D)$:

$$L(1, \chi_D) < 2^{-1} \log D + 1$$

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in Dirichlet's formula ([23], [33]). Another uses D. A. Burgess' upper bound (cf. [5]) for $L(1, \chi_p)$:

$$L(1, \chi_p) < 0.2456 \log p \quad (p > p_0)$$

([15], [6], [33]).

Among all real quadratic fields, especially real quadratic fields of R-D type whose fundamental unit was well-known by Richaud [24] and Degert [7] were first studied ([2], [9], [28, 29], [17, 19], [14]), and later by H. Yokoi [27], T. Nakahara [22] and R. A. Mollin [20] real quadratic fields of non R-D type were studied.

In recent papers [31, 32, 33, 34], we defined some new $p$-invariants for any rational prime $p$ congruent to 1 mod 4, and studied relationships among these new invariants and already known invariants. Above all, the new $p$-invariant $n_p$ defined by

$$|t_p/u_p^2 - n_p| < 1/2$$

through the fundamental unit

$$\varepsilon_p = (t_p + u_p\sqrt{p})/2 \quad (> 1)$$

of real quadratic field $\mathbf{Q}(\sqrt{p})$ with prime discriminant was fundamental as far as $n_p \neq 0$ (i.e. $t_p/u_p^2 > 1/2$). On the other hand, it became clear that the case $n_p = 0$ (i.e. $0 < t_p/u_p^2 < 1/2$) is more important than the case $n_p \neq 0$ for the purpose of solving the class number one problem or Artin's conjecture for real quadratic fields (cf. [1], [32], [34]).

Therefore, one of our purposes in this paper is to define and to study similarly new invariants valuable in case of $n_p = 0$, and another is to generalize $p$-invariants to $D$-invariants for any positive square-free integer $D$.

All results obtained in this paper are valid for any positive square-free integer $D$ provided we add a few unessential modifications, but for the sake of simplicity, we shall restrict integer $D$ to a positive square-free integer satisfying $N\varepsilon_D = -1$ i.e. $t_D^2 - Du_D^2 = -4$.

In order to deal with the case $n_D = 0$ in the same way as the case $n_D \neq 0$, in §1 we shall slightly reform the definition of $n_D$. Namely, we shall newly define $n_D$ by $n_D = [t_D/u_D^2]$ (or $n_D = [D/t_D]$ if $u_D > 2$), and first express uniquely $D$ (resp. $t_D$) as a quadratic (resp. linear) polynomial of $n_D$ with $D$-invariant coefficients, where $[x]$ means the greatest integer less than or equal to $x$. Using the Dirichlet's class number formula, we shall next provide bounds for the class number $h_D$ of real quadratic field $\mathbf{Q}(\sqrt{D})$ in terms of $D$ and $n_D$.

In §2, we shall first define a new invariant $m_D$ by $m_D = [t_D/D]$ (or $m_D = [u_D^2/t_D]$ if $D > 5$), and express uniquely $u_D^2$ (resp. $t_D$) as a quadratic (resp. linear) polynomial of $m_D$ with $D$-invariant coefficients. Next, in terms of new $D$-invariants, we shall give a necessary and sufficient condition for Artin's conjecture on the fundamental unit of real quadratic fields with prime discriminant to be true. Finally, using Dirichlet's formula, we shall estimate the class number $h_D$ of $\mathbf{Q}(\sqrt{D})$ in terms of $D$ and $m_D$.

Throughout this paper, we denote by $\mathbf{N}_0 = \{0, 1, 2, \cdots\}$ the set of all non-negative rational integers and by $[x]$ the greatest integer less than or equal to $x$. Moreover, for any positive square-free integer $D$, we denote by $h_D$ and $\varepsilon_D = (t_D + u_D\sqrt{D})/2$ $(> 1)$ the class number and the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ respectively. Furthermore, we denote by $\mathbf{D}_-$ the set of all positive square free integers such that the norm of the fundamental unit $(t_D + u_D\sqrt{D})/2$ of $\mathbf{Q}(\sqrt{D})$ is equal to $-1$.

## § 1.

For any $D$ in $\mathbf{D}_-$, set

$$\mathbf{A}_D = \{a\colon\ 0 \leqq a < u_D^2,\ a^2 \equiv -4 \bmod u_D^2\}$$

and

$$(A, B)_D = \{(a, b)\colon\ a \in \mathbf{A}_D,\ a^2 + 4 = bu_D^2\}\,.$$

Then, we can first prove the following theorem:

THEOREM 1.1. *For any $D$ in $\mathbf{D}_-$, there are an uniquely determined integer $n_D$ in $\mathbf{N}_0$ and an element $(a_D, b_D)$ in $(A, B)_D$ such that*

$$\begin{cases} t_D = u_D^2 \cdot n_D + a_D\,, \\ D = u_D^2 \cdot n_D^2 + 2a_D \cdot n_D + b_D\,. \end{cases}$$

*Additionally, if $u_D > 2$, then $0 < b_D < a_D < u_D^2$ and $[D/t_D] = n_D$.*

*Proof.* For any $D$ in $\mathbf{D}_-$, put

$$[t_D/u_D^2] = n_D \quad \text{and} \quad t_D = u_D^2 \cdot n_D + a_D\,,$$

then $n_D$ $(\in \mathbf{N}_0)$ and $a_D$ $(\geqq 0)$ are uniquely determined and

$$0 \leqq a_D < u_D^2$$

holds. Moreover, since

$$Du_D^2 = t_D^2 + 4 = u_D^4 \cdot n_D^2 + 2a_D \cdot u_D^2 \cdot n_D + (a_D^2 + 4),$$

we obtain

$$a_D^2 + 4 \equiv 0 \pmod{u_D^2}$$

and so $a_D$ is in $\mathbf{A}_D$.

Now, put

$$a_D^2 + 4 = b_D \cdot u_D^2$$

then $b_D$ $(> 0)$ is also uniquely determined and $(a_D, b_D)$ is in $(A, B)_D$, and moreover

$$D = u_D^2 \cdot n_D^2 + 2a_D \cdot n_D + b_D$$

holds.

Especially, in the case $u_D > 2$, it follows from $0 \leqq a_D < u_D^2$ that

$$b_D \cdot u_D^2 = a_D^2 + 4 < u_D^4 + 4,$$

and so $b_D \leqq u_D^2$. Here, if we assume $b_D = u_D^2$, then $a_D^2 + 4 = u_D^4$ implies $(u_D^2 - a_D)(u_D^2 + a_D) = 4$, which contradicts $u_D > 2$. Hence, we have first $0 < b_D < u_D^2$.

Next, if we put

$$g(x) = -x^2 + u_D^2 x - 4,$$

then $g(1) = g(u_D^2 - 1) = u_D^2 - 5 > 0$, and so $g(b_D) = a_D^2 - b_D^2 > 0$. Therefore, we obtain

$$0 < b_D < a_D < u_D^2.$$

Finally, in the expression

$$D = n_D \cdot t_D + (a_D \cdot n_D + b_D),$$

we have $a_D \cdot n_D + b_D > 0$ and

$$t_D - (a_D \cdot n_D + b_D) = (u_D^2 - a_D)n_D + (a_D - b_D) > 0.$$

Hence we obtain $[D/t_D] = n_D$.

COROLLARY 1.1. *For any* $D$ $(> 2)$ *in* $\mathbf{D}_-$,

(1) *The following are equivalent*:

(i) $n_D = 0$, (ii) $D < t_D$, (iii) $t_D < u_D^2$ *(in this case $u_D > 2$ holds).*

(2) *$a_D = 0$ holds only if $u_D = 1$ or 2.*
*In the special case where $D \not\equiv 2 \pmod 4$ i.e. $D \equiv 1 \pmod 4$,*

$$a_D = 0 \quad \textit{if and only if} \quad u_D = 1 \textit{ or } 2.$$

*Proof.* (1) It is clear from $[t_D/u_D^2] = n_D$ that

$$n_D = 0 \quad \text{if and only if} \quad 0 < t_D < u_D^2.$$

Suppose $n_D = 0$. Then the above implies $u_D > 1$, and if $u_D = 2$, then $t_D \equiv u_D \pmod 2$ yields

$$t_D = 2 \quad \text{i.e.} \quad D = 2.$$

Therefore, $n_D = 0$ implies $u_D > 2$ and by the theorem $D < t_D$. Finally $D < t_D$ implies

$$t_D - D = (u_D^2 n_D + a_D)(1 - n_D) - a_D n_D - b_D > 0$$

and hence $n_D = 0$ holds except for $D = 2$.

(2) It is clear from Theorem 1.1 that

$$a_D = 0 \quad \text{if and only if} \quad t_D \equiv 0 \pmod{u_D^2},$$

which implies easily $u_D = 1$ or $u_D = 2$.

On the other hand, in the case $u_D = 1$, we have $n_D = t_D$, $a_D = 0$, $b_D = 4$, and in the case $u_D = 2$, we have

$$a_D = 0, \quad b_D = 1, \quad D = 4 \cdot n_D^2 + 1 \equiv 1 \pmod 4$$

or

$$a_D = 2, \quad b_D = 2, \quad D = 4 \cdot n_D^2 + 4n_D + 2 \equiv 2 \pmod 4.$$

Hence, in the special case $D \equiv 2 \pmod 4$ or $D \equiv 1 \pmod 4$,

$$a_D = 0 \quad \text{if and only if} \quad u_D = 1 \text{ or } 2.$$

Next, we provide upper bounds for the class number $h_D$ of $\mathbf{Q}(\sqrt{D})$.

THEOREM 1.2. *For any $D$ in $\mathbf{D}_-$,*

(1) $\varepsilon_D > (D - 4)/(n_D + 1)$,

(2) $h_D < \{(1/4)\sqrt{D} \cdot (2 + \log D)\}/\{\log((D - 4)/(n_D + 1))\}$.

*Additionally, in the special case where $D = p$ is prime, there exists a constant $p_0$ such that if a prime $p$ in $\mathbf{D}_-$ satisfies $p > p_0$, then*

(3) $h_p < \{(1/8)\sqrt{p} \cdot \log p\}/\{\log((p-4)/(n_p+1))\}$.

In order to prove this theorem, we need the following two lemmas:

LEMMA 1.1 (L.K. Hau [11]). *For the value $L(1, \chi_d)$ at $s = 1$ of L-function $L(s, \chi_d)$ belonging to the real quadratic field $\mathbf{Q}(\sqrt{d})$,*

$$L(1, \chi_d) < (1/2) \log d + 1$$

*holds.*

LEMMA 1.2 (D.A. Burgess [5]). *For any prime $p$ in $\mathbf{D}_-$, let $L(1, \chi_p)$ be the value at $s = 1$ of L-function $L(s, \chi_p)$ belonging to $\mathbf{Q}(\sqrt{p})$. Then there exists a prime constant $p_0$ such that*

$$L(1, \chi_p) < 0.2456 \log p$$

*holds provided $p > p_0$.*

*Proof of Theorem* 1.2. (1) It follows from $[t_D/u_D^2] = n_D$ that

$$D - 4u_D^{-2} = t_D^2 \cdot u_D^{-2} < t_D(n_D + 1),$$

and so

$$t_D > (D - 4u_D^{-2})/(n_D + 1) \geqq (D - 4)/(n_D + 1).$$

Hence, we get

$$-4 = t_D^2 - Du_D^2 > ((D-4)/(n_D+1))^2 - Du_D^2,$$

and so

$$u_D^2 > D^{-1}((D-4)/(n_D+1))^2 + 4/D > D^{-1}((D-4)/(n_D+1))^2,$$

which implies

$$u_D > ((D-4)/(n_D+1)) \cdot (1/\sqrt{D}).$$

Therefore, we obtain first

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 > (D-4)/(n_D+1).$$

(2) By applying the above assertion (1) and Lemma 1.1 to the Dirichlet's class number formula, we can get immediately the upper bound for $h_p$ in the Theorem.

(3) Using Lemma 1.2 instead of Lemma 1.1, we can prove assertion (3) by the same way as assertion (2).

COROLLARY 1.2. *For any positive $\varepsilon$ and for an arbitrarily fixed invariant $n = n_D$ ($\geqq 0$), there is an effectively determined constant $D_0 = D_0(\varepsilon, n)$, depending on both $\varepsilon$ and $n$, such that if $D$ in $\mathbf{D}_-$ satisfies $D > D_0$ and $n_D = n$, then*

$$h_D < ((1 + \varepsilon)/4)\sqrt{D}\,.$$

*Proof.* For any given and fixed $n = n_D$ ($\geqq 0$), it holds

$$\lim_{D\to\infty} (2 + \log D)/\{\log ((D - 4)/(n + 1))\} = 1\,.$$

Hence, we can get Corollary 1.2 by Theorem 1.2, (2).

COROLLARY 1.3. *For an arbitrarily fixed $n = n_p$ ($\geqq 0$), there is a constant $p_0 = p_0(n)$ depending only on $n$, such that if a prime $p$ in $\mathbf{D}_-$ satisfies $p > p_0$, then for the class number $h_p$ of $\mathbf{Q}(\sqrt{p})$*

$$h_p < (1/8)\sqrt{p}$$

*holds.*

*Proof.* For an arbitrarily fixed $n = n_p$ ($\geqq 0$), we get

$$\lim_{p\to\infty} (\log p)/\{\log ((p - 4)/(n + 1))\} = 1\,.$$

On the other hand, since $(1/8) - (0.2456/2) > 0$, we can choose a positive number $\varepsilon$ satisfying $0.2456/2 = (1/8)(1 - \varepsilon)$. Hence, for this $\varepsilon$, we can choose a prime number $p_0'(n)$ such that

$$(\log p)/\{\log ((p - 4)/(n + 1))\} < 1 + \varepsilon$$

holds provided $p > p_0'(n)$.

Therefore, using Lemma 1.2 for any prime $p$ in $\mathbf{D}_-$ satisfying $p > p_0(n) := \max (p_0'(n), p_0)$ for $p_0$ in Lemma 1.2, we have

$$\begin{aligned} h_p &< \{(0.2456/2)\sqrt{p}\log p\}/\{\log ((p - 4)/(n + 1))\} \\ &< (1/8)(1 - \varepsilon)\sqrt{p}(1 + \varepsilon) \\ &< (1/8)\sqrt{p}\,. \end{aligned}$$

Finally, we provide lower bounds for the class number $h_D$ of $\mathbf{Q}(\sqrt{D})$.

THEOREM 1.3. *For any $D$ in $\mathbf{D}_-$, $n_D \neq 0$ implies the following*:

(1) $\varepsilon_D < (D/n_D) + 1$,

(2) $h_D > (0.3275/s)\cdot D^{(s-2)/2s}/\{\log ((D/n_D) + 1)\}$

*for any* $s \geqq 11.2$ *with one exception of* $D$, *and without any exception under the Generalized Riemann Hypothesis.*

In order to prove this theorem, we need the following lemma:

LEMMA 1.3 (T. Tatuzawa [26], cf. [10]). *For any positive* $c$ *satisfying* $1/2 > c > 0$, *let* $d$ *be any positive integer such that* $d \geqq \max(e^{1/c}, e^{11.2})$, *and let* $\chi_d$ *be any non-principal primitive character with modulus* $d$.

*Then, it holds with one possible exception of* $d$

$$L(1, \chi_d) > 0.655(c/d^c)\,,$$

*where* $L(1, \chi_d)$ *is the value at* $s = 1$ *of the L-function* $L(s, \chi_d)$ *corresponding to* $\chi_d$.

*Proof of Theorem* 1.3. (1) In the case $n_D \neq 0$, it follows from Theorem 1.1 that

$$D > t_D \cdot n_D \quad \text{i.e.} \quad t_D < D/n_D\,.$$

Hence we get

$$-4 = t_D^2 - Du_D^2 < (D/n_D)^2 - Du_D^2\,,$$

and so

$$u_D^2 < D^{-1}((D/n_D)^2 + 4) = Dn_D^{-2} + 4D^{-1}\,,$$

which implies

$$u_D < (\sqrt{D}/n_D) + (2/\sqrt{D})\,.$$

Therefore, we obtain first

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 < (D/n_D) + 1\,.$$

(2) Putting $s = 1/c$ in Lemma 1.3, for any $s \geqq 11.2$ we have

$$L(1, \chi_D) > (0.655/s)D^{-1/s}\,.$$

On the other hand, since under the Generalized Riemann Hypothesis Lemma 1.3 is true without any exception (cf. Kim [13]), we can prove Theorem 1.3, (2) by applying the above assertion (1) to the Dirichlet's class number formula.

From Theorem 1.3, we can obtain the following corollary:

COROLLARY 1.4. *There is an effectively determined* $D_0$ *such that if* $D$ *in* $\mathbf{D}_-$ *satisfies both* $D > D_0$ *and* $n_D \neq 0$, *then* $h_D > 1$ *holds with one more* *possible exception of* $D$.

*In other words, there exist only finitely many* $D$ *in* $\mathbf{D}_-$ *satisfying both* $h_D = 1$ *and* $n_D \neq 0$.

*Proof.* If $n_D \neq 0$, then $\log((D/n_D)+1) < \log 2D$ holds, and Theorem 1.3 yields

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s}/\log 2D$$

for any fixed $s \geqq 11.2$ with one possible exception of $D$.

On the other hand,

$$f_s(x) = x^{(s-2)/2s}/\log 2x$$

is monotone increasing on $[6, \infty)$, and $\lim_{x\to\infty} f_s(x) = \infty$ holds. Hence we can determine $D_0$ effectively such that

$$0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s}/\log 2D > 1$$

holds for any $D \geqq D_0$.

In fact, it is known that there are exactly 54 $D$'s satisfying both $h_D = 1$ and $n_D \neq 0$ (cf. Yokoi [34], Mollin and Williams [21]).

From Corollary 1.4, we can easily derive the following corollary closely related to Gauss' conjecture on class numbers of real quadratic fields:

COROLLARY 1.5. *There exist infinitely many primes* $p$ *in* $\mathbf{D}_-$ *satisfying* $h_p = 1$ *if and only if there exist infinitely many primes* $p$ *in* $\mathbf{D}_-$ *satisfying both* $h_p = 1$ *and* $n_p = 0$.

## § 2.

For any $D$ in $\mathbf{D}_-$, set

$$\tilde{\mathbf{A}}_D = \{a\colon\ 0 \leqq a < D,\ a^2 \equiv -4 \bmod D\}$$

and

$$(\tilde{A}, \tilde{B})_D = \{(a, b)\colon\ a \in \tilde{\mathbf{A}}_D,\ a^2 + 4 = bD\}\,.$$

Then, we can first prove the following theorem:

THEOREM 2.1. *For any* $D$ *in* $\mathbf{D}_-$, *there are uniquely determined* $m_D$ *in* $\mathbf{N}_0$ *and* $(a_D, b_D)$ *in* $(\tilde{A}, \tilde{B})_D$ *such that*

$$\begin{cases} t_D = D \cdot m_D + a_D\,, \\ u_D^2 = D \cdot m_D^2 + 2a_D \cdot m_D + b_D\,. \end{cases}$$

*Additionally, if $D > 5$, then*

$$0 < b_D < a_D < D \quad \text{and} \quad [u_D^2/t_D] = m_D .$$

*Proof.* For any $D$ *in* $\mathbf{D}_-$, put

$$[t_D/D] = m_D \quad \text{and} \quad t_D = D\cdot m_D + a_D ,$$

then $m_D$ ($\in \mathbf{N}_0$) and $a_D$ are uniquely determined and

$$0 \leqq a_D < D .$$

Moreover, since

$$Du_D^2 = t_D^2 + 4 = D^2\cdot m_D^2 + 2a_D\cdot Dm_D + (a_D^2 + 4) ,$$

we obtain

$$a_D^2 + 4 \equiv 0 \pmod{D} ,$$

and so $a_D$ is in $\tilde{\mathbf{A}}_D$.

Now, put

$$a_D^2 + 4 = D\cdot b_D ,$$

then $b_D$ ($> 0$) is also uniquely determined and $(a_D, b_D)$ is in $(\tilde{A}, \tilde{B})_D$, and moreover

$$u_D^2 = D\cdot m_D^2 + 2a_D\cdot m_D + b_D$$

holds.

Suppose $D > 5$. If we assume $b_D \geqq D$, then

$$a_D^2 + 4 = D\cdot b_D \geqq D^2 ,$$

and so

$$4 \geqq D^2 - a_D^2 = (D - a_D)(D + a_D) \geqq D + a_D \geqq D ,$$

which contradicts with $D > 5$. Hence we have first $0 < b_D < D$.

Next, if we put

$$g(x) = -x^2 + Dx - 4 ,$$

then $g(1) = g(D - 1) = D - 5 > 0$, and so $g(b_D) = a_D^2 - b_D^2 > 0$. Therefore we obtain

$$0 < b_D < a_D < D .$$

Finally, in the expression

$$u_D^2 = t_D \cdot m_D + (a_D \cdot m_D + b_D),$$

we have $a_D \cdot m_D + b_D > 0$ and

$$t_D - (a_D \cdot m_D + b_D) = (D - a_D) \cdot m_D + (a_D - b_D) > 0.$$

Hence, we have $[u_D^2/t_D] = m_D$.

COROLLARY 2.1. *For any $D$ in $\mathbf{D}_-$,*

(1) *if $D > 5$, then the following are equivalent:*

(i) $m_D = 0$ (ii) $t_D < D$ (iii) $u_D^2 < t_D$ (iv) $a_D = t_D$, $b_D = u_D^2$,

(2) $a_D = 0$ *if and only if* $D = 2$

*(in this case $t_D = u_D = b_D = 2$, $m_D = 1$).*

*Proof.* (1) In the case $D > 5$, it is clear from $[u_D^2/t_D] = m_D$ that

$$m_D = 0 \quad \text{if and only if} \quad 0 < u_D^2 < t_D.$$

Moreover, it is also clear from $[t_D/D] = m_D$ that

$$m_D = 0 \quad \text{if and only if} \quad 0 < t_D < D.$$

On the other hand, if $m_D = 0$, then by Theorem 2.1 we have

$$t_D = a_D \quad \text{and} \quad u_D^2 = b_D.$$

Conversely, if $t_D = a_D$, then $0 < a_D < D$ implies $0 < t_D < D$, i.e.

$$m_D = [t_D/D] = 0.$$

(2) If we assume $a_D = 0$, then by Theorem 2.1 we have $D \leqq 5$. On the other hand, in the case $D = 5$, we know $t_5 = u_5 = 1$, $m_5 = 0$, $a_5 = b_5 = 1$. In the case $D = 2$, we know also $t_2 = u_2 = 2$, $m_2 = 1$, $a_2 = 0$, $b_2 = 2$.

Hence $a_D = 0$ if and only if $D = 2$.

In the case where $D = p$ is prime congruent to $1 \bmod 4$, the following conjecture is well-known as Artin's conjecture (cf. [1]):

CONJECTURE (Artin). *For the fundamental unit*

$$\varepsilon_p = (t_p + u_p\sqrt{p})/2 > 1$$

*of the quadratic field $\mathbf{Q}(\sqrt{p})$,*

$$u_p \not\equiv 0 \pmod{p}$$

*holds.*

Concerning this conjecture, we are able to prove the following:

THEOREM 2.2. *Artin's conjecture is true if and only if*

$$a_p \cdot b_p \not\equiv 8m_p \pmod{p}.$$

*In particular, if $m_p \equiv 0 \pmod{p}$, Artin's conjecture is true.*

*Proof.* In the special case $p = 5$, ($t_5 = u_5 = 1$)), we get $m_5 = 0$, $a_5 = b_5 = 1$, and hence both $u_5 \not\equiv 0 \pmod 5$ and $a_5 \cdot b_5 \not\equiv 8m_5 \pmod 5$ simultaneously hold. Therefore without loss of generality, we may assume $p > 5$, and so $(a_p, p) = 1$ by Theorem 2.1.

Moreover, it follows immediately from $u_p^2 = p \cdot m_p^2 + 2a_p \cdot m_p + b_p$ that

$$u_p \equiv 0 \pmod{p} \quad \text{if and only if} \quad 2a_p^2 \cdot m_p + a_p b_p \equiv 0 \pmod{p}.$$

On the other hand, $a_p^2 + 4 = p \cdot b_p$ implies

$$2a_p^2 \cdot m_p + a_p b_p \equiv a_p b_p - 8m_p \pmod{p}.$$

Hence, it is clear that

$$u_p \not\equiv 0 \pmod{p} \quad \text{if and only if} \quad a_p b_p \not\equiv 8m_p \pmod{p}.$$

Especially, in the case $p > 5$, we get $a_p b_p \not\equiv 0 \pmod{p}$ by Theorem 2.1, and hence if $m_p \equiv 0 \pmod{p}$, then $a_p b_p \not\equiv m_p \pmod{p}$ i.e. Artin's conjecture is true.

Finally, we provide a lower bound and an upper bound for the class number $h_D$ of $\mathbf{Q}(\sqrt{D})$.

THEOREM 2.3. *For any $D > 5$ in $\mathbf{D}_-$,*

(1) $[\varepsilon_D/D] = m_D$,

(2) *For any $s \geqq 11.2$ and $D \geqq e^s$,*

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log((m_D + 1)D)\}$$

*holds with one possible exception of $D$,*

(3) $h_D < \{(1/4)\sqrt{D} \cdot (2 + \log D)\} / \{\log(D \cdot m_D)\}$.

*Especially, if $m_D > e^2$, then*

$$h_D < (1/4)\sqrt{D}.$$

*Proof.* (1) By Theorem 2.1, we have first $Dm_D \leqq t_D < D(m_D + 1)$. Hence we get also

$$D^2 m_D^2 \leqq t_D^2 = Du_D^2 - 4 < D^2(m_D + 1)^2,$$

i.e.

$$Dm_D^2 + (4/D) \leqq u_D^2 < D(m_D + 1)^2 + (4/D),$$

which implies easily

$$m_D\sqrt{D} < u_D \leqq (m_D + 1)\sqrt{D}$$

because of $D > 5$. Therefore we get

$$Dm_D < \varepsilon_D = (t_D + u_D\sqrt{D})/2 < D(m_D + 1),$$

from which implies $[\varepsilon_D/D] = m_D$.

*Remark.* We can prove also $[u_D/D] = m_D$.

(2) Since $\varepsilon_D < D(m_D + 1)$, we can prove assertion (2) by applying Lemma 1.3 to the Dirichlet's class number formula.

(3) Since $Dm_D < \varepsilon_D$, we can also obtain

$$h_D < \{(1/4)\sqrt{D}\cdot(2 + \log D)\}/\log(D\cdot m_D)$$

by applying Lemma 1.1 to the Dirichlet's class number formula.

Moreover, in the special case $m_D > e^2$,

$$0 < (2 + \log D)/\log(D\cdot m_D) < 1,$$

yields immediately $h_D < (1/4)\sqrt{D}$.

COROLLARY 2.2. *In the special case where $D = p$ is prime, if $m_D > e^2$ then*

$$h_p < (1/8)\sqrt{p}$$

*holds except for at most finite number of $p$.*

*Proof.* Using Lemma 1.2 instead of Lemma 1.1, we can prove Corollary 2.2 in the same way as the proof of Theorem 2.3, (2).

| | | |
|---|---|---|
| $t_D = Dm_D + a_D$ | • | $t_D = u_D^2 n_D + a_D$ |
| $a_D^2 + 4 = b_D D$ | • | $a_D^2 + 4 = b_D u_D^2$ |
| $c_D = a_D m_D + b_D$ | • | $c_D = a_D n_D + b_D$ |
| $u_D^2 = m_D t_D + c_D$ | • | $D = n_D t_D + c_D$ |

| $D$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $n_D$ | $a_D$ | $b_D$ | $c_D$ |
|---|---|---|---|---|---|---|---|---|
| * 2 | 2 | 2 | 1 | 1 | | 0 | 2 | 2 |
| * 5 | 1 | 1 | 1 | | 1 | 0 | 4 | 4 |
| 10 | 6 | 2 | 2 | | 1 | 2 | 2 | 4 |
| * 13 | 3 | 1 | 1 | | 3 | 0 | 4 | 4 |
| * 17 | 8 | 2 | 1 | | 2 | 0 | 1 | 1 |
| 26 | 10 | 2 | 2 | | 2 | 2 | 2 | 6 |
| * 29 | 5 | 1 | 1 | | 5 | 0 | 4 | 4 |
| * 37 | 12 | 2 | 1 | | 3 | 0 | 1 | 1 |
| * 41 | 64 | 10 | 1 | 1 | | 23 | 13 | 36 |
| * 53 | 7 | 1 | 1 | | 7 | 0 | 4 | 4 |
| 58 | 198 | 26 | 2 | 3 | | 24 | 10 | 82 |
| * 61 | 39 | 5 | 1 | | 1 | 14 | 8 | 22 |
| 65 | 16 | 2 | 2 | | 4 | 0 | 1 | 1 |
| * 73 | 2136 | 250 | 1 | 29 | | 19 | 5 | 556 |
| 74 | 86 | 10 | 2 | 1 | | 12 | 2 | 14 |
| 82 | 18 | 2 | 4 | | 4 | 2 | 2 | 10 |
| 85 | 9 | 1 | 2 | | 9 | 0 | 4 | 4 |
| * 89 | 1000 | 106 | 1 | 11 | | 21 | 5 | 236 |
| * 97 | 11208 | 1138 | 1 | 115 | | 53 | 29 | 6124 |
| * 101 | 20 | 2 | 1 | | 5 | 0 | 1 | 1 |
| 106 | 8010 | 778 | 2 | 75 | | 60 | 34 | 4534 |
| * 109 | 261 | 25 | 1 | 2 | | 43 | 17 | 103 |
| * 113 | 1552 | 146 | 1 | 13 | | 83 | 61 | 1140 |
| 122 | 22 | 2 | 2 | | 5 | 2 | 2 | 12 |
| 130 | 114 | 10 | 4 | | 1 | 14 | 2 | 16 |
| * 137 | 3488 | 298 | 1 | 25 | | 63 | 29 | 1604 |
| 145 | 24 | 2 | 4 | | 6 | 0 | 1 | 1 |
| * 149 | 61 | 5 | 1 | | 2 | 11 | 5 | 27 |
| * 157 | 213 | 17 | 1 | 1 | | 56 | 20 | 76 |
| 170 | 26 | 2 | 4 | | 6 | 2 | 2 | 14 |
| * 173 | 13 | 1 | 1 | | 13 | 0 | 4 | 4 |
| * 181 | 1305 | 97 | 1 | 7 | | 38 | 8 | 274 |

| $D$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $n_D$ | $a_D$ | $b_D$ | $c_D$ |
|---|---|---|---|---|---|---|---|---|
| 185 | 136 | 10 | 2 | | 1 | 36 | 13 | 49 |
| * 193 | 3528264 | 253970 | 1 | 18281 | | 31 | 5 | 566716 |
| * 197 | 28 | 2 | 1 | | 7 | 0 | 1 | 1 |
| 202 | 6282 | 442 | 2 | 31 | | 20 | 2 | 622 |
| 218 | 502 | 34 | 2 | 2 | | 66 | 20 | 152 |
| 226 | 30 | 2 | 8 | | 7 | 2 | 2 | 16 |
| * 229 | 15 | 1 | 3 | | 15 | 0 | 4 | 4 |
| * 233 | 46312 | 3034 | 1 | 198 | | 178 | 136 | 35380 |
| * 241 | 142022136 | 9148450 | 1 | 589303 | | 113 | 53 | 66591292 |
| * 257 | 32 | 2 | 3 | | 8 | 0 | 1 | 1 |
| 265 | 12144 | 746 | 2 | 45 | | 219 | 181 | 10036 |
| * 269 | 164 | 10 | 1 | | 1 | 64 | 41 | 105 |
| 274 | 2814 | 170 | 4 | 10 | | 74 | 20 | 760 |
| * 277 | 2613 | 157 | 1 | 9 | | 120 | 52 | 1132 |
| * 281 | 2127064 | 126890 | 1 | 7569 | | 175 | 109 | 1324684 |
| 290 | 34 | 2 | 4 | | 8 | 2 | 2 | 18 |
| * 293 | 17 | 1 | 1 | | 17 | 0 | 4 | 4 |
| 298 | 819114 | 47450 | 2 | 2748 | | 210 | 148 | 577228 |
| * 313 | 253724736 | 14341370 | 1 | 810622 | | 50 | 8 | 40531108 |
| 314 | 886 | 50 | 2 | 2 | | 258 | 212 | 728 |
| * 317 | 89 | 5 | 1 | | 3 | 14 | 8 | 50 |
| * 337 | | | 1 | | | | | |
| 346 | 186 | 10 | 6 | | 1 | 86 | 74 | 160 |
| * 349 | 18420 | 986 | 1 | 52 | | 272 | 212 | 14356 |
| * 353 | 142528 | 7586 | 1 | 403 | | 269 | 205 | 108612 |
| 362 | 38 | 2 | 2 | | 9 | 2 | 2 | 20 |
| 365 | 19 | 1 | 2 | | 19 | 0 | 4 | 4 |
| 370 | 654 | 34 | 4 | 1 | | 284 | 218 | 502 |
| * 373 | 10236 | 530 | 1 | 27 | | 165 | 73 | 4528 |
| * 389 | 2564 | 130 | 1 | 6 | | 230 | 136 | 1516 |
| 394 | 790046070 | 39801946 | 2 | | | | | |
| * 397 | 3447 | 173 | 1 | 8 | | 271 | 185 | 2353 |

| $D$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $n_D$ | $a_D$ | $b_D$ | $c_D$ |
|---|---|---|---|---|---|---|---|---|
| * 401 | 40 | 2 | 5 | | 10 | 0 | 1 | 1 |
| * 409 | | | 1 | | | | | |
| * 421 | 444939 | 21685 | 1 | 1056 | | 363 | 313 | 383641 |
| * 433 | | | 1 | | | | | |
| 442 | 42 | 2 | 8 | | 10 | 2 | 2 | 22 |
| 445 | 21 | 1 | 4 | | 21 | 0 | 4 | 4 |
| * 449 | 378942664 | 17883410 | 1 | 843970 | | 134 | 40 | 113092020 |

* indicates prime number

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