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FIBRATIONS WITH MOVING CUSPIDAL SINGULARITIES

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Let f: $V \to C$ be a fibration from a smooth projective surface onto a smooth projective curve over an algebraically closed field k. In the case of characteristic zero, almost all fibres of f are nonsingular. In the case of positive characteristic, it is, however, known that there exist fibrations whose general fibres have singularities. Moreover, it seems that such fibrations often have pathological phenomena of algebraic geometry in positive characteristic (see M. Raynaud [7], W. Lang [4]).

In the present article, we consider the surfaces with cuspidal fibration which are obtained as the quotients of surfaces with smooth fibration by p-closed rational vector fields. In particular, we shall give a construction of generalized Raynaud surfaces and give a dimensional estimate of the nonzero first cohomology group which appears in counter-examples to the Kodaira Vanishing Theorem in positive characteristic.

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§ 1. Preliminaries

Throughout this article, we assume that k is an algebraically closed field of characteristic $p \geq 3$. Let V be a smooth projective surface over k and let D be a k-derivation of the function field $k(V)$. Then we say that D is a *rational vector field* on V. We call D a *p-closed* rational vector field if there exists a rational function h on V such that $D^p = hD$. For a rational vector field D, let V^D be the scheme whose underlying space is the same as V and whose structure sheaf consists of the germs of sections of $\mathcal{O}_V$ killed by D. We call V^D the *quotient* of V by D. Then the quotient V^D is normal and the canonical projection π: $V \to V^D$ is a purely inseparable morphism. Moreover, if D is p-closed, then the degree of π is p. Let (x, y) be a local coordinate system at a point P of V.

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Then D can be written as

$$D = h_P\left(f_P\frac{\partial}{\partial x} + g_P\frac{\partial}{\partial y}\right),$$

where h_P is a rational function on V and where f_P and g_P are regular functions at P which are relatively prime. The functions $\{h_P\}_{P\in V}$ determine a divisor on V, which we call the divisor of D and which we denote by (D). Therefore, the invertible sheaf $\mathcal{O}((D))$ is generated locally by h_P^{-1}. If D' is a rational vector field on V such that $D' = hD$, where h is a non-zero rational function, then we say that the rational vector fields D and D' are *equivalent*. When one of these equivalent rational vector fields is p-closed, so is the other. Moreover, the quotients V^D and $V^{D'}$ are isomorphic to each other. Consider the rational vector field $(1/h_P)D = f_P\partial/\partial x + g_P\partial/\partial y$, where h_P, f_P and g_P are the same as above. Then $(1/h_P)D$ is equivalent to D and is a k-derivation of the local ring $\mathcal{O}_{V,P}$. If $f_P(P) = g_P(P) = 0$, then we say that P is an *isolated singularity* of D. If the ideal (f_P, g_P) in $\mathcal{O}_{V,P}$ contains the unity, then we say that D *has only divisorial singularities* in a neighbourhood of P. When D has only divisorial singularities in a neighbourhood of any point of V, we say that D has only divisorial singularities on V.

LEMMA 1.1. *Let V be a smooth surface, let D be a p-closed rational vector field on V and let P be a point of V. Then we have the following*:

(1) *The quotient V^D is nonsingular at $\pi(P)$ if and only if D has only divisorial singularities in a neighbourhood of P.*

(2) *Suppose that D has only divisorial singularities in a neighbourhood of P. Then we can choose a local prarameter system (x, y) in the completion $\widehat{\mathcal{O}_{V,P}}$ such that D is equivalent to $\partial/\partial x$. Moreover, (x^p, y) is a local parameter system in the completion $\widehat{\mathcal{O}_{V^D,\pi(P)}}$.*

For the proof of this lemma, we refer to Seshadri [9]. Next, we shall define the notion of integral curves of a rational vector field D. Let Γ be a curve on V and let P be a point of Γ. Denote $(1/h_P)D$ by D_P, where h_P is a local equation of the divisor (D) at P. We call Γ an *integral curve* of D if $D_P(f) \equiv 0$ modulo (f) in $\mathcal{O}_{V,P}$ for any point P of Γ, where f is a local equation of Γ at P.

LEMMA 1.2. *Let D be a p-closed rational vector field on V, let π: $V \to V^D$ be the canonical projection and let Γ be a curve on V. Denote by*

Γ' the image of Γ by π. Then we have the following:

(1) *If Γ is an integral curve of D, then $\pi_*\Gamma = p\Gamma'$ and $\pi^*\Gamma' = \Gamma$ as divisors;*

(2) *If Γ is not an integral curve of D, then $\pi_*\Gamma = \Gamma'$ and $\pi^*\Gamma' = p\Gamma$ as divisors.*

For the proof of this lemma and more detailed discussions on rational vector fields, see Rudakov-Shafarevich [8].

§ 2. p-closed rational vector fields and cuspidal fibrations

Let φ: $Z \to B$ be a fibration from a smooth projective surface onto a smooth projective curve such that general fibres of φ are smooth curves. Let D be a p-closed rational vector field on Z. Suppose that $k(B)$ is not contained in the invariant field $k(Z)^D = \{h \in k(Z) \mid D(h) = 0\}$. Then at most finitely many fibres are integral curves of the rational vector field D. Indeed, let P be a general point of Z. We may assume that φ is smooth in a neighbourhood of P. Take a local coordinate t on an open neighbourhood of $\varphi(P)$. By the smoothness of φ near P, the image of t in $\mathcal{O}_Z$ by φ^* forms part of a local coordinate system near P. So, let (z, t) be a local coordinate system on an open neighbourhood W of P. Then, choosing W small enough, we can express D on W as $D = h(f\partial/\partial z + g\partial/\partial t)$, where h is a local equation of the divisor of D and where f and g are regular functions on W. Let Q be a point of $\varphi(W)$. By the definition of integral curves, we know that the fibre $\varphi^{-1}(Q)$ is an integral curve of D if and only if g is zero on $\varphi^{-1}(Q)$. Hence either at most finitely many fibres over $\varphi(W)$ are integral curves or $g = 0$ in $\mathcal{O}_Z$, i.e., $D(t) = 0$. The latter case implies that $k(B) \subset k(Z)^D$.

We can regard the rational vector field D as a global section of $\Theta_Z \otimes \mathcal{O}(-(D))$, where Θ_Z is the tangent bundle of Z. So, we have an injection α: $\mathcal{O}((D)) \to \Theta_Z$. Meanwhile, there is a natural homomorphism β: $\Theta_Z \to \varphi^*\Theta_B$. Consider the composite of these homomorphisms $\beta \circ \alpha$: $\mathcal{O}((D)) \to \varphi^*\Theta_B$. We have

LEMMA 2.1. *Under the above assumptions and notations, the homomorphism $\beta \circ \alpha$: $\mathcal{O}((D)) \to \varphi^*\Theta_B$ is injective.*

Proof. Let P be a general point of Z and take a local coordinate t at $\varphi(P)$. Let (ξ, η) be a local coordinate system at P. Then we can write D as $D = h(f\partial/\partial\xi + g\partial/\partial\eta)$, where h is a rational function on Z and where

f and g are regular functions at P and they are relatively prime. Hence the image of α in $\Theta_{Z,P}$ is $\mathcal{O}_{Z,P}(f\partial/\partial\xi + g\partial/\partial\eta)$ over P. Denote the regular function $\varphi^*(t)$ by $u(\xi, \eta)$. Then the images of $\partial/\partial\xi$ and $\partial/\partial\eta$ by β are $(\partial u/\partial\xi)\varphi^*(\partial/\partial t)$ and $(\partial u/\partial\eta)\varphi^*(\partial/\partial t)$ respectively. Hence the image of $\beta\circ\alpha$ is $\mathcal{O}_{Z,P}(f\partial u/\partial\xi + g\partial u/\partial\eta)\varphi^*(\partial/\partial t)$. Since $k(B)$ is not contained in $k(Z)^D$, we know that $f\partial u/\partial\xi + g\partial u/\partial\eta$ is not zero in $k(Z)$. So, $\beta\circ\alpha$ is not a zero-map. Therefore, it follows that $\beta\circ\alpha$ is injective. □

Let $\mathscr{R}$ be the cokernel of $\beta\circ\alpha$. Then we have an exact sequence on Z:

$$0 \longrightarrow \mathcal{O}((D)) \longrightarrow \varphi^*\Theta_B \longrightarrow \mathscr{R} \longrightarrow 0\,.$$

$\mathscr{R}$ has the support of dimension 1. Write $\operatorname{Supp}\mathscr{R} = T_1 + \cdots + T_r + l_1 + \cdots + l_s$, where T_i's and l_j's are the irreducible components of $\operatorname{Supp}\mathscr{R}$ such that T_i's are horizontal components, i.e., T_i's meet general fibres transversally, and each l_j is a component contained in fibres of φ. By tensoring $\mathcal{O}(\varphi^*K_B)$ to the above exact sequence, we have

$$0 \longrightarrow \mathcal{O}((D) + \varphi^*K_B) \longrightarrow \mathcal{O}_Z \longrightarrow \mathscr{R}\otimes\mathcal{O}(\varphi^*K_B) \longrightarrow 0\,.$$

Hence there are positive integers $a_1, \cdots, a_r, b_1, \cdots, b_s$ such that $a_1T_1 + \cdots + a_rT_r + b_1l_1 + \cdots + b_sl_s$ is linearly equivalent to $-(D) - \varphi^*K_B$. We call this divisor $a_1T_1 + \cdots + a_rT_r + b_1l_1 + \cdots + b_sl_s$ the *tangent locus* of the rational vector field D with respect to the fibration φ and we call $a_1T_1 + \cdots + a_rT_r$ the horizontal part of the tangent locus. Let X be the quotient of Z by D and denote the canonical projection $Z\to X$ by π. Let $\mathbf{F}_B\colon B\to B'$ be the k-Frobenius morphism of B, namely, $\mathcal{O}_{B'} = \mathcal{O}_B\otimes_k(k, [p])$ with $[p]\colon k\to k$ defined by $[p](\lambda) = \lambda^p$ for $\lambda\in k$ and $\mathbf{F}_B^*\colon \mathcal{O}_{B'} = \mathcal{O}_B\otimes_k(k, [p]) \to \mathcal{O}_B$ given by $f\otimes\lambda\mapsto f^p\lambda$. Then there is a natural morphism $\psi\colon X\to B'$ so that the following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ B & \xrightarrow{\mathbf{F}_B} & B'\,. \end{array}$$

Moreover, we have the following fundamental lemma:

LEMMA 2.2. *With the same assumptions and notations as above, we have*:

(1) *ψ is a fibration, i.e., almost all fibres of ψ are reduced and irreducible curves. Moreover, the fibres of ψ are the images of the fibres of φ by π.*

(2) *Let F be a smooth fibre of φ which does not contain any l_j and let P be a point of F. Suppose, furthermore, X is nonsingular at $\pi(P)$. Then $\pi(F)$ is singular at $\pi(P)$ if and only if P is contained in the horizontal part of the tangent locus of D.*

(3) *The fibres of ψ have the arithmetic genus $p_a(F)-(p-1)(F,(D))/2$, where F is a general fibre of φ.*

Proof. (1) Let Q' be a general point of B'. Write $\psi^*(Q')=nF'$, where n is an integer and F' is the set-theoretic inverse image of Q' by ψ. Let Q be a point of B such that $\mathbf{F}_B(Q)=Q'$ and let F be a fibre of φ over Q. Then $\varphi^*\circ\mathbf{F}_B^*(Q')=\varphi^*(pQ)=pF$ as divisors and F is the set-theoretic inverse image of F'. Since F is not an integral curve of D, we have $\pi^*(F')=pF$ by Lemma 1.2.(2). So, we have $\pi^*\circ\psi^*(Q')=npF$ as divisors. Hence we obtain $npF=pF$. This implies $n=1$. Therefore, it follows that ψ is a fibration. Now, the second assertion is clear.

(2) Since D is p-closed and X is nonsingular at $\pi(P)$, D has only divisorial singularities in a neighbourhood of P. Hence, by Lemma 1.1, we can choose a local parameter system (ξ,η) at P in the completion $\widehat{\mathcal{O}_{Z,P}}$ such that D is equivalent to $\partial/\partial\xi$. The quotient X is locally at $\pi(P)$ the same as the quotient of Z by $\partial/\partial\xi$. So, we may assume that $D=\partial/\partial\xi$. Let t be a local parameter at $\varphi(P)$ on B and write $\varphi^*(t)=u(\xi,\eta)$. Then the image of the injection $\mathcal{O}((D))\to\varphi^*\Theta_B$ at P is $\mathcal{O}_P(\partial u/\partial\xi)\varphi^*(\partial/\partial t)$. Hence P is lying on $\operatorname{Supp}\mathscr{R}$ if and only if $\partial u/\partial\xi(P)=0$. On the other hand, (ξ^p,η) forms a local parameter at $\pi(P)$ on X and $\pi(F)$ is defined by $u^p=0$. We know $\partial(u^p)/\partial(\xi^p)=(\partial u/\partial\xi)^p$ and $\partial(u^p)/\partial\eta=0$. Therefore, $\pi(F)$ is singular at $\pi(P)$ if and only if $\partial u/\partial\xi(P)=0$. So, we have the stated result.

(3) Since F is not an integral curve of D, we obtain that $\pi^*(F')=pF$ as divisors by Lemma 1.2.(2), where F' is the set-theoretic image of F by π. Meanwhile, by the canonical divisor formula due to Rudakov and Shafarevich, we have $\pi^*K_X=K_Z-(p-1)(D)$ (see [8, § 2]). So, we compute as $(F',K_X)=(1/p)(\pi^*F',\pi^*K_X)=(F,K_Z-(p-1)(D))$. This implies the stated formula. □

In order to specify the singularities of fibres of $\psi\colon X\to B'$, we shall consider the vector field and the fibres locally. Let P be a point of Z

which is not an isolated singular point of D and is not lying on any singular fibre of φ. Take a local parameter system (ξ, η) at P such that D is equivalent to $\partial/\partial\xi$. Then the curve C defined by the equation $\eta = 0$ is an integral curve of D through P. Let t be a local coordinate on B at $\varphi(P)$ and write $\varphi^*(t) = u \in \widehat{\mathcal{O}_{Z,P}}$. Let T be the closed subscheme defined by the equation $\partial u/\partial\xi = 0$, which is none other than the tangent locus of D at P. We have the following lemma:

LEMMA 2.3. *In addition to the above assumptions and notations, we suppose, furthermore, that the reduced scheme T_{red} associated to T is nonsingular at P and meets the integral curve C at P transversally and that $T = n(T_{\text{red}})$ as divisors, where n is a positive integer with $n \not\equiv p - 1 \pmod p$. Then we have the following*:

(1) *There exists a local parameter system (x, z) in the completion $\widehat{\mathcal{O}_{Z,P}}$ such that D is equivalent to $\partial/\partial z$ and $u = x + z^{n+1}$.*

(2) *The fibre $\psi^{-1}(\mathbf{F}_B \circ \varphi(P))$ is defined by the equation $x^p + y^{n+1} = 0$ at $\pi(P)$.*

Proof. (1) By the assumptions for T, we can choose a local parameter ξ in the completion $\widehat{\mathcal{O}_{Z,P}}$ such that (ξ, η) forms a local parameter system at P and T_{red} is defined by $\xi = 0$. Write $u = \alpha_0(\eta) + \xi\alpha_1(\eta) + \xi^2\alpha_2(\eta) + \cdots$, where $\alpha_i(\eta) \in k[[\eta]]$ and $\alpha_0(0) = 0$. Since D is equivalent to $\partial/\partial\xi$, we have $\partial u/\partial\xi(P) = 0$. So, $\alpha_1(0) = 0$. By the assumption that the fibre $\{u = 0\}$ is nonsingular, we have $\partial\alpha_0/\partial\eta(P) \neq 0$. Changing η to α_0, we may assume that $u = \eta + \xi\alpha_1(\eta) + \xi^2\alpha_2(\eta) + \cdots$ and $\alpha_1(0) = 0$. From $T = n(T_{\text{red}})$, it follows that ξ^n divides $\partial u/\partial\xi$ and $(1/\xi^n)\partial u/\partial\xi$ is a unit. Therefore, we can write $u = \eta + \xi^{n+1}\beta(\xi, \eta)$ with $\beta(\xi, \eta) \in k[[\xi, \eta]]$ and $\beta(0, 0) \neq 0$. Since $n + 1 \not\equiv 0 \pmod p$, there exists $\gamma(\xi, \eta) \in k[[\xi, \eta]]$ such that $\gamma(\xi, \eta)^{n+1} = \beta(\xi, \eta)$. Set $x = \eta$ and $z = \xi\gamma(\xi, \eta)$. Then we have the required expression.

(2) Let $y = z^p$. Then (x, y) forms a local parameter system at $\pi(P)$ on X. Meanwhile, t^p is a local parameter at $\mathbf{F}_B \circ \varphi(P)$ on B'. Hence the fibre $\psi^{-1}(\mathbf{F}_B \circ \varphi(P))$ is defined by $t^p = 0$, i.e., $x^p + y^{n+1} = 0$. $\square$

Remark 2.4. Keep the same assumptions and notations as above. Since $\varphi^*(t) = u = x + z^{n+1}$, we can regard (z, t) as a local parameter system at P on Z. Then D is equivalent to

$$\frac{\partial}{\partial z} + (n+1)z^n \frac{\partial}{\partial t}.$$

By virtue of the above lemmas, we conclude the following:

THEOREM 2.5. *Let Z, B, φ, D, ψ and B' be as above and let $n_1T_1 + \cdots + n_rT_r$ be the horizontal part of the tangent locus of D. Then the following assertions hold:*

(1) *ψ is a fibration whose general fibres have the arithmetic genus $p_a(F) - (p-1)(F, (D))/2$, where F is a general fibre of φ.*

(2) *The singular loci of the generic fibre of ψ consist of the images of all T_i's.*

(3) *Let G be a general fibre of ψ and let Q be a point of G. If Q is lying on $\pi(T_i)$ and $n_i \not\equiv p-1 \pmod{p}$, then G has a cusp at Q of type $x^p + y^{n_i+1} = 0$.*

§ 3. Construction of generalized Raynaud surfaces

In this section, we shall construct fibrations whose fibres are all irreducible rational curves with one cusp. Certain surfaces with such fibration were constructed by M. Raynaud for the case of index 2 in our terms, and by P. Russell, H. Kurke and W. Lang for the case of index $mp-1$ ($m \in \mathbf{N}$). Our construction is based on their methods, especially the one by Russell and Kurke. To begin with, we recall generalized Tango curves. Let C be a smooth projective curve over k and let $\mathcal{N}$ be an invertible sheaf on C with positive degree. Suppose that there exist local sections $\{\zeta_i \in \Gamma(U_i, \mathcal{O}_C)\}_{i\in I}$ whose differentials $\{d\zeta_i\}$ are local generators of Ω_C^1 satisfying $d\zeta_i = a_{ij}^{pn} d\zeta_j$, where $\{a_{ij}\}$ are transition functions of $\mathcal{N}$ for an affine open covering $\{U_i\}_{i\in I}$ and where n is a positive integer with $n \not\equiv 0 \pmod{p}$ and $n > 1$. Then we call the triplet $(C, \mathcal{N}, \{d\zeta_i\})$ a *generalized Tango curve of index n*. The following lemma is useful for constructing generalized Tango curves.

LEMMA 3.1 (Kurke [2]). *Let ω be an exact differential on a smooth projective curve C. Suppose that the divisor of ω has the form pnH, where H is a non-zero effective divisor and n is a positive integer. Then there exist an affine open covering $\{U_i\}_{i\in I}$ of C and local sections $\{\zeta_i \in \Gamma(U_i, \mathcal{O}_C)\}_{i\in I}$ such that $d\zeta_i = a_{ij}^{pn} d\zeta_j$ on $U_i \cap U_j$, where $\{a_{ij}\}_{i,j\in I}$ are transition functions of $\mathcal{O}_C(H)$. In other words, $(C, \mathcal{O}_C(H), \{d\zeta_i\})$ is a generalized Tango curve provided $n \not\equiv 0 \pmod{p}$ and $n > 1$.*

Proof. Since ω is exact, we can write $\omega = d\zeta_0$ with a rational function ζ_0 on C. Let V_0 be an affine open subset of C such that $d\zeta_0$ is nonzero everywhere on V_0 and $\mathrm{Supp}(H) \cap V_0 = \emptyset$. Then we have $\Omega_C^1|_{V_0} = \mathcal{O}_{V_0} d\zeta_0$. Write $H = r_1P_1 + \cdots + r_lP_l$, where P_λ's are points of C and r_λ's are positive integers. For $1 \le \lambda \le l$, let η_λ be a local coordinate at P_λ and let V_λ be an affine open subset such that $d\eta_\lambda$ is nonzero everywhere on V_λ and $\mathrm{Supp}(H - r_\lambda P_\lambda) \cap V_\lambda = \emptyset$. Then we have $\Omega_C^1|_{V_\lambda} = \mathcal{O}_{V_\lambda} d\eta_\lambda = \mathcal{O}_{V_\lambda} d\zeta_0/\eta_\lambda^{r_\lambda pm}$. Hence it follows that $\varepsilon_\lambda \eta_\lambda^{r_\lambda pm} d\eta_\lambda = d\zeta_0$, where $\varepsilon_\lambda \in \Gamma(V_\lambda, \mathcal{O}_C)^*$. Consider the derivation $\partial/\partial\eta_\lambda$. We know that $(\partial/\partial\eta_\lambda)^p = 0$ and that $\partial\zeta_0/\partial\eta_\lambda = \varepsilon_\lambda \eta_\lambda^{r_\lambda pm}$. Therefore, we have $(\partial/\partial\eta_\lambda)^{p-1}(\varepsilon_\lambda) = 0$. So, we can write $\varepsilon_\lambda = \alpha_{\lambda,0}^p + \alpha_{\lambda,1}^p\eta_\lambda + \cdots + \alpha_{\lambda,p-2}^p\eta_\lambda^{p-2}$, where $\alpha_{\lambda,\nu} \in \Gamma(V_\lambda, \mathcal{O}_C)$ and $\alpha_{\lambda,0} \ne 0$. Let $\zeta_\lambda = \alpha_{\lambda,0}^p\eta_\lambda + \alpha_{\lambda,1}^p\eta_\lambda^2/2 + \cdots + \alpha_{\lambda,p-2}^p\eta_\lambda^{p-1}/(p-1)$. Then ζ_λ is a local coordinate at P_λ and $d\zeta_\lambda = \varepsilon_\lambda d\eta_\lambda$, so, $\eta_\lambda^{r_\lambda pm} d\zeta_\lambda = d\zeta_0$. Since $\{\eta_\mu^{r_\mu}\eta_\lambda^{-r_\lambda}\}$ are transition functions of $\mathcal{O}_C(H)$, we have the required result. □

Remark 3.2. By the proof of the previous lemma, we can choose local sections $\{\zeta_i\}_{i\in I}$ so that each ζ_i is a local coordinate on U_i.

We shall give two examples of generalized Tango curves.

EXAMPLE 3.3 ([7]). Let C be an affine plane curve defined by an equation $y^{np} - y = x^{np-1}$, where n is a positive integer with $n \not\equiv 0 \pmod p$ and $n > 1$. Since the genus of C is $(np-1)(np-2)/2$, the divisor of an exact differential dx is $np(np-3)P_\infty$, where P_∞ is the point at infinity. By Lemma 3.1, we know that $(C, \mathcal{O}((np-3)P_\infty), dx)$ is a generalized Tango curve of index n.

EXAMPLE 3.4. ([4]). Consider an affine plane curve C defined by an equation $y^2 = x^m - 1$, where m is an odd integer. Suppose that p does not divide m. Then C is a hyperelliptic curve of genus $(m-1)/2$. The divisor associated to a differential $(1/y)dx$ is $(m-3)P_\infty$, where P_∞ is the point at infinity. Let $m = pn + 3$, where n is a positive integer with $n \not\equiv 0 \pmod p$ and $n > 1$. By virtue of Lemma 3.1, if $(1/y)dx$ is exact, then $(C, \mathcal{O}(P_\infty), (1/y)dx)$ is a generalized Tango curve of index n. We shall compute $\mathscr{C}((1/y)dx)$, where $\mathscr{C}$ is the Cartier operator. Note that

$$y\mathscr{C}\left(\frac{dx}{y}\right) = \mathscr{C}(y^{p-1}dx) = \mathscr{C}((x^m-1)^{(p-1)/2}dx)\,.$$

Meanwhile, by the definition of the Cartier operator, we have $\mathscr{C}(x^\nu dx) = 0$ unless $\nu \equiv p-1 \pmod p$. Hence it is easy to verify that $\mathscr{C}((1/y)dx) = 0$

if and only if $p \equiv 2 \pmod 3$. Therefore, we obtain a generalized Tango curve $(C, \mathcal{O}(P_\infty), (1/y)dx)$ of index n provided $p \equiv 2 \pmod 3$.

Now, we shall construct a generalized Raynaud surface. Let $(B', \mathcal{N}, \eta)$ be a generalized Tango curve of index n. Then there exist an affine open covering $\{U'_i\}_{i\in I}$ of B' and local sections $\eta_i \in \Gamma(U'_i, \mathcal{O}_{B'})$ satisfying $d\eta_i = a_{ij}^{pn} d\eta_j$, where a_{ij} is the transition function of $\mathcal{N}$ on $U'_i \cap U'_j$. Consider the k-Frobenius morphism $\mathbf{F}_B$: $B \to B'$ (see § 2). We know $\mathbf{F}_B^*\mathcal{N}^n \cong \Omega_B^1$. Moreover, there exist local sections $\xi_i \in \Gamma(U_i, \mathcal{O}_B)$ such that $\xi_i^p = \mathbf{F}_B^*(\eta_i)$ and $d\xi_i = \tilde{a}_{ij}^n d\xi_j$, where U_i is the inverse image of U'_i by $\mathbf{F}_B$ and $\tilde{a}_{ij} = \mathbf{F}_B^*(a_{ij})$. Let $\mathcal{E} = \mathcal{O}_B \oplus \mathbf{F}_B^*\mathcal{N}$ and consider the $\mathbf{P}^1$-fibration φ: $\mathbf{P}(\mathcal{E}) \to B$. Denote $\mathbf{P}(\mathcal{E})$ by Z. Set $z_i = e_i/e$ and $w_i = e/e_i$, where e_i is a generator of $\mathbf{F}_B^*\mathcal{N}|_{U_i}$ and e is the image of $1 \in \mathcal{O}_B$ in $\mathcal{E}$. Then, by Remark 3.2, (z_i, ξ_i) and (w_i, ξ_i) are inhomogeneous coordinate systems on $\varphi^{-1}(U_i) \cong \mathbf{P}^1 \times U_i$ such that $z_i = 1/w_i$ and $z_i = \tilde{a}_{ij} z_j$. Choose one arbitrary open set U_{i_0} among $\{U_i\}_{i \in I}$. Consider a p-closed rational vector field

$$D = \frac{\partial}{\partial z_{i_0}} + n z_{i_0}^{n-1} \frac{\partial}{\partial \xi_{i_0}}$$

on $\varphi^{-1}(U_{i_0})$ (cf. Remark 2.4). Then D is regarded as a rational vector field on Z. Namely, we have

$$D = -w_{i_0}^{-(n-1)} \left(w_{i_0}^{n+1} \frac{\partial}{\partial w_{i_0}} - n \frac{\partial}{\partial \xi_{i_0}} \right)$$

and

$$D = \tilde{a}_{i i_0} \left(\frac{\partial}{\partial z_i} + n z_i^{n-1} \frac{\partial}{\partial \xi_i} \right).$$

Note that D has only divisorial singularities. We see that $\mathcal{O}_Z((D)) = \mathcal{O}_Z(-(n-1)S) \otimes \varphi^* \circ \mathbf{F}_B^*\mathcal{N}^{-1}$, where S is a cross-section defined locally by $w_i = 0$ and $\mathcal{O}_Z(S)$ is the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Moreover, S is an integral curve of D. Meanwhile, since $\mathcal{O}(S) \cong \mathcal{O}_Z(T) \otimes \varphi^* \circ \mathbf{F}_B^*\mathcal{N}$, we know that the tangent locus is $(n-1)T$, where T is a cross-section defined locally by $z_i = 0$. Let X be the quotient of Z by D. Then X is non-singular and, by virtue of Theorem 2.5, there exists a fibration ψ: $X \to B'$ such that all fibres are rational curves with one cusp of type $x^p + y^n = 0$ and that the arithmetic genus of a fibre is $(p-1)(n-1)/2$. Furthermore, the locus of cusps is the image of T. We call this surface X a *generalized Raynaud surface* over a generalized Tango curve $(B', \mathcal{N}, \eta)$. Let $\mathcal{E}' =$

$\mathcal{O}_{B'}\oplus\mathcal{N}^p$ and let $Z' = \mathbf{P}(\mathcal{E}')$. Denote by φ' the canonical projection $Z'\to B'$. We know that Z' is the image of Z by the k-Frobenius morphism $\mathbf{F}_Z$ of Z which is defined similarly to the case of curves. Hence there exists a natural projection $\rho\colon X\to Z'$ such that the composite $\rho\circ\pi\colon Z\to Z'$ is the k-Frobenius morphism $\mathbf{F}_Z$. We have the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{\pi} & X & \xrightarrow{\rho} & Z' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi'} \\ B & \xrightarrow[\mathbf{F}_B]{} & B' & \xrightarrow[\mathrm{id}_{B'}]{} & B' \end{array} .$$

From now on, we regard X as a purely inseparable covering of Z'. Set $\mathbf{F}_Z^*(y_i) = z_i^p$ and $\mathbf{F}_Z^*(t_i) = w_i^p$. Then (y_i, η_i) and (t_i, η_i) are inhomogeneous coordinate system on $\varphi'^{-1}(U_i')$ such that $y_i = 1/t_i$ and $y_i = a_{ij}^p y_j$. Let q and h be positive integers such that $n+q = hp$ and $0<q<p$. Consider the line bundle $\mathbf{L}$ on Z' which is associated to $\mathcal{O}(hS')\otimes\varphi'^*\mathcal{N}^{-n}$, i.e., $\mathbf{L}$ is the spectrum of the symmetric $\mathcal{O}_{Z'}$-algebra generated by the dual of $\mathcal{O}(hS')\otimes\varphi'^*\mathcal{N}^{-n}$, where S' is the cross-section of φ' defined locally by $t_i = 0$, and $\mathcal{O}_{Z'}(S')$ is the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)$. Let $\mathbf{F}_{\mathbf{L}}\colon \mathbf{L}\to\mathbf{L}^p$ be the Frobenius homomorphism of a group scheme $\mathbf{L}$ over Z' and let $\alpha_{\mathbf{L}}$ be the kernel of $F_{\mathbf{L}}$. Set $V_i = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[y_i]$ and set $W_i = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[t_i]$. Then $\{V_i, W_i\}_{i\in I}$ is an affine open covering of Z'. Consider the following trivialization of $\mathbf{L}$:

$$\mathbf{L}|_{V_i} = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[y_i, \theta_i]\,,$$
$$\mathbf{L}|_{W_i} = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[t_i, \tau_i]\,,$$

where $\theta_i = t_i^{-h}\tau_i$, $\theta_i = a_{ij}^n\theta_j$ and $\tau_i = a_{ij}^{-q}\tau_j$. Similarly, we can express $\mathbf{L}^p$ as

$$\mathbf{L}^p|_{V_i} = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[y_i, \theta_i^p]\,,$$
$$\mathbf{L}^p|_{W_i} = \operatorname{Spec}\Gamma(U_i', \mathcal{O}_{B'})[t_i, \tau_i^p]\,,$$

where θ_i and τ_i are the same as above. Consider the local sections: $\mathbf{L}^p|_{V_i}\supset\{\theta_i^p - y_i^n + \eta_i = 0\}$, $\mathbf{L}^p|_{W_i}\supset\{\tau_i^p - t_i^q + t_i^{hp}\eta_i = 0\}$, in terms of the above expression of $\mathbf{L}^p$. Take the pull-backs of these local sections by $\mathbf{F}_{\mathbf{L}}$. Then they glue together to define an $\alpha_{\mathbf{L}}$-torsor $\sigma\colon Y\to Z'$ (see Ekedahl [1] for the definition of $\alpha_{\mathbf{L}}$-torsor). Y is not normal in general, but the normalization of Y is nothing but X because $D(z_i^n - \xi_i) = 0$ for every $i\in I$. Concerning Y, we have the following lemma:

LEMMA 3.5 (cf. [1, Proposition 1.7]). *With the same assumptions and notations as above, we have*:

(1) *There is a filtration of $\mathcal{O}_{Z'}$-modules*

$$\mathcal{O}_{Z'} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} = \sigma_* \mathcal{O}_Y$$

with successive quotients $\mathcal{F}_l/\mathcal{F}_{l-1} = \mathcal{O}_{Z'}(-lhS') \otimes \varphi'^* \mathcal{N}^{ln}$.

(2) *Y has the dualizing sheaf*

$$\omega_Y = \sigma^*(\omega_{Z'} \otimes \mathcal{O}_{Z'}((p-1)hS') \otimes \varphi'^* \mathcal{N}^{-(p-1)n}) .$$

Proof. (1) By the construction of Y, we know that $\sigma^{-1}(V_i) = \operatorname{Spec} \Gamma(U'_i, \mathcal{O}_{B'})[\theta_i, y_i]/(\theta_i^p - y_i^n + \eta_i)$ and $\sigma^{-1}(W_i) = \operatorname{Spec} \Gamma(U'_i, \mathcal{O}_{B'})[\tau_i, t_i]/(\tau_i^p - t_i^q + t_i^{hp}\eta_i)$. Consider the $\mathcal{O}_{Z'}$-submodule $\mathcal{F}_l$ of $\sigma_*\mathcal{O}_Y$ generated by $1, \theta_i, \cdots, \theta_i^l$ over V_i and by $1, \tau_i, \cdots, \tau_i^l$ over W_i. It is easy to verify that these $\mathcal{O}_{Z'}$-modules $\{\mathcal{F}_l\}$ give the required filtration.

(2) Take $b_{ij} \in \Gamma(U'_i \cap U'_j, \mathcal{O}_{B'})$ such that $\eta_i = a_{ij}^{np}\eta_j - b_{ij}^p$. Consider an $\mathbf{A}^1$-bundle Φ: $\mathfrak{A} \to Z'$ such that

$$\mathfrak{A}|_{V_i} = \operatorname{Spec} \Gamma(U'_i, \mathcal{O}_{B'})[y_i, x_i] ,$$
$$\mathfrak{A}|_{W_i} = \operatorname{Spec} \Gamma(U'_i, \mathcal{O}_{B'})[t_i, s_i] ,$$

where $x_i = t_i^{-h}s_i$, $x_i = a_{ij}^n x_j + b_{ij}$ and $s_i = a_{ij}^{-q}s_j + a_{ij}^{-hp}t_j^h b_{ij}$. It is easy to verify that there exists a closed immersion ι: $Y \hookrightarrow \mathfrak{A}$. The image of Y is written as $\{x_i^p - y_i^n + \eta_i = 0\}$ on $\mathfrak{A}|_{V_i}$ and $\{s_i^p - t_i^q + t_i^{hp}\eta_i = 0\}$ on $\mathfrak{A}|_{W_i}$. Since $x_i = a_{ij}^n x_j + b_{ij}$, we have

$$\begin{aligned} x_i^p - y_i^n + \eta_i &= a_{ij}^{np}x_j^p + b_{ij}^p - a_{ij}^{np}y_j^n + a_{ij}^{np}\eta_j - b_{ij}^p \\ &= a_{ij}^{np}(x_j^p - y_j^n + \eta_j) . \end{aligned}$$

Meanwhile, since $x_i = t_i^{-h}s_i$ and $y_i = t_i^{-1}$, we have

$$\begin{aligned} x_i^p - y_i^n + \eta_i &= t_i^{hp}s_i^p - t_i^{-n} + \eta_i \\ &= t_i^{-hp}(s_i^p - t_i^{hp-n} + t_i^{hp}\eta_i) \\ &= t_i^{-hp}(s_i^p - t_i^q + t_i^{hp}\eta_i) . \end{aligned}$$

Hence it follows that $\iota^*(\mathcal{O}_{\mathfrak{A}}(Y) \otimes \mathcal{O}_Y) \cong \sigma^*(\mathcal{O}(hpS') \otimes \varphi'^*\mathcal{N}^{-np})$. On the other hand, we have $\omega_{\mathfrak{A}} = \Phi^*(\omega_{Z'} \otimes \mathcal{O}(-hS') \otimes \varphi'^*\mathcal{N}^n)$. Applying the adjunction formula, we obtain the stated formula. □

We know that X is the normalization of Y. We shall express the structure sheaf $\mathcal{O}_X$ and the dualizing sheaf ω_X in terms of those of Y.

THEOREM 3.6. *Retain the same assumptions and notations as above. Let $\mathscr{H}$ be the cokernel of the natural injection $\sigma_*\mathcal{O}_Y \to \rho_*\mathcal{O}_X$. Then the following assertions hold*:

(1) $\operatorname{Supp}\mathscr{H} = S'$ *and* $\mathscr{H}$ *is a locally free* $\mathcal{O}_{S'}$*-module.*

(2) $\mathscr{H}$ *has a filtration* $0 = \mathscr{H}_0 \subset \mathscr{H}_1 \subset \cdots \subset \mathscr{H}_{q-1} = \mathscr{H}$ *with successive quotients* $\mathscr{H}_\alpha/\mathscr{H}_{\alpha-1} \cong \oplus_\beta \mathscr{N}^{\alpha p-\beta q}$ *for* $0 < \alpha < q$ *if* S' *is identified with* B', *where* β *ranges over all integers such that* $0 < \beta < p$ *and* $-\alpha p + \beta q > 0$.

Proof. (1) With the same notations as in the proof of Lemma 3.5, consider the normalization of the equation $s_i^p = t_i^q(1 - t_i^{hp-q}\eta_i)$. Since p and q are relatively prime, there are integers λ and μ such that $\lambda q - \mu q = 1$. We may assume that $0 < \lambda < q$ and $0 < \mu < p$. Set $u_i = t_i^\lambda s_i^{-\mu}$. Then we have $u_i^p = t_i(1 - t_i^{hp-q}\eta_i)^{-\mu}$ and $u_i^q = s_i(1 - t_i^{hp-q}\eta_i)^{-\lambda}$. Let $\widehat{\sigma_*\mathcal{O}_Y}$ and $\widehat{\rho_*\mathcal{O}_X}$ be the completions of $\sigma_*\mathcal{O}_Y$ and $\rho_*\mathcal{O}_X$ along S', respectively. Then we obtain that $\widehat{\sigma_*\mathcal{O}_Y}|_{W_i\cap S'} \cong \mathcal{O}_{U_i'}[[u_i^p, u_i^q]]$ and $\widehat{\rho_*\mathcal{O}_X}|_{W_i\cap S'} \cong \mathcal{O}_{U_i'}[[u_i]]$. Therefore, it is easy to verify that $\mathscr{H}|_{W_i\cap S'} \cong \oplus_{\alpha,\beta}\mathcal{O}_{U_i'}u_i^{-\alpha p+\beta q} = \oplus_{\alpha,\beta}\mathcal{O}_{U_i'}t_i^{-\alpha}s_i^\beta$, where (α, β) ranges over all pairs of integers such that $0 < \alpha < q$, $0 < \beta < p$ and $-\alpha p + \beta q > 0$. So, we know that $\mathscr{H}$ is a locally free $\mathcal{O}_{S'}$-module.

(2) Consider the $\mathcal{O}_{S'}$-submodule $\mathscr{H}_\alpha$ of $\mathscr{H}$ for $0 < \alpha < q$ which is generated by $\{t_i^{-\alpha'}s_i^\beta\}_{\alpha'\beta}$ over $W_i \cap S'$, where α' and β range over all integers such that $0 < \alpha' \leq \alpha$, $0 < \beta < p$ and $-\alpha' p + \beta q > 0$. We have

$$\begin{aligned} t_i^{-1}s_i^\beta &= a_{ij}^p t_j^{-1}(a_{ij}^{-q}s_j + a_{ij}^{-hp}t_j^h b_{ij})^\beta \\ &\equiv a_{ij}^{p-\beta q}t_j^{-1}s_j^\beta \pmod{\widehat{\sigma_*\mathcal{O}_Y}}. \end{aligned}$$

This implies that $\mathscr{H}_1 \cong \oplus_\beta \mathscr{N}^{p-\beta q}$, where $\beta \in \mathbf{Z}$, $0 < \beta < p$ and $-p + \beta q > 0$. By a similar computation, we have

$$t_i^{-\alpha}s_i^\beta = a_{ij}^{\alpha p-\beta q}t_j^{-\alpha}s_j^\beta + (\text{terms of } t_j^{-\alpha+hl}s_j^{\beta-l} \text{ for } 1 \leq l \leq \beta).$$

Note that $t_j^{-\alpha+hl}s_j^{\beta-l}$ for $1 \leq l \leq \beta$ is contained either in $\mathscr{H}_{\alpha-1}$ or in $\widehat{\sigma_*\mathcal{O}_Y}$. So, we have $\mathscr{H}_\alpha/\mathscr{H}_{\alpha-1} \cong \oplus_\beta \mathscr{N}^{\alpha p-\beta q}$, where β ranges over all integers such that $0 < \beta < p$ and $-\alpha p + \beta q > 0$. □

Remark 3.7. With the same notations as in the proof of the previous theorem, we know that $t_i^{-\alpha+hl}s_j^{\beta-l}$ is contained in $\widehat{\sigma_*\mathcal{O}_Y}$ if $h \geq \alpha$. Hence, if $h \geq \alpha_0$, then we obtain that $\mathscr{H}_{\alpha_0} \cong \oplus_{\alpha,\beta}\mathscr{N}^{\alpha p-\beta q}$, where α and β range over all integers such that $0 < \alpha \leq \alpha_0$, $0 < \beta < p$ and $-\alpha p + \beta q > 0$. In particular, we know $\mathscr{H} \cong \oplus_{\alpha,\beta}\mathscr{N}^{\alpha p-\beta q}$ if $h \geq q$, where (α, β) ranges over all pairs of integers such that $0<\alpha<q$, $0<\beta<p$ and $-\alpha p+\beta q>0$.

In the rest of this section, we give the dualizing sheaf of a generalized Raynaud surface.

PROPOSITION 3.8. *Under the same assumptions and notations as above, X has the dualizing sheaf*

$$\begin{aligned}\omega_X &= \rho^*(\omega_{Z'}\otimes\mathcal{O}((p-1)h)\otimes\varphi'^*\mathcal{N}^{-n(p-1)})\otimes\mathcal{O}_X(-(p-1)(q-1)E)\\ &= \mathcal{O}_X((np-p-n-1)E)\otimes\psi^*\mathcal{N}^{p+n},\end{aligned}$$

where E is the set-theoretic inverse image of S' by ρ.

Proof. By Lemma 3.5, we know that $\omega_Y=\sigma^*(\omega_{Z'}\otimes\mathcal{O}_{Z'}((p-1)hS')\otimes\varphi'^*\mathcal{N}^{-(p-1)n})$. Hence we can write $\omega_X=\rho^*(\omega_{Z'}\otimes\mathcal{O}_{Z'}((p-1)hS')\otimes\varphi'^*\mathcal{N}^{-pn+n})\otimes\mathcal{O}(-rE)$ with $r\in\mathbf{Z}$, where $\mathcal{O}(-rE)$ is the contribution coming from the conductor ideal for the extension $\sigma_*\mathcal{O}_Y\subset\rho_*\mathcal{O}_X$. We have only to determine r. Let F' be a general fibre of φ'. Then ρ^*F' is a general fibre of ψ. Moreover, since $\mathcal{O}((D))=\mathcal{O}_Z(-(n-1)S)\otimes\varphi^*\circ\mathbf{F}_B^*\mathcal{N}^{-1}$, we have $p_a(\rho^*F')=(p-1)(n-1)/2$ by Theorem 2.5.(2). On the other hand, E is the image by π of the integral curve S of D. Therefore, E is a cross-section of ψ and $\rho^*S'=pE$ as divisors by Lemma 1.2.(1). Applying the adjunction formula, we obtain $(p-1)(n-1)-2=(\omega_X,\rho^*F')=-2p+p(p-1)h-r$. From this, it follows that $r=(p-1)(q-1)$. □

§ 4. Kodaira non-Vanishing on generalized Raynaud surfaces

In this section, we consider certain ample invertible sheaves on generalized Raynaud surfaces. Especially, we shall give a lower bound of the first cohomology group of their dual sheaves. Throughout this section, we assume that X is a generalized Tango curve $(B',\mathcal{N},\eta)$ of index n. We begin with the following general theorem, which is proved by applying a Leray spectral sequence. For the reader's convenience, we shall give a proof.

THEOREM 4.1 (Mumford [6], Szpiro [10]). *Let $f\colon V\to C$ be a fibration from a smooth projective surface onto a smooth projective curve such that all fibres are reduced and irreducible. Let Γ be a cross-section of f. Assume that the fibres of f have positive arithmetic genus. If the self-intersection number of Γ is positive, then we have*:

(1) *$\mathcal{O}_V(\Gamma)\otimes f^*\mathcal{L}$ is ample on V, where $\mathcal{L}$ is an invertible sheaf on C isomorphic to $\mathcal{O}_V(\Gamma)|_\Gamma$.*

(2) $H^1(V, \mathcal{O}_V(-\Gamma)\otimes f^*\mathcal{L}^{-1}) \neq 0$.

Proof. (1) Apply the Nakai criterion.

(2) Consider a Leray spectral sequence $H^i(C, \mathbf{R}^j f_*\mathcal{O}_V(-\Gamma)\otimes\mathcal{L}^{-1}) \Rightarrow H^{i+j}(V, \mathcal{O}_V(-\Gamma)\otimes f^*\mathcal{L}^{-1})$. Then we have

$$\begin{aligned} 0 &\longrightarrow H^1(C, f_*\mathcal{O}(-\Gamma)\otimes\mathcal{L}^{-1}) \longrightarrow H^1(V, \mathcal{O}(-\Gamma)\otimes f^*\mathcal{L}^{-1}) \\ &\longrightarrow H^0(C, \mathbf{R}^1 f_*\mathcal{O}(-\Gamma)\otimes\mathcal{L}^{-1}) \longrightarrow 0\,. \end{aligned}$$

Since $H^1(C, f_*\mathcal{O}(-\Gamma)\otimes\mathcal{L}^{-1}) = 0$, we obtain $H^1(V, \mathcal{O}(-\Gamma)\otimes f^*\mathcal{L}^{-1}) \cong H^0(C, \mathbf{R}^1 f_*\mathcal{O}(-\Gamma)\otimes\mathcal{L}^{-1})$. By using an exact sequence

$$0 \longrightarrow \mathcal{O}_V(-\Gamma) \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0\,,$$

we have

$$f_*\mathcal{O}_V \longrightarrow f_*\mathcal{O}_\Gamma \longrightarrow \mathbf{R}^1 f_*\mathcal{O}_V(-\Gamma) \longrightarrow \mathbf{R}^1 f_*\mathcal{O}_V \longrightarrow 0\,.$$

Noting that $f_*\mathcal{O}_V = f_*\mathcal{O}_\Gamma = \mathcal{O}_C$, we know that $\mathbf{R}^1 f_*\mathcal{O}_V(-\Gamma) \cong \mathbf{R}^1 f_*\mathcal{O}_V$. Now using an exact sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(\Gamma) \longrightarrow \mathcal{O}_V(\Gamma)|_\Gamma \longrightarrow 0\,,$$

we find

$$0 \longrightarrow f_*\mathcal{O}_V \longrightarrow f_*\mathcal{O}_V(\Gamma) \longrightarrow f_*\mathcal{O}_V(\Gamma)|_\Gamma \longrightarrow \mathbf{R}^1 f_*\mathcal{O}_V\,.$$

Since the genus of a fibre is positive, we have $f_*\mathcal{O}_V(\Gamma) = \mathcal{O}_C$. Hence $f_*\mathcal{O}_V(\Gamma)|_\Gamma = \mathcal{L} \to \mathbf{R}^1 f_*\mathcal{O}_V$ is injective. By tensoring $\mathcal{L}^{-1}$, we get an injection $\mathcal{O}_C \to \mathbf{R}^1 f_*\mathcal{O}_V\otimes\mathcal{L}^{-1} = \mathbf{R}^1 f_*\mathcal{O}_V(-\Gamma)\otimes\mathcal{L}^{-1}$. Hence $H^1(V, \mathcal{O}_V(-\Gamma)\otimes f^*\mathcal{L}^{-1}) = H^0(C, \mathbf{R}^1 f_*\mathcal{O}_V(-\Gamma)\otimes\mathcal{L}^{-1}) \neq 0$. □

We return to the subject. Let φ', ψ, S' and E be the same as in § 3. Since $pE = \rho^*S'$, we have $(E^2) = (S'^2)/p = \deg\mathcal{N} > 0$. By virtue of the previous theorem, the invertible sheaf $\mathcal{O}_X(E)\otimes\psi^*\mathcal{O}_E(E)$ gives a counter-example to the Kodaira Vanishing Theorem. We notice that the normal sheaf of E is isomorphic to $\mathcal{N}$. Indeed, let $\mathcal{M}$ be the normal sheaf of E. From $pE = \rho^*S'$, it follows that $\mathcal{M}^p \cong \mathcal{O}_{S'}(S') \cong \mathcal{N}^p$. Meanwhile, using an exact sequence

$$0 \longrightarrow \mathcal{O}_X((np-p-n-1)E) \longrightarrow \mathcal{O}_X((np-p-n)E) \longrightarrow \mathcal{M}^{np-p-n} \longrightarrow 0\,,$$

we have an exact sequence on B':

$$\mathcal{M}^{np-p-n} \longrightarrow \mathbf{R}^1\psi_*\mathcal{O}_X((np-p-n-1)E) \longrightarrow \mathbf{R}^1\psi_*\mathcal{O}_X((np-p-n)E)\,.$$

Since $\omega_{X/B'} = \omega_X \otimes \psi^* \omega_{B'}^{-1} = \mathcal{O}((np - p - n - 1)E) \otimes \psi^* \mathcal{N}^{(1-n)p+n}$ (see Proposition 3.8), we obtain $\mathbf{R}^1\psi_* \mathcal{O}((np - p - n - 1)E) \cong (\psi_* \mathcal{O}_X \otimes \mathcal{N}^{(1-n)p+n})^\vee = \mathcal{N}^{np-p-n}$ and $\mathbf{R}^1\psi_* \mathcal{O}(np - p - n)E) \cong (\psi_* \mathcal{O}_X(-E) \otimes \mathcal{N}^{(1-n)p+n})^\vee = 0$ by the Serre duality. Hence $\mathcal{M}^{np-p-n} \to \mathcal{N}^{np-p-n}$ is surjective. Hence this is an isomorphism. Therefore, we have $\mathcal{M} \cong \mathcal{N}$ because p and n are relatively prime. In the sequel of this section, we consider the ample invertible sheaf $\mathcal{O}_X(E) \otimes \psi^* \mathcal{N}$ and the first cohomology group $H^1(X, \mathcal{O}(-E) \otimes \psi^* \mathcal{N}^{-1})$. There is an exact sequence on X:

$$0 \longrightarrow \mathcal{O}(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0 .$$

By tensoring $\psi^* \mathcal{N}^{-1}$, we have

$$0 \longrightarrow \mathcal{O}(-E) \otimes \psi^* \mathcal{N}^{-1} \longrightarrow \psi^* \mathcal{N}^{-1} \longrightarrow \mathcal{N}^{-1} \longrightarrow 0 .$$

Since $H^0(B', \mathcal{N}^{-1}) = 0$, we obtain

$$0 \longrightarrow H^1(X, \mathcal{O}(-E) \otimes \psi^* \mathcal{N}^{-1}) \longrightarrow H^1(X, \psi^* \mathcal{N}^{-1}) \longrightarrow H^1(B', \mathcal{N}^{-1}) .$$

Hence it follows that

$$\dim H^1(X, \mathcal{O}(-E) \otimes \psi^* \mathcal{N}^{-1}) \geq \dim H^1(X, \psi^* \mathcal{N}^{-1}) - \dim H^1(B', \mathcal{N}^{-1}) .$$

By virtue of Theorem 3.6, there is the following exact sequence on Z':

$$0 \longrightarrow \sigma_* \mathcal{O}_Y \longrightarrow \rho_* \mathcal{O}_X \longrightarrow \mathcal{H} \longrightarrow 0 ,$$

where $\mathcal{H}$ is the same as in the theorem. By tensoring $\varphi'^* \mathcal{N}^{-1}$, we have

$$(1) \quad 0 \longrightarrow \sigma_* \mathcal{O}_Y \otimes \varphi'^* \mathcal{N}^{-1} \longrightarrow \rho_* \mathcal{O}_X \otimes \varphi'^* \mathcal{N}^{-1} \longrightarrow \mathcal{H} \otimes \varphi'^* \mathcal{N}^{-1} \longrightarrow 0 .$$

Consider $H^0(S', \mathcal{H} \otimes \varphi'^* \mathcal{N}^{-1})$. Theorem 3.6 implies that $\mathcal{H} \otimes \varphi'^* \mathcal{N}^{-1}$ has a filtration $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{q-1} = \mathcal{H} \otimes \varphi'^* \mathcal{N}^{-1}$ with successive quotients $\mathcal{G}_\alpha / \mathcal{G}_{\alpha-1} \cong \oplus_\beta \mathcal{N}^{\alpha p - \beta q - 1}$, where β is the same as in the theorem. We know that $H^0(B', \mathcal{N}^{\alpha p - \beta q - 1}) = 0$. Hence we have $H^0(S', \mathcal{H} \otimes \varphi'^* \mathcal{N}^{-1}) = 0$. So, taking the cohomology exact sequence associated with (1), we obtain

$$0 \longrightarrow H^1(Z', \sigma_* \mathcal{O}_Y \otimes \varphi'^* \mathcal{N}^{-1}) \longrightarrow H^1(Z', \rho_* \mathcal{O}_X \otimes \varphi'^* \mathcal{N}^{-1}) .$$

Therefore, we know that

$$\dim H^1(X, \mathcal{O}(-E) \otimes \psi^* \mathcal{N}^{-1}) \geq \dim H^1(Z', \sigma_* \mathcal{O}_Y \otimes \varphi'^* \mathcal{N}^{-1}) - \dim H^1(B', \mathcal{N}^{-1}) .$$

We shall compute $H^1(Z', \sigma_* \mathcal{O}_Y \otimes \varphi'^* \mathcal{N}^{-1})$. By Lemma 3.5, there is the following exact sequence on Z' for $1 \leq l \leq p - 1$:

$$0 \longrightarrow \mathscr{F}_{l-1} \longrightarrow \mathscr{F}_l \longrightarrow \mathcal{O}_{Z'}(-lh) \otimes \varphi'^* \mathscr{N}^{ln} \longrightarrow 0\,,$$

where $\mathscr{F}_0 = \mathcal{O}_{Z'}$ and $\mathscr{F}_{p-1} = \sigma_* \mathcal{O}_Y$. Tensoring $\varphi'^* \mathscr{N}^{-1}$, we have

$$(2) \quad 0 \longrightarrow \mathscr{F}_{l-1} \otimes \varphi'^* \mathscr{N}^{-1} \longrightarrow \mathscr{F}_l \otimes \varphi'^* \mathscr{N}^{-1} \longrightarrow \mathcal{O}_{Z'}(-lh) \otimes \varphi'^* \mathscr{N}^{ln-1} \longrightarrow 0\,.$$

Since $H^0(Z', \mathcal{O}_{Z'}(-lh) \otimes \varphi'^* \mathscr{N}^{ln-1}) = 0$ for $1 \leq l \leq p-1$, we know that

$$\dim H^1(Z', \mathscr{F}_{l-1} \otimes \varphi'^* \mathscr{N}^{-1}) \leq \dim H^1(Z', \mathscr{F}_l \otimes \varphi'^* \mathscr{N}^{-1})$$

for $1 \leq l \leq p-1$. When $l = 1$, we have

$$0 \longrightarrow \varphi'^* \mathscr{N}^{-1} \longrightarrow \mathscr{F}_1 \otimes \varphi'^* \mathscr{N}^{-1} \longrightarrow \mathcal{O}_{Z'}(-h) \otimes \varphi'^* \mathscr{N}^{n-1} \longrightarrow 0\,.$$

Hence we have

$$\begin{aligned} 0 &\longrightarrow H^1(Z', \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^1(Z', \mathscr{F}_1 \otimes \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^1(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) \\ &\longrightarrow H^2(Z', \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^2(Z', \mathscr{F}_1 \otimes \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^2(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) \\ &\longrightarrow 0\,. \end{aligned}$$

Since $\omega_{Z'/B'} = \mathcal{O}_{Z'}(-2) \otimes \varphi'^* \mathscr{N}^p$, we know that $\mathbf{R}^1 \varphi'_*(\varphi'^* \mathscr{N}^{-1}) = \mathbf{R}^1 \varphi'_* \mathcal{O}_{Z'} \otimes \mathscr{N}^{-1} = (\varphi'_* \mathcal{O}_{Z'}(-2) \otimes \mathscr{N}^p)^\vee \otimes \mathscr{N}^{-1} = 0$ by the Serre duality. Hence we obtain that $H^1(Z', \varphi'^* \mathscr{N}^{-1}) = H^1(B', \mathscr{N}^{-1})$ and $H^2(Z', \varphi'^* \mathscr{N}^{-1}) = 0$ by using a Leray spectral sequence $H^i(B', \mathbf{R}^j \varphi'_*(\varphi'^* \mathscr{N}^{-1})) \Rightarrow H^{i+j}(Z', \varphi'^* \mathscr{N}^{-1})$. Therefore, we have

$$\begin{aligned} 0 &\longrightarrow H^1(B', \mathscr{N}^{-1}) \longrightarrow H^1(Z', \mathscr{F}_1 \otimes \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^1(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) \\ &\longrightarrow 0 \end{aligned}$$

and

$$0 \longrightarrow H^2(Z', \mathscr{F}_1 \otimes \varphi'^* \mathscr{N}^{-1}) \longrightarrow H^2(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) \longrightarrow 0\,.$$

Moreover, a Leray spectral sequence $H^i(B', \mathbf{R}^j \varphi'_*(\mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1})) \Rightarrow H^{i+j}(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1})$ implies that

$$H^2(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) = H^1(B', \mathbf{R}^1 \varphi'_* \mathcal{O}(-h) \otimes \mathscr{N}^{n-1})$$

and

$$H^1(Z', \mathcal{O}(-h) \otimes \varphi'^* \mathscr{N}^{n-1}) = H^0(B', \mathbf{R}^1 \varphi'_* \mathcal{O}(-h) \otimes \mathscr{N}^{n-1})$$

since $\varphi'_* \mathcal{O}(-h) \otimes \mathscr{N}^{n-1} = 0$.

Suppose that $h \geq 2$, whence $n \geq p+1$. Then, by the Serre duality, we have $\mathbf{R}^1 \varphi'_* \mathcal{O}(-h) \otimes \mathscr{N}^{n-1} \cong (\varphi'_* \mathcal{O}(h-2) \otimes \mathscr{N}^p)^\vee \otimes \mathscr{N}^{n-1} = (\varphi'_* \mathcal{O}(h-2))^\vee \otimes \mathscr{N}^{n-p-1}$. Note that there exists a surjection $\varphi'_* \mathcal{O}(h-2) = S^{h-2}(\mathscr{E}') \rightarrow \mathcal{O}_{B'}$. So, we have an injection $\mathscr{N}^{n-p-1} \rightarrow (\varphi'_* \mathcal{O}(h-2))^\vee \otimes \mathscr{N}^{n-p-1}$. Hence

$$\dim H^0(B', \mathbf{R}^1\varphi'_*\mathcal{O}(-h)\otimes\mathcal{N}^{n-1}) = \dim H^0(B', (\varphi'_*\mathcal{O}(h-2))^\vee\otimes\mathcal{N}^{n-p-1}) \geq \dim H^0(B', \mathcal{N}^{n-p-1}).$$

Thus we obtain that

$$\dim H^1(Z', \mathcal{F}_1\otimes\varphi'^*\mathcal{N}^{-1}) \geq \dim H^1(B', \mathcal{N}^{-1}) + \dim H^0(B', \mathcal{N}^{n-p-1}).$$

From these observations, it follows that

$$\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) \geq \dim H^0(B', \mathcal{N}^{n-p-1}).$$

Next, suppose $h = 1$, whence $2 \leq n \leq p-1$. Then, by the Serre duality, we have $\mathbf{R}^1\varphi'_*\mathcal{O}(-1)\otimes\mathcal{N}^{n-1} = (\varphi'_*\mathcal{O}(-1)\otimes\mathcal{N}^p)^\vee\otimes\mathcal{N}^{n-1} = 0$. Hence $H^1(Z', \mathcal{O}(-1)\otimes\varphi'^*\mathcal{N}^{n-1}) = 0$ and $H^2(Z', \mathcal{O}(-1)\otimes\varphi'^*\mathcal{N}^{n-1}) = 0$. These imply that $H^1(Z', \mathcal{F}_1\otimes\varphi'^*\mathcal{N}^{-1}) = H^1(B', \mathcal{N}^{-1})$ and $H^2(Z', \mathcal{F}_1\otimes\varphi'^*\mathcal{N}^{-1}) = 0$. Recall the exact sequence (2) and let $l = 2$. Then we have

$$0 \longrightarrow \mathcal{F}_1\otimes\varphi'^*\mathcal{N}^{-1} \longrightarrow \mathcal{F}_2\otimes\varphi'^*\mathcal{N}^{-1} \longrightarrow \mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1} \longrightarrow 0.$$

Taking the cohomology exact sequence, we have

$$0 \longrightarrow H^1(B', \mathcal{N}^{-1}) \longrightarrow H^1(Z', \mathcal{F}_2\otimes\varphi'^*\mathcal{N}^{-1}) \longrightarrow H^1(Z', \mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1}) \longrightarrow 0.$$

Since $\varphi'_*(\mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1}) = 0$ and $\mathbf{R}^1\varphi'_*(\mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1}) \cong (\varphi'_*\mathcal{O}_{Z'}\otimes\mathcal{N}^p)^\vee \otimes\mathcal{N}^{2n-1} = \mathcal{N}^{2n-p-1}$, we know that $H^1(Z', \mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1}) = H^0(B', \mathcal{N}^{2n-p-1})$ by considering a Leray spectral sequence $H^i(B', \mathbf{R}^j\varphi'_*\mathcal{O}(-2)\otimes\mathcal{N}^{2n-1}) \Rightarrow H^{i+j}(Z', \mathcal{O}(-2)\otimes\varphi'^*\mathcal{N}^{2n-1})$. Hence we have

$$\dim H^1(Z', \mathcal{F}_2\otimes\varphi'^*\mathcal{N}^{-1}) = \dim H^1(B', \mathcal{N}^{-1}) + \dim H^0(B', \mathcal{N}^{2n-p-1}).$$

From this, it follows that

$$\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) \geq \dim H^0(B', \mathcal{N}^{2n-p-1}).$$

By virtue of the above results, we obtain the following

THEOREM 4.2. *With the same assumptions and notations as above, $\mathcal{O}(E)\otimes\psi^*\mathcal{N}$ is ample and we have*:

(1) *If $n \geq p+1$, then*

$$\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) \geq \dim H^0(B', \mathcal{N}^{n-p-1});$$

(2) *If $2 \leq n \leq p-1$, then*

$$\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) \geq \dim H^0(B', \mathcal{N}^{2n-p-1}).$$

To close this article, we shall give two examples.

EXAMPLE 4.3. Let X be the generalized Raynaud surface over a generalized Tango curve $(B', \mathcal{N}, \eta)$ which is given in Example 3.3. Then there is a finite covering $\varpi\colon B' \to \mathbf{P}^1$ of degree np which is totally ramified over the point at infinity of $\mathbf{A}^1$. Since $\mathcal{N} = \mathcal{O}((np-3)P_\infty)$, we have $\varpi^*\mathcal{O}_{\mathbf{P}^1}(r) \subset \mathcal{N}^{n-p-1}$ provided that $(n-p-1)(np-3) \geq rnp$, where r is a positive integer. Therefore, we know that $\dim H^0(B', \mathcal{N}^{n-p-1}) \geq \dim H^0(\mathbf{P}^1, \mathcal{O}(r))$. Note that $\lim_{n\to\infty}(n-p-1)(np-3)/np = \infty$ and that $\lim_{r\to\infty}\dim H^0(\mathbf{P}^1, \mathcal{O}(r)) = \infty$. Hence, by the above theorem, we have

$$\lim_{n\to\infty}\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) = \infty\ .$$

EXAMPLE 4.4. Let $(B', \mathcal{N}, \eta)$ be the same generalized Tango curve as in Example 3.4. Then B' is a hyperelliptic curve and $\mathcal{N} = \mathcal{O}(P_\infty)$, where P_∞ is the point at infinity. Consider the generalized Raynaud surface over $(B', \mathcal{N}, \eta)$. By the same arguments as in the previous example, we have

$$\lim_{n\to\infty}\dim H^1(X, \mathcal{O}(-E)\otimes\psi^*\mathcal{N}^{-1}) = \infty\ .$$

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