S. Böcherer and R. Schulze-Pillot
Nagoya Math. J.
Vol. 121 (1991), 35–96

SIEGEL MODULAR FORMS AND THETA SERIES ATTACHED TO QUATERNION ALGEBRAS

SIEGFRIED BÖCHERER AND RAINER SCHULZE-PILLOT

Contents

Introduction

The two main problems in the theory of the theta correspondence or lifting (between automorphic forms on some adelic orthogonal group and on some adelic symplectic or metaplectic group) are the characterization of kernel and image of this correspondence. Both problems tend to be particularly difficult if the two groups are approximately the same size.

Eichler's famous solution of the basis problem for elliptic modular forms [E4] (and its representation theoretic versions by Shimizu and

Received February 22, 1990.

Jacquet/Langlands [Shz, J-L]) characterizes the image for the lifting from $O(4)$ (the orothogonal group of the norm form of a definite quaternion algebra over $\mathbb{Q}$) to $Sp(1)$, the work of Waldspurger [Wa1, Wa2, Wa3] on the Shimura correspondence characterizes kernel and image for the correspondence between $O(2, 1)$ resp. $O(3)$ (the orthogonal groups of the trace zero parts of the norm form of a split or division quaternion algebra respectively) and $\widetilde{Sp(1)}$. Rallis proved fairly general results for the lifting from $\widetilde{Sp(1)}$ to a rank 1 orthogonal group $O(1, m)$ [Ra1] and for the lifting from $Sp(n)$ to a "large" orthogonal group [Ra4].

Howe and Piatetski-Shapiro [H-PS] could prove injectivity of the lifting for the pair $(O(2, 2), Sp(2))$ (using Whittaker model techniques). They could not generalize this to the lifting from $O(4)$ to $Sp(2)$ (in this anisotropic case Whittaker models do not exist). Restricting attention to forms on $O(4)$ whose lifting to $Sp(2)$ corresponds to a holomorphic Siegel modular form of degree 2, Yoshida [Y1, Y2] treated the same problem and conjectured injectivity of the lifting. He could, however, prove his conjecture only for special D and even for these only for special forms on the orthogonal group. Part of his motivation was the conjecture that the (one dimensional part of the) zeta function of a certain abelian variety should be the spinor L-function of a Siegel modular form. For the particular varieties he considered, the lifting (if nonzero) of a suitable form on the orthogonal group would provide a Siegel modular form with the required spinor L-function (for details see [Y1, Y2]).

In the present article, we take up this problem in the same semiclassical spirit as Yoshida and prove that his lifting is almost injective. More precisely, we can show that the forms Ψ on $O(4)$ in the kernel of the lifting to $Sp(2)$ are characterized by a condition on a special value of their standard L-function. From the work of Waldspurger [Wa1, Wa2, Wa3] and Rallis [Ra1, Ra3, Ra4] it is not surprising that a condition of this type characterizes the kernel of the lifting. However, using the special situation we associate (as in [Y1, Y2]) to the form Ψ on $O(4)$ a pair of elliptic modular forms of weight 2 whose symmetric L-function is the standard L-function of Ψ. With the help of a result of Ogg [O] on this symmetric L-function we can then show that the vanishing of the lifting is indeed a quite exceptional case, thus proving the "almost injectivity" mentioned above. In particular, the lifting does not vanish in the cases needed for Yoshida's geometric conjecture. This sharp result is surprising and depends very much on the special situation considered here. That the exceptional case does indeed occur has been shown in [SP].

In classical terms our result gives a precise description of the linear dependence relations between the theta series of degree two of the integral quaternary quadratic forms attached to normal ideals in definite quaternion algebras over $\mathbb{Q}$. As a consequence we can show that the classes of these quadratic forms are distinguished by their theta series of degree 2. We now sketch the organization of this paper.

In [Bö3] the first named author gave a characterization of those Siegel modular forms of level 1 which are linear combinations of theta series attached to even unimodular positive definite quadratic forms. Part 1 can be viewed as an attempt to make the method of [Bö3] applicable to higher levels N (at least for small weights and with emphasis on the case of squarefree level and trivial character). Our basic objects are Eisenstein series of type

$$E_n^k(Z, s, N) = \sum_{\{C,D\}} \frac{\det(Y)^s}{\det(CZ+D)^k |\det(CZ+D)|^{2s}},$$

where $\{C, D\}$ runs over all non associated coprime symmetric pairs with $C \equiv 0 \bmod N$.

In section 1 we review the "pullback machinery" created mainly by P. Garrett [Ga, Böl]. The integration of a Siegel cusp form of degree n (in the sense of the Petersson scalar product) against such an Eisenstein series of degree $n+n'$ (restricted to a block-type diagonal) can be described in terms of Hecke operators and Eisenstein series of Klingen type. The Hecke operators decompose into a contribution from the "good primes" (this part is of the same type as for level 1) and a "bad part". In section 2 we show that the bad part also has a formal Euler product expansion-for the good primes this is well known [Bö2]. In section 3 we build a bridge between the theory of singular modular forms (as created mainly by Freitag [Fre2]) and the fact (due to Shimura/Feit [Shi2, Fe]) that $\mathrm{Res}_{s=((n+1)/2)-k}\, E_n^k(Z, s, N)$ is a holomorphic Siegel modular form (for "small" weights $k \leq (n+1)/2$). We describe those residues very explicitly as linear ocmbinations of theta series, in fact "all" theta series of quadratic forms of levels dividing N occur in that residue. The results obtained so far are combined in section 4. Theorem 4.1 gives a sufficient condition for an eigenform F (of sufficiently small weight) to be a linear combination of theta series. We can at present prove the necessity of this condition only under an additional restriction on F (see Remark 4.1 for more details). A second delicate point is that we cannot generally assume the existence of a basis of eigenforms of our Hecke algebra for the space of cusp forms. This seems to be due to the lack of a theory of newforms in our situation. Fortunately both difficulties play no role in the application to Yoshida's lifting which are the main goal of this article. We hope to clarify them in the general situation in future work.

In part 2, we come to the main problem of our work, the investigation of the injectivity properties of Yoshida's lifting. Section 5 reviews the ideal theory of an Eichler order R of squarefree level N in the definite quaternion algebra D over $\mathbb{Q}$. For a pair (φ, ψ) of automorphic forms on $D_{\mathbb{A}}^{\times}$ which are right invariant under the adelic group of units $R_{\mathbb{A}}^{\times}$ we define (following [Y1, Y2]) Yoshida's lifting of degree n, $Y^{(n)}(\varphi, \psi)$. This is a holomorphic modular form of degree n and weight 2 which can be written as a linear combination of the theta series of degree n of the ideals of the Eichler orders of level N in D. It can be viewed as the result of applying the theta lifting to an automorphic form $\Psi(\varphi, \psi)$ on the adelic orthogonal group $O_{\mathbb{A}}(D)$ of the norm from of D that is derived from the pair (φ, ψ) via the embedding $SO(D) \hookrightarrow (D^{\times} \times D^{\times})/\mathbb{Q}^{\times}$.

From section 6 on we assume φ and ψ to be newforms (or "essential" in the sense of [Hi-Sa]) and eigenforms for the Hecke algebra of $D^{\times}$ (whose action is represented by Brandt matrices). In sections 6 and 7 we show that $Y^{(n)}(\varphi, \psi)$ is then an eigenform of the Hecke operators arising from the pullback machinery of section 1 and compute the Satake parameters. For the good primes this is a straightforward application of results of Rallis [Ra2] and Kudla [Ku] (generalized Eichler commutation relation), for the primes ramified in D it is almost trivial, but some work is required for the remaining primes dividing the level N. In section 8 we sketch a proof of the fact that $Y^{(n)}(\varphi, \psi) = 0$ implies that $Y^{(n+1)}(\varphi, \psi)$ is cuspidal.

We are then in the position to prove our nonvanishing theorem for Yoshida's lifting (Corollary 9.1). To be more specific, we split the space $\Theta^{(n)}$ generated by the $Y^{(n)}(\varphi, \psi)$ for newforms φ, ψ on $D_{\mathbb{A}}^{\times}$ as above into a direct sum of subspaces $\Theta^{(n,j)}$ where $\Theta^{(n,j)}$ is annihilated by $n-j+1$-fold application of Siegel's Φ-operator (but not by $n-j$-fold application). By a result of Kitaoka [Ki] the theta series of degree $m-1$ of quadratic forms of rank m having the same discriminant are linearly independent. This implies that all information on the above splitting is contained in the case $m = 3$. We concentrate on that case and characterize the subspaces $\Theta^{(3,j)}$ by conditions on special values of the reduced standard L-function of elements of these spaces (Theorem 9.1). This is done by applying Theorem 4.1 and using the explicit computation of the contribution from the bad primes performed in section 7. By the computations of section 6 the reduced standard L-function of $Y^{(n)}(\varphi, \psi)$ is related to the symmetric L-function associated to φ, ψ. Since φ and ψ correspond to elliptic modular forms of weight 2 with the same Hecke eigenvalues (by the results of Eichler [E4], Shimizu [Shz], Jacquet/Langlands [J-L]) we can apply a theorem of Ogg [O] on the value at $s = 2$ of the symmetric L-function of a pair of elliptic modular forms. The results obtained on the analytic properties of the reduced standard L-function of $Y^{(3)}(\varphi, \psi)$ suffice to determine the subspace $\Theta^{(3,j)}$ in which $Y^{(3)}(\varphi, \psi)$ lies by a comparison with the analytic characterization of these subspaces obtained in Theorem 9.1. Our nonvanishing result is then an easy consequence. The final section 10 collects relations between the Petersson norms of the various forms on orthogonal and symplectic groups that appeared so far. These are interesting in their own right [Ra1] and allow to express the Eisenstein series of Klingen type of degree n attached to a cuspidal $Y^{(n_0)}(\varphi, \psi)$ $(n_0 < n)$ by the Yoshida lifting of degree n. This last result will be needed in a forthcoming paper where we give a new proof of Waldspurger's formula relating special values of twisted L-functions to Fourier coefficients of modular forms of half integral weight.

Preliminaries

For generalities on Siegel modular forms we refer to [Fre1]. For $\mathrm{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we denote by $(\mathrm{M}, Z) \mapsto \mathrm{M}\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ the usual action of the group $G^+Sp(n, \mathbb{R})$ of proper symplectic similitudes on Siegel's upper half space $\mathbb{H}_n$.

For any function $f\colon \mathbb{H}_n \to \mathbb{C}$, any $\mathrm{M} \in G^+Sp(n, \mathbb{R})$ and any "weight" k we write

$$(f|_k \mathrm{M})(Z) = (\det \mathrm{M})^{k/2} j(\mathrm{M}, Z)^{-k} f(\mathrm{M}\langle Z\rangle)$$

with $j(\mathrm{M}, Z) = \det(CZ+D)$.

We shall mainly be concerned with Siegel modular forms for congruence subgroups of type

$$\Gamma_0^{(n)}(N)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{Z})\mid C\equiv 0 \bmod N\right\}.$$

The space of Siegel modular forms (and cusp forms respectively) of degree n and weight k for $\Gamma_0^{(n)}(N)$ will be denoted by $M_n^k(N)(S_n^k(N))$; by $\langle\, ,\rangle$ we denote the Petersson scalar product. Φ denotes Siegel's operator $\Phi\colon M_n^k(N)\to M_{n-1}^k(N)$.

Our normalization of Hecke operators is as follows:

For $f\in M_n^k(N)$ and $\mathrm{M}\in GSp(n,\mathbb{Q})$ with $\Gamma_0^{(n)}(N)\mathrm{M}\Gamma_0^{(n)}(N)=\bigcup_i \Gamma_0^{(n)}(N)\mathrm{M}_i$ we put $f|_k\Gamma_0^{(n)}(N)\mathrm{M}\Gamma_0^{(n)}(N)=\sum_i f|_k\mathrm{M}_i$ We shall freely use the adelic interpretation of Siegel modular forms and Hecke operators (see e.g. [Y3]. In particular, let

$$K_p(N)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in GSp(n,\mathbb{Z}_p)\mid C\equiv 0 \bmod N\mathbb{Z}_p\right\},$$

denote for $\mathrm{M}\in G^+Sp(n,\mathbb{R})$ by $(\mathrm{M},1,\cdots)$ the adele with ∞-component M and all other components 1_{2n}. Then to the Siegel modular form $f\in M_n^k(N)$ there corresponds a unique automorphic form Ψ_f on $GSp(n,\mathbb{A})$ with trivial central character, right invariant under $K(N)=\prod_p K_p(N)$ and satisfying

$$f(\mathrm{M}\langle i1_n\rangle)j(\mathrm{M},i1_n)^{-k}=\Psi_f(\mathrm{M},1,\cdots)$$

for all $\mathrm{M}\in G^+Sp(n,\mathbb{R})$.

By $M_n(\mathbb{Z})^*$ we denote the set of nonsingular integral $n\times n$ matrices, $\mathbb{Z}_{\mathrm{sym}}^{(n,n)}$ denotes the set of symmetric integral $n\times n$ matrices. For an integer a, $a\mid N^\infty$ means that $p\mid a$ only if $p\mid N$.

Part I. Eisenstein Series and Theta Series

§1. Pullbacks of Eisenstein series

In the case of the full modular group it is well known how Siegel's Eisenstein series behaves when restricted to the diagonal–see [Böl], [Ga]. In this section we describe how these results generalize to groups of type $\Gamma_0^{(n)}(N)$.

Let m, n be natural numbers with $m\geq n$. The "small" symplectic groups $Sp(n)$ and $Sp(m)$ can be embedded into $Sp(n+m)$ by means of

$$M^{\uparrow} := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_m & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_m \end{pmatrix}, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n)\,,$$

$$M^{\downarrow} := \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_n & 0 \\ 0 & C & 0 & D \end{pmatrix}, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(m)\,.$$

We have to understand the double coset decomposition of

$$\Gamma_0^{(n+m)}(N)_\infty \backslash \Gamma_0^{(n+m)}(N) / \Gamma_0^{(n)}(N)^{\uparrow} \cdot \Gamma_0^{(m)}(N)^{\downarrow})\,, \tag{1.1}$$

where, as usual, for any subgroup G of $Sp(n)$, we denote by G_∞ the subgroup of all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in G with $C = 0$.

THEOREM 1.1. *A complete set of representatives for the double cosets in* (1.1) *is given by*

$$\bigcup_{i=0}^{n} \mathscr{M}_i$$

with

$$\mathscr{M}_i = \left\{ g_{\widetilde{\mathrm{M}}} := \begin{bmatrix} & 1_{n+m} & & 0 \\ 0_n & & \widetilde{\mathrm{M}} & \\ & & & 1_{n+m} \\ \widetilde{\mathrm{M}} & & 0_m & \end{bmatrix} \;\middle|\; \begin{array}{l} \widetilde{\mathrm{M}} = \begin{pmatrix} \mathrm{M} & 0 \\ 0 & 0 \end{pmatrix} \\ \mathrm{M} = \operatorname{diag}(m_1, \cdots, m_i) \\ \mathrm{M} \equiv 0 \bmod N \\ 1 \le m_1 | m_2 \cdots | m_i \end{array} \right\}.$$

In the sequel we shall call such a matrix M an elementary divisor matrix of size i.

THEOREM 1.2. *For $g_{\widetilde{\mathrm{M}}} \in \mathscr{M}_n$, a complete set of representatives of the $\Gamma_0^{(n+m)}(N)_\infty$-left cosets in $\Gamma_0^{(n+m)}(N)_\infty g_{\widetilde{\mathrm{M}}} \Gamma_0^{(n)}(N)^{\uparrow} \times \Gamma_0^{(m)}(N)^{\downarrow}$ is given by*

$$\{g_{\widetilde{\mathrm{M}}} \cdot g^{\uparrow} l(h)^{\downarrow} \hat{g}^{\downarrow} \mid g \in \Gamma_0^{(n)}(N),\ \hat{g} \in C_{m,n}(N) \backslash \Gamma_0^{(m)}(N),\ h \in \Gamma_0^{(n)}(\mathrm{M}) \backslash \Gamma_0^{(n)}(N)\}\,.$$

Here $C_{m,n}(N)$ denotes the parabolic subgroup

$$C_{m,n}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(m)}(N) \,|\, (C, D) = \begin{pmatrix} * & * \\ 0^{(m-n,m+n)} & * \end{pmatrix} \right\}$$

and $\Gamma_0^{(n)}(\mathrm{M})$ is the (*congruence*) *subgroup of $\Gamma_0^{(n)}(N)$ given by*

$$\Gamma_0^{(n)}(\mathrm{M}) := \Gamma_0^{(n)}(N) \cap \begin{pmatrix} 0 & \mathrm{M}^{-1} \\ -M & 0 \end{pmatrix} \Gamma_0^{(n)}(N) \begin{pmatrix} 0 & -\mathrm{M}^{-1} \\ \mathrm{M} & 0 \end{pmatrix}.$$

Furthermore $l = l_{m,n}$ *is the embedding* $Sp(n) \hookrightarrow Sp(m)$ *given by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_{m-n} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_{m-n} \end{pmatrix}.$$

Proofs of these theorems are in [Ga] for $N = 1$. The results for arbitrary N can be proved in essentially the same way or may be deduced from the corresponding statement for $N = 1$ (we omit details).

For $\mathscr{Z} \in \mathbb{H}_{m+n}$ and even integral weight k we consider the Eisenstein series

$$\begin{aligned} E^k_{n+m}(\mathscr{Z}, s, N) &= \sum_{M \in \Gamma^{(n+m)}_\infty \backslash \Gamma^{(n+m)}_0(N)} j(M, \mathscr{Z})^{-k} \det(\operatorname{Im}(M\langle \mathscr{Z} \rangle))^s \\ &= \sum_{M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} \det(C\mathscr{Z} + D)^{-k} |\det(C\mathscr{Z} + D)|^{-2s} \det(\mathscr{Y})^s . \end{aligned}$$

This series converges for $2\operatorname{Re}(s) + k > n + m + 1$ and has a meromorphic continuation to the whole complex plane.

Now we restrict the argument to a "diagonal" of type $\mathscr{Z} = \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}$ with $W \in \mathbb{H}_n$, $Z \in \mathbb{H}_m$.

According to Theorem 1 and Theorem 2 we may split our Eisenstein series into subseries:

$$E^k_{m+n}\left(\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}, s, N\right) = \sum_{i=0}^{n} \omega_i(W, Z, s, N)$$

and

$$\omega_n(W, Z, s, N) = \sum_{\mathrm{M}} \omega_{n,\mathrm{M}}(W, Z, s, N) \tag{1.2}$$

summing over all elementary divisor matrices M of size n with $\mathrm{M} \equiv 0 \bmod N$.

Each of the $\omega_i(\cdots)$ and $\omega_{n,\mathrm{M}}(\cdots)$ behaves like a modular form of weight k for $\Gamma^n_0(N)$ with respect to the variable W; moreover, these functions are of "slow growth", therefore we may look at their Petersson scalar product against cusp forms.

To describe these scalar products precisely we need certain Hecke operators $T_N(\mathrm{M})$, given by the double cosets

$$\Gamma^{(n)}_0(N) \begin{pmatrix} 0 & -\mathrm{M}^{-1} \\ \mathrm{M} & 0 \end{pmatrix} \Gamma^{(n)}_0(N)$$

where again M is an elementary divisor matrix of size n with $M \equiv 0 \bmod N$.

We let $T_N(\mathrm{M})$ act on modular forms from $M_n^k(N)$ by

$$F \mid T_N(\mathrm{M}) = \sum_j F|_k g_j$$

where

$$\Gamma_0^{(n)}(N)\begin{pmatrix} 0 & -\mathrm{M}^{-1} \\ \mathrm{M} & 0 \end{pmatrix}\Gamma_0^{(n)}(N) = \bigcup_j \Gamma_0^{(n)}(N) g_j$$

is a decomposition of the double coset into disjoint left cosets.

Theorem 1.3. *For $F \in S_n^k(N)$ and any* M *like in* (1.2) *we have* (*with* $k + 2\,\mathrm{Re}\,(s) > n + m + 1$, $m \geq n$):

a) $\langle F, \omega_i(*, -\bar{Z}, \bar{s}, N)\rangle = 0$ *for* $i < n$.

b) $\langle F, \omega_{n,\mathrm{M}}(*, -\bar{Z}, \bar{s}, N)\rangle = \mu(n, k, s) \det(\mathrm{M})^{-k-2s} E_{m,n}^k(F \mid T_N(\mathrm{M}), s, N)(Z)$.

Here $E_{m,n}^k(F, s, N)$ is an Eisenstein series of Klingen type [Kli], *defined by*

$$E_{m,n}^k(F, s, N)(Z) = \sum_{M \in C_{m,n}(N)\backslash\Gamma_0^{(m)}(N)} j(M, Z)^{-k} F(M\langle Z\rangle^*)\left(\frac{\det(\mathrm{Im}\,(M\langle Z\rangle)}{\det(\mathrm{Im}\,(M\langle Z\rangle)^*)}\right)^s$$

and Z^* denotes the n-rowed submatrix of Z in the upper left corner. The factor $\mu(n, k, s)$ is equal to

$$\mu(n, k, s) = 2^{(n^2+3n)/2-2ns-nk+1}(-1)^{nk/2}\pi^{n(n+1)/2}\frac{\Gamma_n\left(k + s - \dfrac{n+1}{2}\right)}{\Gamma_n(k+s)}$$

with

$$\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{\nu=0}^{n-1}\Gamma\left(s - \frac{\nu}{2}\right).$$

For $N = 1$ one can find a proof in [Böl], Satz 1. Therefore we only give a sketch of proof for b).

According to Theorem 1.2 a typical summand of $\omega_{n,\mathrm{M}}$ is of the form

$$(1.3)\qquad j\left(g_{\tilde{\mathrm{M}}} g^{\uparrow} l(h)^{\downarrow} \hat{g}^{\downarrow}, \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}\right)^{-k} \det\left(\mathrm{Im}\, g_{\tilde{\mathrm{M}}} g^{\uparrow} l(h)^{\downarrow} \hat{g}^{\downarrow}\left\langle\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}\right\rangle\right)^s.$$

Using the (elementary) formula

$$j\left(g_{\tilde{\mathrm{M}}}, \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}\right) = \det\nolimits_{m+n}\begin{pmatrix} 1_n & \widetilde{\mathrm{M}}Z \\ \widetilde{M}W & 1_m \end{pmatrix}$$
$$= \det\nolimits_n(1_n - \mathrm{M}Z^*\mathrm{M}W),$$

one can write (1.3) in the form

$$H_M(g\langle W\rangle, (l(h)\hat{g})\langle Z\rangle^*, s)j(g, W)^{-k}j(l(h)\hat{g}, Z)^{-k}\det \mathrm{Im}\,((l(h)\hat{g})\langle Z\rangle)^s$$

with

$$\begin{aligned} H_M(W, Z^*, s) &= \det(1_n - MZ^*MW)^{-k} \\ &\quad\times |\det(1_n - MZ^*MW)|^{-2s}\det(\mathrm{Im}\,(W))^s . \end{aligned}$$

The standard unfolding argument leads to

$$\begin{aligned} (1.4)\qquad &\langle F, \omega_{n,M}(*, -\bar{Z}, \bar{s}, N)\rangle \\ &= \sum_{\hat{g},h}\int_{H_n} F(W)\overline{H_M(W, l(h)\hat{g}\langle -\bar{Z}\rangle^*, \bar{s})}\det(V)^{k-n-1}dUdV \\ &\quad\times \overline{j(l(h)\hat{g}, -\bar{Z})^{-k}}\det(\mathrm{Im}\,(l(h)\hat{g}\langle -\bar{Z}\rangle))^s \end{aligned}$$

with $W = U + i\mathrm{V}$.

Now we use an integral formula (attributed to Selberg):

$$\begin{aligned} (1.5)\qquad &\int_{H_n} F(W)\overline{H_M(W, -\bar{Z}^*, \bar{s})}\det(V)^{k-n-1}dUdV \\ &= \mu(n, k, s)F(-(MZ^*M)^{-1})\det(MZ^*M)^{-k}\det\mathrm{Im}\,(MZ^*M)^{-s} \\ &= \mu(n, k, s)F|_k\begin{pmatrix} 0 & -M^{-1} \\ M & 0 \end{pmatrix}(Z^*)\det(M)^{-k-2s}\det(\mathrm{Im}\,(Z^*))^{-s} \end{aligned}$$

(compare [Böl], 2.2).

To combine (1.4) and (1.5) we need the automorphism $\sharp$ of $Sp(n, \mathbb{R})$ given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\sharp = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$, which satisfies

$$M\langle -\bar{Z}\rangle = -\overline{M^\sharp\langle Z\rangle} .$$

We obtain

$$\begin{aligned} (1.6)\qquad &\langle F, \omega_{n,M}(*, -\bar{Z}, \bar{s}, N)\rangle \\ &= \mu(n, k, s)\det(M)^{-k-2s}\sum_{\hat{g},h} F|_k\begin{pmatrix} 0 & -M^{-1} \\ M & 0 \end{pmatrix}(l(h)^\sharp\hat{g}^\sharp\langle Z\rangle)^* \\ &\quad\times j(l(h)^\sharp\hat{g}^\sharp, Z)^{-k}\left(\frac{\det(\mathrm{Im}\,(l(h)^\sharp\circ\hat{g}^\sharp\langle Z\rangle))}{\det(\mathrm{Im}\,(l(h)^\sharp\hat{g}^\sharp\langle Z\rangle)^*)}\right)^s, \end{aligned}$$

Now we may omit $\sharp$, since $\sharp$ stabilizes all the relevant groups. Moreover we have

$$\begin{aligned} j(l(h)\hat{g}, Z) &= j(h, \hat{g}\langle Z\rangle^*)j(\hat{g}, Z) \\ (l(h)\circ\hat{g})\langle Z\rangle^* &= h\langle\hat{g}\langle Z\rangle^*\rangle \end{aligned}$$

which means that (1.6) is equal to

$$\mu(n,k,s)\det(\mathrm{M})^{-k-2s}\sum_{\hat g}\Bigl(\sum_h F\Bigl|_k\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M} & 0\end{pmatrix}\Bigr|_k h\Bigr)(\hat g\langle Z\rangle^*)$$
$$\times j(\hat g, Z)^{-k}\Bigl(\frac{\det(\mathrm{Im}(\hat g\langle Z\rangle))}{\det(\mathrm{Im}(\hat g\langle Z\rangle^*))}\Bigr)^s .$$

Now we are done, observing that

$$\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M} & 0\end{pmatrix}\cdot h, \qquad h\in \Gamma_0^{(n)}(\mathrm{M})\backslash\Gamma_0^{(n)}(N)$$

runs over a complete set of representatives of the left cosets in the double coset

$$\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M} & 0\end{pmatrix}\Gamma_0^{(n)}(N). \qquad \square$$

§ 2. An Euler product

All the Hecke operators $T_N(\mathrm{M})$ of § 1 are hermitian with respect to the Petersson inner product on $S_n^k(N)$, since

$$\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M}' & 0\end{pmatrix}\Gamma_0^{(n)}(N)=\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M}' & 0\end{pmatrix}^{-1}\Gamma_0^{(n)}(N)$$

but unfortunately they do *not* commute in general unless $N=1$. (For $n=1$ they do commute as operators acting on newforms, but if N is not squarefree, the $T_N(\mathrm{M})$ act as zero operators on $S_1^k(N)^{\mathrm{new}}$, see [Li], Theorem 3).

In this section we investigate the algebraic properties of the $T_N(\mathrm{M})$ by considering them as elements of the big abstract Hecke algebra $\mathscr{H}(\Gamma_0^{(n)}(N), G^+Sp(n,\mathbb{Q}))$ over $\mathbb{C}$ associated to the Hecke pair $(\Gamma_0^{(n)}(N), G^+Sp(n,\mathbb{Q}))$. For generalities on Hecke algebras we refer to the books [An1], [Fre1]. The case $N=1$ was investigated in [Bö2].

Each double coset $T_N(\mathrm{M})$ factors into an "ordinary part" and an N-component:

Lemma 2.1. *Let* $\mathrm{M}\in M_n(\mathbb{Z})^*$ *with* $\mathrm{M}\equiv 0 \bmod N$ *be factorized in* $M_n(\mathbb{Z})^*$ *as* $\mathrm{M}=\mathrm{M}_0\cdot\mathrm{M}_1$ *with* $\mathrm{M}_0\equiv 0\bmod N$ *and* $(\det(\mathrm{M}_1), N)=1$. *Then*

$$\begin{aligned}&\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}^{-1}\\ \mathrm{M}' & 0\end{pmatrix}\Gamma_0^{(n)}(N)\\ &=\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}_0^{-1}\\ \mathrm{M}_0' & 0\end{pmatrix}\Gamma_0^{(n)}(N)\cdot\Gamma_0^{(n)}(N)\begin{pmatrix}\mathrm{M}_1' & 0\\ 0 & \mathrm{M}_1^{-1}\end{pmatrix}\Gamma_0^{(n)}(N)\\ &=\Gamma_0^{(n)}(N)\begin{pmatrix}\mathrm{M}_1^{-1} & 0\\ 0 & \mathrm{M}_1'\end{pmatrix}\Gamma_0^{(n)}(N)\cdot\Gamma_0^{(n)}(N)\begin{pmatrix}0 & -\mathrm{M}_0^{-1}\\ \mathrm{M}_0' & 0\end{pmatrix}\Gamma_0^{(n)}(N).\end{aligned}$$

Moreover the double cosets

$$\Gamma_0^{(n)}(N)\begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_1' \end{pmatrix}\Gamma_0^{(n)}(N) \quad \textit{and} \quad \Gamma_0^{(n)}(N)\begin{pmatrix} M_1' & 0 \\ 0 & M_1^{-1} \end{pmatrix}\Gamma_0^{(n)}(N)$$

are equal.

The proof is straightforward, so we omit it.

The ordinary part of $T_N(M)$ is an element of the Hecke algebra $\mathscr{H}(\Gamma_0^{(n)}(N), Sp(n, \mathbb{Q}_{(N)}))$ with $\mathbb{Q}_{(N)} = \{a/b \in \mathbb{Q} \mid (b, N) = 1\}$.

Its algebraic structure is well known:

$$\begin{aligned}\mathscr{H}(\Gamma_0^{(n)}(N), Sp(n, \mathbb{Q}_{(N)})) &\simeq \mathscr{H}(Sp(n, \mathbb{Z}), Sp(n, \mathbb{Q}_{(N)})) \\ &\simeq \bigotimes_{p \nmid N} \mathscr{H}\left(Sp(n, \mathbb{Z}), Sp\left(n, \mathbb{Z}\left[\frac{1}{p}\right]\right)\right).\end{aligned}$$

Here we should restrict to $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Q}_{(N)})$ with $C \in NM_n(\mathbb{Q}_{(N)})$. (The tensor product being restricted in the usual sense). Moreover we have (by means of the Satake-isomorphism)

$$\mathscr{H}\left(Sp(n, \mathbb{Z}), Sp\left(n, \mathbb{Z}\left[\frac{1}{p}\right]\right)\right) \simeq \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{W_n}$$

where W_n is a certain finite group (Weylgroup).

The N-part of $T_N(M)$ can be further decomposed according to

LEMMA 2.2. *For* $M = N \cdot \overline{M}$, $\overline{M} \in M_n(\mathbb{Z})^*$ *with* $\det(\overline{M}) \mid N^\infty$ *we have*

$$T_N(M) = \Gamma_0^{(n)}(N)\begin{pmatrix} \overline{M}^{-1} & 0 \\ 0 & \overline{M}' \end{pmatrix}\Gamma_0^{(n)}(N) \cdot T_N(N \cdot 1_n).$$

This statement follows directly from the decomposition of the double cosets involved into left cosets:

(a) $$\begin{aligned} T_N(N \cdot 1_n) &= \Gamma_0^{(n)}(N)\begin{pmatrix} 0 & -N^{-1} \cdot 1_n \\ N \cdot 1_n & 0 \end{pmatrix}\Gamma_0^{(n)}(N) \\ &= \bigcup_{A \in \mathbf{Z}_{\mathrm{sym}}^{n,n} \bmod N} \Gamma_0^{(n)}(N)\begin{pmatrix} A & -N^{-1} \cdot 1_n \\ N \cdot 1_n & 0 \end{pmatrix}.\end{aligned}$$

(b) $$\begin{aligned} T_N(M) &= \Gamma_0^{(n)}(N)\begin{pmatrix} 0 & -M^{-1} \\ M' & 0 \end{pmatrix}\Gamma_0^{(n)}(N) \\ &= \bigcup_{w, A} \Gamma_0^{(n)}(N)\begin{pmatrix} w^{-1}A & -w^{-1} \\ w' & 0 \end{pmatrix}.\end{aligned}$$

Here A runs through

$$\{\mathbf{A} \in \mathbb{Z}_{\mathrm{sym}}^{(n,n)} \mid \mathbf{A} \equiv 0 \bmod N\} \text{ modulo } w\mathbb{Z}_{\mathrm{sym}}^{(n,n)}w'$$

and w' runs through the left cosets $GL(n, \mathbb{Z})w'$ contained in $GL(n, \mathbb{Z})\mathrm{M}'GL(n, \mathbb{Z})$.

$$\text{(c)} \quad \Gamma_0^{(n)}(N)\begin{pmatrix} \overline{\mathrm{M}}^{-1} & 0 \\ 0 & \overline{\mathrm{M}}' \end{pmatrix}\Gamma_0^{(n)}(N) = \bigcup_{w,\mathbf{A}} \Gamma_0^{(n)}(N)\cdot\begin{pmatrix} w^{-1} & w^{-1}\mathbf{A} \\ 0 & w' \end{pmatrix}.$$

Here w' runs through the left cosets $GL(n, \mathbb{Z})w'$ contained in $GL(n, \mathbb{Z})\overline{\mathrm{M}}'GL(n, \mathbb{Z})$ and $\mathbf{A}$ runs through $\mathbb{Z}_{\mathrm{sym}}^{(n,n)}$ modulo $w\mathbb{Z}_{\mathrm{sym}}^{(n,n)}w'$. Again we omit the elementary proofs.

COROLLARY 2.1. *The mapping*

$$GL(n, \mathbb{Z})\overline{\mathrm{M}}GL(n, \mathbb{Z}) \longmapsto \Gamma_0^{(n)}(N)\begin{pmatrix} \overline{\mathrm{M}}^{-1} & 0 \\ 0 & \overline{\mathrm{M}}' \end{pmatrix}\Gamma_0^{(n)}(N)$$

induces an isomorphism φ between the Hecke algebra

$$\mathscr{H}\left(GL(n, \mathbb{Z}),\ GL\left(n, \mathbb{Z}\left[\frac{1}{q_1}, \cdots, \frac{1}{q_l}\right]\right) \cap M_n(\mathbb{Z})\right)$$

and the Hecke algebra generated by the

$$\Gamma_0^{(n)}(N)\begin{pmatrix} \overline{\mathrm{M}}^{-1} & 0 \\ 0 & \overline{\mathrm{M}}' \end{pmatrix}\Gamma_0^{(n)}(N) \qquad \textit{with } \overline{\mathrm{M}} \textit{ integral, } \det\overline{\mathrm{M}} \mid N^\infty ;$$

here $q_1, \cdots, q_l$ are the primes dividing N.

It is known that

$$\begin{aligned}
&\mathscr{H}\left(GL(n, \mathbb{Z}),\ GL\left(n, \mathbb{Z}\left[\frac{1}{q_1}, \cdots, \frac{1}{q_l}\right]\right) \cap M_n(\mathbb{Z})\right) \\
&\qquad \simeq \bigotimes_{q \mid N} \mathscr{H}\left(GL(n, \mathbb{Z}),\ GL\left(n, \mathbb{Z}\left[\frac{1}{q}\right]\right) \cap M_n(\mathbb{Z})\right)
\end{aligned}$$

and (again via Satake-isomorphism):

$$\mathscr{H}\left(GL(n, \mathbb{Z}),\ GL\left(n, \mathbb{Z}\left[\frac{1}{q}\right]\right) \cap \mathrm{M}_n(\mathbb{Z})\right) \simeq \mathbb{C}[X_1, \cdots, X_n]^{S_n}.$$

Suppose now that $F \in M_n^k(N)$ is an eigenfunction of all the $T_N(\mathrm{M})$ with $\mathrm{M} \equiv 0 \bmod N$, $\mathrm{M} \in M_n(\mathbb{Z})^*$:

$$F \mid T_N(\mathrm{M}) = \lambda_F(\mathrm{M})\cdot F.$$

In the sequel we need an important additional assumption, namely

$$\lambda_F(N\cdot 1_n) \neq 0 .$$

Then Lemma 2.1 implies that F is also an eigenform for all the double cosets corresponding to $M_1 \in M_n(\mathbb{Z})^*$, $(\det M_1, N) = 1$:

$$F|\Gamma_0^{(n)}(N)\begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_1' \end{pmatrix}\Gamma_0^{(n)}(N) = \lambda_F^{(1)}(M_1)\cdot F ,$$

and for M as in Lemma 2.1 we get

$$\lambda_F(M) = \lambda_F(M_0)\cdot\lambda_F^{(1)}(M_1) .$$

In [Bö2] it was proven that the Dirichlet series

$$\sum_{M_1} \lambda_F^{(1)}(M_1) \det (M_1)^{-s}$$

(where M_1 runs over all integral elementary divisor matrices with $(\det (M_1), N) = 1$) is equal to the Euler product

$$\frac{1}{\zeta^{(N)}(s)\prod_{i=1}^{n}\zeta^{(N)}(2s-2i)} D_F^{(N)}(s-n)$$

where $D_F^{(N)}(s)$ is the "standard L-function" attached to F:

$$D_F^{(N)}(s) = \prod_{p\nmid N}\left(\frac{1}{(1-p^{-s})}\prod_{i=1}^{n}\frac{1}{(1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s})}\right);$$

here the $\alpha_{i,p}$ are the "Satake-parameters" of F—we use the same normalisation of the Satake isomorphism as in [Bö1], [Fre1].

We want to prove a similar result for the Dirichlet series formed by the $\lambda_F(M_0)$. To do this we observe that—using the Corollary—a 1-dimensional representation $\tilde{\lambda}$ of the Hecke algebra

$$\mathscr{H}\left(GL(n, \mathbb{Z}), GL\left(n, \mathbb{Z}\left[\frac{1}{q_1}, \cdots, \frac{1}{q_l}\right]\right) \cap M_n(\mathbb{Z})\right)$$

is given by

$$\begin{aligned} (2.1)\qquad & GL(n, \mathbb{Z})M_0GL(n, \mathbb{Z}) \\ & \longmapsto F\,|\,T_N(N\cdot 1_n)^{-1}\left|\Gamma_0^{(n)}(N)\begin{pmatrix} M_0^{-1} & 0 \\ 0 & M_0' \end{pmatrix}\Gamma_0^{(n)}(N)\right|T_N(N\cdot 1_n) \\ & = \frac{1}{\lambda_F(N\cdot 1_n)}\lambda_F(NM_0)\cdot F = \tilde{\lambda}(M_0)\cdot F \end{aligned}$$

(In (2.1) $T_N(N\cdot 1_n)^{-1}$ makes sense when considered as endomorphism of $\mathbb{C}\cdot F$).

Now we can use a result of Tamagawa [Tam] which implies that

$$\text{(2.2)} \qquad \sum_{\mathrm{M}_0} \tilde{\lambda}(\mathrm{M}_0) \det(\mathrm{M}_0)^{-s} = \prod_{q|N} \frac{1}{\sum_{i=0}^{n} (-1)^i q^{(i(i-1))/2} \tilde{\lambda}(\prod_i) q^{-is}}$$

$$= \prod_{q|N} \frac{1}{\prod_{i=1}^{n} (1 - \beta_{i,q} q^{-s+n})}$$

where $\prod_i$ denotes the $GL(n, \mathbf{Z})$-double coset given by

$$\begin{pmatrix} 1_{n-i} & 0 \\ 0 & q \cdot 1_i \end{pmatrix}$$

and the $\beta_{i,q}$ are the Satake-parameters of the representation $\tilde{\lambda}$ (normalized such that $\beta_{i,q} = q^{-i}$ if $\tilde{\lambda}$ is the trivial homomorphism counting the number of left cosets in a double coset). (From the point of view of GL_n-Hecke algebras the "Hecke series"

$$\sum_{\mathrm{M}_0} \Gamma_0^{(n)}(N) \begin{pmatrix} \mathrm{M}_0^{-1} & 0 \\ 0 & \mathrm{M}_0' \end{pmatrix} \Gamma_0^{(n)}(N) \det(\mathrm{M}_0)^{-s}$$

looks more natural than the one considered here. However, in the applications we have in mind (see § 1) we must consider representations $\tilde{\lambda}$ like in (2.1).) Summing up, we get

THEOREM 2.1. *Suppose that* $F \in M_n^k(N)$ *is an eigenfunction of all the operators* $T_N(\mathrm{M})$, $\mathrm{M} \in M_n(\mathbf{Z})^*$, $\mathrm{M} \equiv 0 \bmod N$ *and assume that the eigenvalue* $\lambda_F(N \cdot 1_n)$ *is different from zero. Then we have for* $\mathrm{Re}(s) \gg 0$ *an Euler product expansion*

$$\sum_{\mathrm{M}} \lambda_F(\mathrm{M}) \det(\mathrm{M})^{-s} = \frac{\lambda_F(N \cdot 1_n)}{N^{n \cdot s}} \frac{1}{\zeta^{(N)}(s) \prod_{i=1}^{n} \zeta^{(N)}(2s - 2i)}$$

$$\times \Lambda_N(s-n) \cdot D_F^{(N)}(s-n)$$

where

$$\Lambda_N(s) = \prod_{q|N} \frac{1}{\prod_{i=1}^{n} (1 - \beta_{i,q} q^{-s})}$$

and the $\beta_{i,q}$ *are the Satake-parameters of the representation* $\tilde{\lambda}$ *given by* (2.1).

Finally we mention a very special type of eigenvalues:

Remark. Suppose $F \in M_n^k(N)$ is an eigenform of all the $T_N(\mathrm{M})$, $\mathrm{M} \equiv 0 \bmod N$, $\det \mathrm{M} \mid N^\infty$ and assume that the constant term a_0 of the Fourier expansion of F is different from zero. Then

(a) $\lambda_F(N \cdot 1_n) = \dfrac{a_0}{b_0} N^{n(n+1)/2 - nk/2}$

and for M as above one has

(b) $\lambda_F(\mathrm{M}) = \lambda_F(N \cdot 1_n) \cdot \det\left(\dfrac{1}{N}\mathrm{M}\right)^{-k+n+1}$

$$\times \text{ number of left costes in } GL(n, \mathbb{Z})\left(\frac{1}{N} \cdot \mathrm{M}\right) GL(n, \mathbb{Z}).$$

Here $b_0 \neq 0$ is the constant term in the Fourier expansion of $F|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Moreover one has

$$\sum \frac{\lambda_F(\mathrm{M})}{\det(\mathrm{M})^s} = \frac{\lambda_F(N \cdot 1_N)}{N^{ns}} \Lambda_N^{\mathrm{triv}}(s - n)$$

with

$$\Lambda_N^{\mathrm{triv}}(s) = \prod_{q \mid N} \prod_{j=1}^{n} \frac{1}{1 - q^{-s-k+j}}$$

The proof is elementary.

Finally we define $M_n^k(N)^{\mathrm{triv}}$ and $S_n^k(N)^{\mathrm{triv}}$ to be the eigenspaces of the $T_N(\mathrm{M})$, $\mathrm{M} \equiv 0 \bmod N$, $\det \mathrm{M} \mid N^\infty$ with eigenvalues given by (a) and (b).

§ 3. Eisenstein series

In this section k is again an even positive number and $N > 1$. We collect here the properties of the Eisenstein series $E_n^k(Z, s, N)$ needed later on. It is quite useful to consider at the same time another Eisenstein series of similar type:

$$F_n^k(Z, s, N) = \sum_{\{C, D\}} \frac{\det(Y)^s}{\det(CZ + D)^k |\det(CZ + D)|^{2s}}, \tag{3.1}$$

where $\{C, D\}$ runs over all non associated coprime symmetric pairs with $(\det(C), N) = 1$. In particular, the Fourier expansion of $F_n^k(Z, s, N)$ is somewhat easier to handle, since in (3.1) the rank of C is always maximal (thanks to $N > 1$). Both Eisenstein series are closely related to each other by means of the Fricke involution

$$E_n^k(-, s, N)|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = N^{(-(k/2)-s)n} \cdot F_n^k(-, s, N). \tag{3.2}$$

For generalities on such Eisenstein series we refer to the very explicit results of Shimura [Shi1], [Shi2] and Feit [Fe]. For us the behavior of Eisenstein series at $s=(n+1)/2-k$ is of special interest (for "small" weights, so we are outside the range of convergence):

THEOREM 3.1. *For $0\leq k<(n+1)/2$ we have*

(a) *The Eisenstein series $E_n^k(-,s,N)$ and $F_n^k(-,s,N)$ have poles of (at most) first order in $s_{n,k}:=(n+1)/2-k$; the corresponding residues $\mathscr{E}_n^k(-,N)$ and $\mathscr{F}_n^k(-,N)$ are holomorphic modular forms.*

(b) *The Fourier coefficients of $\mathscr{E}_n^k(-,N)$ and $\mathscr{F}_n^k(-,N)$ do not depend on the exponent matrices T themselves but only on their genera.*

(c) *The modular forms $\mathscr{E}_n^k(-,N)$ and $\mathscr{F}_n^k(-,N)$ do not vanish identically.*

(d) *Assume that $2k=n$.*

Then for all $l\geq 1$ there is a constant $d_{n,l}(N)\neq 0$ such that

$$\Phi\mathscr{F}^k_{n+l}(-,N)=d_{n,l}(N)\cdot\mathscr{F}^k_{n+l-1}\,. \tag{3.3}$$

Proof. Part (a) is contained in [Shi2], Theorem 2.7. To prove b)–d) we look at the Fourier expansion of $F_n^k(-,s,N)$.

According to [Ma], § 18—and using his notation—we have

$$F_n^k(Z,s,N)=\sum_T a_n^k(Y,T,s,N)\exp(2\pi i\operatorname{trace}(TX)) \tag{3.4}$$

where T runs over all symmetric half integral matrices of size n and

$$a_n^k(Y,T,s,N)=A_n^k(s)S_n^{(N)}(k+2s,T)h^{(n)}_{k+s,s}(Y,T)\det(Y)^s \tag{3.5}$$

with

$$A_n^k(s)=\frac{(-1)^{nk/2}2^n\pi^{n(k+2s)}}{\Gamma_n(k+s)\Gamma_n(s)}\,. \tag{3.6}$$

The "singular series" is defined by

$$S_n^{(N)}(s,T)=\sum_{\substack{R=R'\bmod 1\\(\nu(R),N)=1}}\nu(R)^{-s}e^{2\pi i\operatorname{trace}(RT)} \tag{3.7}$$

where $\nu(R)$ is the product of the denominators of the elementary divisors of the rational matrix R. The confluent hypergeometric function $h^{(n)}_{k+s,s}(Y,T)$ was investigated very precisely in [Shi1].

To prove c) we consider the "constant Fourier coefficient" $a_n^k(Y,0_n,s,N)$. Here the singular series is a product of Riemann zeta functions and $h^{(n)}_{k+s,s}(Y, 0_n)$ is a product of gamma factors. The relevant part of (3.5) for $T = 0_n$ is

$$\frac{\Gamma_n\left(k+2s-\frac{n+1}{2}\right)}{\Gamma_n(k+s)\Gamma_n(s)}\;\frac{\zeta^{(N)}(k+2s-n)}{\zeta^{(N)}(k+2s)}\prod_{\nu=1}^{n}\frac{\zeta^{(N)}(2k+4s-n-\nu)}{\zeta^{(N)}(2k+4s-2\nu)}\,.$$

In fact, this function has a first order pole in $s = s_{n,k}$, which proves c). Assertion b) may (at least in the case of $\mathscr{F}^k_n(-, N)$ and T positive definite) be read off directly from (3.5) when combined with a). For general T and also for $\mathscr{E}^k_n(-, N)$ it is easier to use Theorem 2 below.

Concerning d) we prove here only a much weaker statement, from which by the theory of singular modular forms [Fr2] the assertion d) follows in a straightforward way:

PROPOSITION 3.1. *If $2k = n$ with k even, we have for all $l \geq 1$: There exists a constant $d \neq 0$ such that for all half integral $T^{(n)} > 0$:*

$$\operatorname*{Res}_{s=s_{n+l,k}} a^k_{n+l}\left(\begin{pmatrix} Y^{(n)} & 0 \\ 0 & y^{(l)} \end{pmatrix}, \begin{pmatrix} T^{(n)} & 0 \\ 0 & 0^{(l)} \end{pmatrix}, s, N\right) = d\cdot \operatorname*{Res}_{s=s_{n,k}} a^k_n(Y, T, s, N)\,.$$

Proof. The factors on the right hand side of (3.5) satisfy nice recursion formulas with respect to s and n for $\begin{pmatrix} Y^{(n)} & 0 \\ 0 & y^{(l)} \end{pmatrix}$ and $\begin{pmatrix} T^{(n)} & 0 \\ 0 & 0^{(l)} \end{pmatrix}$ and any $l \geq 1$:

$$A^k_{n+l}(s) = \frac{(-1)^{lk/2}2^l\pi^{l(k+2s)}}{\Gamma_l(k+s)\Gamma_l(s)}A^k_n\left(s-\frac{l}{2}\right) \tag{3.8}$$

$$\begin{aligned} &S^{(N)}_{n+l}\left(s, \begin{pmatrix} T^{(n)} & 0 \\ 0 & 0^{(l)} \end{pmatrix}\right) \\ &\quad = \frac{\zeta^{(N)}(s-l)}{\zeta^{(N)}(s)}\prod_{\nu=1}^{l}\frac{\zeta^{(N)}(2s-n-l-\nu)}{\zeta^{(N)}(2s-2\nu)}S^{(N)}_n(s-l, T) \end{aligned} \tag{3.9}$$

$$\begin{aligned} &h^{(n+l)}_{k+s,s}\left(\begin{pmatrix} Y^{(n)} & 0 \\ 0 & y^{(l)} \end{pmatrix}, \begin{pmatrix} T^{(n)} & 0 \\ 0 & 0^{(l)} \end{pmatrix}\right) \\ &\quad = 2^{-nl/2}\Gamma_l\left(k+2s-\frac{n+l+1}{2}\right)\det(Y)^{-l/2}\det(2\pi y)^{k-2s+(n+l+1)/2} \\ &\quad\quad \times h^{(n)}_{k+s-l/2,s-l/2}(Y, T)\,. \end{aligned} \tag{3.10}$$

Here (3.8) is elementary, (3.9) follows from [Ki4], Theorem 1 and (3.10) is essentially Proposition 4.1 of [Shi1].

Putting (3.8)–(3.10) together we get

$$a_{n+l}^k\left(\begin{pmatrix} Y^{(n)} & 0 \\ 0 & y^{(l)} \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & 0^{(l)} \end{pmatrix}, s, N\right) = \varphi(y, s) a_n^k\left(Y, T, s - \frac{l}{2}, N\right)$$

where $\varphi(y, s)$ is (up to powers of 2 and π) equal to

$$\frac{\det(y)^{-k-s+(n+l+1)/2}}{\Gamma_l(k+s)\Gamma_l(s)} \frac{\zeta^{(N)}(k+2s-l)}{\zeta^{(N)}(k+2s)} \prod_{\nu=1}^{l} \frac{\zeta^{(N)}(2k+4s-n-l-\nu)}{\zeta^{(N)}(2k+4s-2\nu)} .$$

In particular, $\varphi(y, s)$ is of order zero in $s = (n + l + 1)/2 - k$ (and independent of y).

From the formulas above—applied to $l = 1$, $n = 2k - 1$ and $s = 1/2$—one easily obtains (observing that in that case $\varphi(y, s)$ has a first order pole at $s = 1/2$):

Corollary 3.1. *For $n = 2k - 1$ the Eisenstein series $E_n^k(-, s, N)$ and $F_n^k(-, s, N)$ are regular in $s = 0$; their values at $s = 0$ are holomorphic nonvanishing Siegel modular forms, denoted by $\mathscr{F}_n^k(-, N)$ and $\mathscr{E}_n^k(-, N)$. There is a nonzero constant $d_k(n)$ with*

$$\Phi(\mathscr{F}_{n+1}^k(-, N)) = d_k(n)\mathscr{F}_n^k(-, N) .$$

All the weights which occur in Theorem 3.1 are "singular weights" except the extreme case $k = n/2$, therefore we expect $\mathscr{E}_n^k(-, N)$ and $\mathscr{F}_n^k(-, N)$ to be interesting linear combinations of theta series.

So let (for $m = 2k$ divisible by 4 and $N > 1$) $\mathscr{S}(m, N)$ be the set of all even integral positive definite quadratic forms in m variables whose level divides N and whose determinant is a square.

We consider, for $S \in \mathscr{S}(m, N)$, the degree n theta series

$$\vartheta_S^{(n)}(Z) = \sum_{X \in \mathbf{Z}^{(m,n)}} e^{\pi i \operatorname{trace}(X'SXZ)}$$

and the corresponding genus theta series

$$\vartheta_{\mathrm{gen}\,S}^{(n)}(Z) = \sum_{i=1}^{h} \frac{1}{A(S_i)} \vartheta_{S_i}^n(Z)$$

where $S_1, \cdots, S_h$ runs over representatives of the $GL_n(\mathbb{Z})$-classes of forms in the genus of S and $A(S)$ is the number of units of S. (We omit the usual normalizing factor $(\sum 1/A(S_i))^{-1}$ in order to save notation in the sequel). All these theta series define elements of $M_n^k(N)$. We have the following (very weak) version of Siegel's main theorem:

Theorem 3.2. *Let $m = 2k$ be divisible by 4 and let $\mathscr{S}_1, \cdots, \mathscr{S}_t$ be the genera in $\mathscr{S}(m, N)$.*

a) *Then for all* $n \geq m$ *there exist numbers* $\alpha_i^{(n)} = \alpha^{(n)}(\mathscr{S}_i)$, $\beta_i^{(n)} = \beta^{(n)}(\mathscr{S}_i)$, $1 \leq i \leq t$ *such that*

$$\mathscr{E}_n^k(-, N) = \sum_{i=1}^{t} \alpha_i^{(n)} \vartheta_{\mathscr{S}_i}^{(n)}, \tag{3.11}$$

$$\mathscr{F}_n^k(-, N) = \sum_{i=1}^{t} \beta_i^{(n)} \vartheta_{\mathscr{S}_i}^{(n)}. \tag{3.12}$$

b) *The numbers* $\alpha_i^{(n)}$, $\beta_i^{(n)}$ *are not all equal to zero.*

c) *The* $\beta_i^{(n)}$ *are essentially independent of* n:

$$\beta_i^{(n)} = c_n \beta_i \qquad (1 \leq i \leq t)$$

with a suitable constant $c_n \neq 0$.

d) $\alpha_i^{(n)} = N^{(-n(n+1))/2} \det(\mathscr{S}_i)^{n/2} \beta^{(n)}(\mathscr{S}_i^*)$ *where* $\mathscr{S}_i^*$ *is the genus of* $N \cdot S^{-1}$, $S \in \mathscr{S}_i$, *(for the exact value of the* $\beta_i^{(n)}$ *in case* $n = m$, N *squarefree see the following corollary).*

Proof. All we have really to prove here is (3.12) for all $n \geq m$. Everything else will then follow from Theorem 3.1 d) or by applying the Fricke involution $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and using (3.2) and

$$\vartheta_S^{(n)} |_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = N^{nm/4} \det(S)^{-(n/2)} \vartheta_{NS^{-1}}^{(n)}. \tag{3.13}$$

For $n > m$ the theory of singular modular forms [Fre2] asserts that $\mathscr{F}_n^k(-, N)$ is indeed a linear combination of theta series $\vartheta_S^{(n)}$, $S \in \mathscr{S}(m, N)$. Thanks to (3.3) the same holds for $m = n$.

Since Siegel's Φ-operator is injective for $n > m$ we may now restrict ourselves to $n = m$. The Eisenstein series $\mathscr{F}_m^k(-, N)$ can be written as

$$\mathscr{F}_m^k(-, N) = \sum_{\{S\}} \frac{a(S)}{A(S)} \vartheta_S^{(m)} \tag{3.14}$$

where S runs over representatives of the $GL(m, \mathbb{Z})$-classes in $\mathscr{S}(m, N)$. The coefficients $a(S)$ are given by the equations

$$a_m(T) = \sum_{\{S\}} \frac{A(S, T)}{A(S)} a(S), \qquad T \in \mathscr{S}(m, N). \tag{3.15}$$

Here $A(S, T)$ is the rumber of integral nepresentations of T by S and $a_m(T)$ is a Fourier coefficient of $\mathscr{F}_m^k(-, N)$; in the terminology of [Bö-Ra] equation (3.15) says that $a(S)$ is a "primitive Fourier coefficient" of $\mathscr{F}_n^k(-, N)$.

Using $GL(n)$-Hecke operators one can express $a(S)$ in terms of the $a_m(T)$—see [Ki1]. Since those $GL(n)$-Hecke operators commute with Andrianov's "genus operator" [An3] we see that (3.15) together with Theorem 1 b) indeed implies that the coefficients $a(S)$ depend only on the genus of S. Therefore we may arrange (3.14) into genera, which proves (3.12) for $n = m$. The Fourier coefficients $a_m(S)$, $S \in \mathscr{S}(m, N)$ are much easier to understand than the "primitive Fourier coefficients" $a(S)$:

PROPOSITION 3.2.

$$a_m(S) = \frac{2^{m/2}\pi^{m(m+1)/2}}{\Gamma_m\left(\frac{m+1}{2}\right)} \frac{1}{\zeta^{(N)}\left(\frac{m}{2}+1\right)} \prod_{i=1}^{m/2} \zeta^{(N)}(m+2-2i)^{-1} \prod_{p \mid N} (1-p^{-1}) .$$

In particular, $a_m(S)$ *depends only on* m *and* N *(not on* $S \in \mathscr{S}(m, N)$*!).*

Proof (sketch). According to (3.5) $a_m(S)$ is essentially a product of a "singular series" part and a confluent hypergeometric part. The singular series in question only depends on the p-adic class of S with $p \nmid N$, so it is independent of the individual $S \in \mathscr{S}(m, N)$. The results of [Ki4] imply that (for $T \in \mathscr{S}(m, N)$)

$$S_m^{(N)}(s, T) = \frac{\zeta^{(N)}\left(s - \frac{m}{2}\right)}{\zeta^{(N)}(s)} \prod_{i=1}^{m/2} \zeta^{(N)}(2s-2i)^{-1} ,$$

in particular $S_m^{(N)}(s, T)$ has a simple pole in $s = m/2 + 1$.

Concerning the hypergeometric part we need

$$\left(\frac{h_{m/2+s,s}^{(m)}(Y, T)}{\Gamma_m(s)}\right)_{|s=1/2} = \det(2\pi Y)^{-1/2} e^{-2\pi \operatorname{trace}(YT)} .$$

This easily follows from [Shi1], 4.35.K.

The primitive Fourier coefficients $a(S)$ can be calculated from the $a_m(S) = a_m(N)$ with the help of a generalized Möbious inversion formula. To see this, let $V = \mathbb{Q}^m$ be equipped with the quadratic form given by S, let L be a lattice on V corresponding to the matrix S, K a lattice corresponding to T. Then $A(S, T)/A(S)$ is just the number of lattices $L' \supseteq K$ that are isometric to L.

We write $a(L) := a(S)$, $a(K) := a(T)$ and obtain $a_m(T) = \sum_{L \supseteq K} a(L)$. We restict attention to the case of square free N (see Remark 3 of § 4 below).

By Theorem 1.7.2 of [Ki1] this implies

$$a(K) = \sum_{L \supseteq K} \pi(L, K) a_m(L)$$

with

$$\pi(L, K) = \prod_{p \mid N} (-1)^{h_p} p^{h_p(h_p-1)/2} = \prod_{p \mid N} \pi_p(L, K)$$

if L is integral, $\pi(L, K) = 0$ otherwise, and where $h_p = h_p(L, K)$ is the $\mathbb{Z}/p\mathbb{Z}$-dimension of L_p/K_p (since our lattices are of square-free level, L_p/K_p is of type $\mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$). Since by Proposition 3.2 $a_m(L)$ has the same value $a_m(N)$ for all integral $L \supseteq K$, we are left with the task of determining for each $p \mid N$ the number of integral $L_p \supseteq K_p$ with L_p/K_p of $\mathbb{Z}_p$-dimension h_p:

$$\begin{aligned} a(K) &= a_m(N) \prod_{p \mid N} \sum_{h_p} (-1)^{h_p} p^{h_p(h_p-1)/2} \,\#\{L \supseteq K \mid h_p(L, K) = h_p\} \\ &= a_m(N) \prod_{p \mid N} \alpha_p(K) \end{aligned}$$

We fix p and omit the index p in the following, meaning the $\mathbb{Z}_p$-lattices when we write K, L and assume all lattices to be of square free level; this implies that they are orthogonal sums of unimodular and p-modular "unramified" [Pf] lattices.

We say that L is of type L_i ($L \sim L_i$) if it is of index p^i in a $\mathbb{Z}_p$-maximal lattice on V. For $i \leq j$, $L \sim L_i$, $K \sim L_j$ we have then $h_p(L, K) = j - i$ (if $L \supseteq K$) and thus for $K \sim L_t$

$$\alpha_p(K) = \sum_{i=0}^{t} (-1)^{t-i} p^{(t-i)(t-i-1)/2} \cdot \#\{L \supseteq K,\, L \sim L_i\}\,.$$

We put $r = \begin{Bmatrix} t & \text{if } V \text{ is split at } p \\ t+1 & \text{otherwise} \end{Bmatrix}$ (so that $\det K$ is exactly divisible by p^{2r}; since the quadratic form has square discriminant, V_p is either a sum of hyperbolic planes (split case) or such a sum plus a four-dimensional anisotropic $\mathbb{Q}_p$-space).

Put further $s_p = s_p(V)$ the Hasse-symbol of the quadratic space V at p ([OM], § 63), $\tilde{s}_p = (-1, -1)_p^{m/4} s_p$ (so that $\prod_{p\,\mathrm{finite}} \tilde{s}_p = (-1)^{m/4}$ and for p finite $\tilde{s}_p = 1$ iff V is split at p.) We fix a maximal unimodular sublattice of K; evidently it splits off orthogonally in each integral $L \supseteq K$.

By an elementary divisor argument (similar to [OM], 82: 23) it is not hard to show that the lattices $L \sim L_i$ with $L \supseteq K$ are in bijective correspondence with the $(t - i)$-dimensional totally isotropic subspaces of the regular quadratic space of dimension $2r$ over $\mathbb{F}_p$ that is split iff V is split at p. The number of these subspaces is computed to be

$$\binom{t}{t-i}_p \cdot \begin{cases} \prod_{j=0}^{t-i-1} (p^{t-j-1}+1) & \text{if } V \text{ is split at } p \\ \prod_{j=0}^{t-i-1} (p^{t-j+1}+1) & \text{otherwise} \end{cases}$$

where $\binom{t}{s}_p$ is the number of s-dimensional subspaces of $\mathbb{F}_p^t$.

By a combinatorial identity attributed to Cauchy ([G-R], p. 242) we get

$$\alpha_p(K) = (-1)^t p^{r(r-1)} = \tilde{s}_p(-1)^r p^{r(r-1)}$$

if the determinant of K is exactly divisible by p^{2r}.

Our final result is then (since $4|m$ and $\det S$ being a square imply $\prod_{\substack{p\nmid N \\ p \text{ finite}}} \tilde{s}_p = 1$).

$$a(S) = (-1)^{m/4} a_m(N) \prod_{p|N} (-1)^{r_p} p^{r_p(r_p-1)} \qquad \text{if } \det S = \prod_{p|N} p^{2r_p}.$$

We have proved:

COROLLARY 3.2. *Let N be square free. Then*

$$\mathscr{F}_m^k(-, N) = \sum_{i=1}^{t} \beta_i^{(m)} \vartheta_{\mathscr{S}_i}^{(m)}$$

with

$$\beta_i^{(m)} = (-1)^{m/4} a_m(N) \prod_{p|N} (-1)^{r_p(i)} p^{r_p(i)(r_p(i)-1)}$$

where the $\mathscr{S}_i$ are the genera in $\mathscr{S}(m, N)$ and for $S_i \in \mathscr{S}_i$ we have $\det S_i = \prod_{p|N} p^{2r_p(i)}$. *In particular, all theta series of forms in $\mathscr{S}(m, N)$ appear in $\mathscr{F}_n^k$ and $\mathscr{E}_n^k$ with non-zero coefficient.*

§ 4. The basis problem for small weights

We are going to characterize (under some additional conditions) those modular forms of small—but nonsingular—weight, which are represented by theta series ("basis problem"). Let $\Theta^{(n)}(m, N) \subseteq M_n^{m/2}(N)$ be the $\mathbb{C}$-vectorspace spanned by those theta series $\vartheta_S^{(n)}$, $S \in \mathscr{S}(m, N)$, for which the coefficient $\alpha^{(n)}(\text{gen } S)$ in (3.11) is different from zero (for square free N we have seen that these are all the $\vartheta_S^{(n)}$, $S \in \mathscr{S}(m, N)$).

THEOREM 4.1. *Let $m = 2k$ be divisible by 4 and $n/2 \leq k \leq n$ and assume that $F \in S_n^k(N)$ is an eigenfunction of all the $T_N(\mathrm{M})$, $\mathrm{M} \equiv 0 \bmod N$*

(a) *For all $n' \geq n$ we have*

$$\begin{aligned}
&\operatorname*{Res}_{s=(n+n'+1)/2-k} \mu(n, k, s) \\
(4.1)\quad &\times \frac{\lambda_F(N)}{(N^n)^{k+2s}} \frac{\Lambda_N(2s+k-n) D_F^{(N)}(2s+k-n)}{\zeta^{(N)}(2s+k) \prod_{i=1}^{n} \zeta^{(N)}(4s+2k-2i)} E_{n',n}^k(F, s) \\
&= \sum_{\{S\}} \alpha^{(n+n')}(\operatorname{gen} S) \frac{\langle F, \vartheta_S^{(n)} \rangle}{A(S)} \vartheta_S^{(n')}
\end{aligned}$$

where $\{S\}$ runs over a set of representatives of $GL_m(\mathbb{Z})$-classes of quadratic forms in $\mathscr{S}(m, N)$ with $\alpha^{(n)}(\operatorname{gen} S) \neq 0$.

(b) *The equation (4.1) implies that $F \in \Theta^{(n)}(m, N)$ if $\lambda_F(N) \cdot \Lambda_N(s) D_F^{(N)}(s)$ has a (simple) pole at $s = n + 1 - k$.*

(c) *For $F \in S_n^k(N)^{\mathrm{triv}}$ one has $F \in \Theta^{(n)}(m, N)$ if $D_F^{(N)}(s)$ has a (simple) pole at $s = n + 1 - k$.*

(d) *If conversely $F \in \Theta^{(n)}(m, N) \cap S_n^k(N)$ is such that*

$$\langle F, \vartheta_S \rangle \neq 0 \quad \text{only for } S \in \mathscr{S}(m, N), \quad \det S = \prod_{p \mid N} p^{2r_p(S)}$$

with $\sum_{p \mid N} r_p(S)$ of fixed parity, then the converse of b) *and* c) *is true. (This apparently unnatural condition will be satisfied in the applications in part* II).

Proof. Statement a) can easily be obtained by combining Theorem 1.2 and Theorem 3.2. We remark here that the gamma factor $\mu(n, k, s)$ as well as the Riemann zeta factors in (4.1) are all of order zero at the arguments in question. Furthermore c) follows from b), since for $F \in S_n^k(N)^{\mathrm{triv}}$, the function $\Lambda_N(s)$ is of order zero at $s = n + 1 - k$. The "if"-direction in b) easily follows from (4.1) for $n = n'$ since $E_{n,n}^k(F, s) = F$.

For the (partial) converse of d) we simply note that the computation of the β_i in § 3 (for square-free N) and the parity condition of d) imply that

$$\left\langle \sum_{\{S\}} \alpha^{(2n)}(\operatorname{gen} S) \frac{\langle F, \vartheta_S^{(n)} \rangle}{A(S)} \vartheta_S^{(n)}, F \right\rangle \neq 0$$

(for N not square-free see Remark 2 below).

Remark. 1) There are two points in our theorem which are not quite satisfactory. The first is the parity condition in d). The only way we see to get around it is to work out the pullback for the genus theta series of each genus in $\mathscr{S}(m, N)$ (which is an Eisenstein series by [Ku-Ra]). When one does this, everything looks the same as above for the $p\nmid N$ (which is no surprise from the adelic point of view), but for $p\mid N$ we have to handle a far more complicated Hecke-algebra than above. It appears that under suitable conditions on the local $Sp_n(\mathbf{Q}_p)$-representation generated by F we can still conclude that the pole of $\tilde{\Lambda}_N(s)D_F^{(n)}(s)$ does not come from $\tilde{\Lambda}_N(s)$ and thus conclude that the converse of c) holds without the parity conditions (where $\tilde{\Lambda}_N(s)$ is the contribution of the above mentioned Hecke-algebra for $p\mid N$).

The second unsatisfactory point is that we cannot choose a basis of $S_k^n(N)$ consisting of eigenforms of all the $T_N(\mathrm{M})$, $\mathrm{M}\equiv 0 \bmod N$ and also do not know anything about $\lambda_F^0(N)$. Of course, this defect becomes even worse when we try to work with the pullback of the genus theta series. May be this point can be clarified by a good theory of newforms for Siegel modular forms (to our knowledge such a theory has not yet been worked out). Again, this will probably demand a detailed study of the local representations.

Curiously, none of these difficulties come into play for the applications of our results to Yoshida's lifting in part II (which were the starting point of our investigations). This may be caused by the fact that in this case we are considering theta liftings of forms on the orthogonal group that are "new" in a natural sense.

2) Our Theorem was formulated for arbitrary level N, but it is only the squarefree case which is really interesting. This is explained by the fact that

$$E_n^k(Z, s, N) = (N'')^{-ns}E_n^k(N''\cdot Z, s, N') \quad \text{if} \quad N' = \prod_{q\mid N} q \quad \text{and} \quad N = N'N''\,.$$

3) The assertion c) of the Theorem above is also true for level 1 ([Bö3], [We]).

Part II. Yoshida's Lifting

§ 5. Automorphic forms on the quaternion algebra and its orthogonal group

Let D be a definite quaternion algebra over $\mathbf{Q}$, ramified at the primes $p_1, \cdots, p_r$, split at all other primes, $N_1 = p_1 \cdots p_r$.

Let R be an Eichler order of level $N = N_1N_2$ ($(N_1, N_2) = 1$, N_2 squarefree), i.e. R_p is a maximal order in D_p for all $p \nmid N_2$ and is conjugate to $\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathbb{Z}_p) \mid c \equiv 0 \bmod p\right\}$ for $p \mid N_2$ (where D_p has been identified with $M_2(\mathbb{Q}_p)$). The arithmetic of these orders has been investigated by Eichler in [E 3].

An ideal I is a $\mathbb{Z}$-lattice of rank 4 on D, it is called an order if it is a subring (with 1) of D. All ideals considered here shall have the following properties:

i) the left order $\{x \in D \mid xI \subseteq I\}$ of I is an Eichler order of level N.

ii) I is locally principal, i.e. for each p there is $x_p \in I_p$ with $I_p = R'_px_p$, where R' is the left order of I.

Obviously, the right order of such an ideal is again an Eichler order of level N and any ideal I with left order R is of the type $I = (R_{\mathbb{A}} \cdot x) \cap D = Rx$ for some $x \in D_{\mathbb{A}}^\times$.

Two ideals I_1 and I_2 are said to be right equivalent or to belong to the same (right) class if there exists $x \in D^\times$ with $I_2 = I_1x$.

With a double coset decomposition

$$D_{\mathbb{A}}^\times = \bigcup_{i=1}^{h} R_{\mathbb{A}}^\times y_i^{-1} D^\times = \bigcup_{i=1}^{h} D^\times y_i R_{\mathbb{A}}^\times \quad (n(y) = 1)$$

we then have a set of representatives Ry_i^{-1} of the classes of ideals with left order R (whose number is known to be finite and denoted by h). On D there are the involution $x \longmapsto \bar{x}$, the norm $n(x) = x\bar{x}$ and the trace $\mathrm{tr}(x) = x + \bar{x}$. The group of proper similitudes of the quadratic form $n(x)$ is isomorphic to $(D^\times \times D^\times)/\mathbb{Q}^\times$ (and accordingly for the completions and the adelization) via

$$(x_1, x_2) \longmapsto \sigma_{x_1,x_2} \quad \text{with} \quad \sigma_{x_1,x_2}(y) = x_1yx_2^{-1}$$

([E 1], § 5), with the special orthogonal group $SO(D, n)$ being the image of

$$\{(x_1, x_2) \in (D^\times \times D^\times)/\mathbb{Q}^\times \mid n(x_1) = n(x_2)\}.$$

Let $I_{ij} = y_iRy_j^{-1}$ $(i, j = 1, \cdots, h)$, $R_i = I_{ii}$, $e_i = |R_i^\times|$, and consider the theta series of degree n of the quadratic lattice I_{ij},

$$\vartheta_{ij}^{(n)}(Z) := \vartheta^{(n)}(I_{ij}, Z) = \sum_{\mathrm{x} = (x_1, \ldots, x_n) \in I_{ij}^n} \exp(2\pi i \ \mathrm{trace}(Q(\mathrm{x})Z))$$

with $Q(x)_{\nu\mu} = \frac{1}{2}\mathrm{tr}(x_\nu \bar{x}_\mu)$, $Z \in \mathbb{H}_n$.

Linear combinations of these theta series are conveniently dealt with by collecting their coefficients in an automorphic form

$$\begin{aligned}\psi &\in \mathscr{A}(D_{\mathbb{A}}^{\times} \times D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times} \times R_{\mathbb{A}}^{\times}) \\ &= \{f\colon D_{\mathbb{A}}^{\times} \times D_{\mathbb{A}}^{\times} \longrightarrow \mathbb{C} \mid f \text{ is left invariant under } D^{\times} \times D^{\times} \text{ and right} \\ &\qquad \text{invariant under } R_{\mathbb{A}}^{\times} \times R_{\mathbb{A}}^{\times}\}\colon\end{aligned}$$

Any such automorphic form f is determined by its values at the (y_i, y_j) and, conversely, can be given by prescribing these values arbitrarily.

DEFINITION 5.1. *Let*

$$\varphi, \psi \in \mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}) = \{f\colon D_{\mathbb{A}}^{\times} \longrightarrow \mathbb{C} \mid f(\gamma x u) = f(x) \text{ for all } \gamma \in D^{\times}, u \in R_{\mathbb{A}}^{\times}\}.$$

The n-th Yoshida-lifting $Y^{(n)}(\varphi, \psi)$ *of* (φ, ψ) *is defined by*

$$Y^{(n)}(\varphi, \psi) := \sum_{i,j=1}^{h} \frac{\varphi(y_i)\psi(y_j)}{e_i e_j} \vartheta_{ij}^{(n)} \in M_n^2(N).$$

$\varphi \in \mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ *is called cuspidal (by abuse of language) if* $\int_{D^{\times}\backslash D_{\mathbb{A}}^{\times}} \varphi(x)dx = 0$.

Remark. This lifting has been studied by H. Yoshida in [Y1], [Y2] from the adelic point of view. In that setting, which is also more convenient for the computation of the action of the Hecke operators on $Y^{(n)}(\varphi, \psi)$ in subsequent sections, it is defined as follows:

Denote by ω the Weil- (or oscillator-) representation of $(Sp_n)_{\mathbb{A}}$ on the space of Schwartz-Bruhat functions $S(D_{\mathbb{A}}^n)$ attached to the norm form on $D_{\mathbb{A}}$ and with respect to the standard additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ (see e.g. [Y2] for formulas for the action of generators of the local components $Sp_n(\mathbb{Q}_p)$), let $f = \prod_v f_v \in \mathscr{S}(D_A^n)$ be given by $f_p =$ characteristic function of R_p^n for p finite, $f_\infty(\mathbf{x}) = \exp(-2\pi \operatorname{trace} Q(\mathbf{x}))$, put

$$\theta_f(g, (x_1, x_2)) = \sum_{\mathbf{z}\in D^n} \omega(g) f(x_1^{-1}\mathbf{z}x_2) \quad (g \in (Sp_n)_{\mathbb{A}},\ x_1, x_2 \in D_{\mathbb{A}}^{\times},\ n(x_1) = n(x_2) = 1).$$

Then $F = Y^{(n)}(\varphi, \psi)$ corresponds to $\Psi_F =: \tilde{Y}^{(n)}(\varphi, \psi, f)\colon (Sp_n)_{\mathbb{A}} \longrightarrow \mathbb{C}$ given by

$$\tilde{Y}^{(n)}(\varphi, \psi, f)(g) = \int_{D^{\times}\times D^{\times}\backslash D_{\mathbb{A}}^{\times}\times D_{\mathbb{A}}^{\times}} \varphi(x_1)\psi(x_2)\theta_f(g, (x_1, x_2))dx_1 dx_2.$$

Here the integral is restricted to x_1, x_2 with $n(x_1) = n(x_2) = 1$. (Alternatively, one can extend the oscillator representation to the group of similitudes (see e.g. [Vi2]) and then integrate over all of $D_{\mathbb{A}}^{\times} \times D_{\mathbb{A}}^{\times}$).

LEMMA 5.1. *Let $\tilde{N}$ be squarefree. The space $\theta^{(n)}(4, \tilde{N})$ generated by theta series of degree n of integral quaternary quadratic forms of level $N|\tilde{N}$ and square discriminant is spanned by the*

$$Y^{(n)}(\varphi, \psi), \quad \mathscr{S}, \psi \in \mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times}),$$

where D ranges over the definite quaternion algebras over $\mathbf{Q}$ unramified outside $\tilde{N}$ and R ranges over the Eichler order of level $N|\tilde{N}$ in D.

Proof. An inspection of the possible Jordan splittings of the completions of such a quadratic form shows that it must be in the genus of one of the Eichler orders and thus, observing the identification of $(D^{\times} \times D^{\times})/\mathbf{Q}^{\times}$with the group of proper similitudes of (D, n), is isometric to one of the (I_{ij}, n). This proves the assertion since obviously

$$\mathscr{A}(D_{\mathbf{A}}^{\times} \times D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times} \times R_{\mathbf{A}}^{\times}) \cong \mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times}) \otimes \mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times}).$$

On $\mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ we have for $p\nmid N$ Hecke operators $\tilde{T}(p)$ defined by $\tilde{T}(p)\varphi(x) = \int_{D_p^{\times}} \varphi(xy^{-1})\tau_p(y)dy$ where τ_p is the characteristic function of $\{y \in R_p \mid n(y) \in p\mathbf{Z}_p^{\times}\}$. They are given explicitly by

$$\tilde{T}(p)\varphi(y_i) = \sum_{j=i}^{h} B_{ij}(p)\varphi(y_j),$$

where the Brandt-matrix entry $B_{ij}(p)$ is the number of ideals of norm p in the class of I_{ij} that are integral (i.e., contained in their left order $R_i = y_i R y_i^{-1}$).

For $N'|N$ there is an involution $w_{N'}$ on $\mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ given as follows:

For $p|N$ choose $\pi_p \in D_p^{\times}$ to be a nontrivial representative of $N(R_p) = \{x \in D_p^{\times} \mid xR_p x^{-1} = R_p\}$ modulo $R_p^{\times}\mathbf{Q}_p^{\times}$ (one has $(N(R_p)\colon R_p^{\times}\mathbf{Q}_p^{\times}) = 2$, see e.g. [Vil]), e.g., for $p|N_1$ choose π to be a prime element of R_p, for $p|N_2$ choose

$$\pi_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$

For $N'|N$ let $\pi_{N'} \in R_{\mathbf{A}}$ be given by

$$(\pi_{N'})_p = \begin{cases} \pi_p & \text{if } p|N' \\ 1 & \text{otherwise} \end{cases}$$

and put

$$w_{N'}\varphi(y) := \varphi(y\pi_{N'}).$$

LEMMA 5.2. *Let $N_2 = p_{r+1} \cdots p_t$, let $\varepsilon : \mathbb{Z}_2^t \longrightarrow \{\pm 1\}$ be a character and let $\eta_{N'} = (\eta_{N'}(p_1), \cdots, \eta_{N'}(p_t)) \in \mathbb{Z}_2^t$ for $N' | N$ be given by*

$$\eta_{N'}(p) = \begin{cases} +1 & \text{if } p | N' \\ 0 & \text{if } p \nmid N' . \end{cases}$$

Let $\mathscr{A}^{\varepsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}) := \{\varphi \in \mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}) \,|\, w_{N'}\varphi = \varepsilon(\eta_{N'})\varphi$ for all $N' | N\}$. Then $\mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}) = \bigoplus_{\varepsilon \in \hat{\mathbb{Z}}_2^t} \mathscr{A}^{\varepsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$.

Proof. Obvious.

LEMMA 5.3 ("Stable non-vanishing"). *Let $\varphi \in \mathscr{A}^{\varepsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$, $\psi \in \mathscr{A}^{\varepsilon'}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$. Then*

$$Y^{(3)}(\varphi, \psi) \neq 0 \qquad \textit{if and only if } \varepsilon = \varepsilon' .$$

Proof. For the proof we have to check when the quadratic lattices (I_{ij}, n), (I_{kl}, n) are isometric. This is easily seen to be the case if and only if either

i) $y_i^{-1} \in R_{\mathbb{A}}^{\times}\pi_{N'} y_k^{-1} D^{\times}$ and $y_j^{-1} \in R_{\mathbb{A}}^{\times}\pi_{N'} y_l^{-1} D^{\times}$ for some $N' | N$

or

ii) (i) holds with i and j interchanged.

Let $\mathbb{Z}_2^t$ act on $\{1, \cdots, h\}$ by putting $R_{\mathbb{A}}^{\times} y_{\eta_{N'}(i)}^{-1} D^{\times} = R_{\mathbb{A}}^{\times} y_i^{-1} \pi_{N'} D^{\times}$ $(N' | N)$, and let $\mathrm{Fix}(i) = \{\eta \in \mathbb{Z}_2^t \,|\, \eta(i) = i\}$.

Writing $Y^{(n)}(\varphi, \psi)$ as a linear combination $\sum \alpha_\nu^{(n)} \vartheta^{(n)}(K_\nu)$ of theta series of pairwise non-isometric lattices in the genus of (R, n) we find for $K_\nu \cong I_{ij}$:

$$\begin{aligned}
\alpha_\nu^{(n)} &= \sum_{\eta \bmod (\mathrm{Fix}(i) \cap \mathrm{Fix}(j))} \frac{\varphi(y_{\eta(i)})\psi(y_{\eta(j)}) + \psi(y_{\eta(i)})\varphi(y_{\eta(j)})}{e_i e_j (1 + \delta_{ij})} \\
&= \sum_{\eta \bmod (\mathrm{Fix}(i) \cap \mathrm{Fix}(j))} \varepsilon\varepsilon'(\eta) \frac{\varphi(y_i)\psi(y_j) + \psi(y_i)\varphi(y_j)}{e_i e_j (1 + \delta_{ij})} \\
&= \begin{cases} 0 & \text{if } \varepsilon \neq \varepsilon' \\ \#\eta \dfrac{\varphi(y_i)\psi(y_j) + \psi(y_i)\varphi(y_j)}{e_i e_j (1 + \delta_{ij})} & \text{if } \varepsilon = \varepsilon' . \end{cases}
\end{aligned}$$

The assertion now follows from Kitaoka's [Ki2] result that the theta series of degree $m - 1$ of inequivalent quadratic forms in m variables having the same discriminant are linearly independent, since $\varphi(y_i)\psi(y_j) + \psi(y_i)\varphi(y_j) = 0$ for all i, j implies $\varphi = 0$ or $\psi = 0$.

Remark. The assertion of Lemma 3 can also be formulated in terms of functions on the orthogonal group $O(D, n)$:

Let $\Psi\colon O_{\mathbb{A}}(D, n) \longrightarrow \mathbb{C}$ be defined by

$$\Psi(\sigma) = \sum_{(y_i, y_j)} \varphi(y_i)\psi(y_j)$$

where the summation is over all (y_i, y_j) with

$$\sigma \in O(D, n)\sigma_{y_i, y_j} O_{\mathbb{A}}(R) \quad \text{and} \quad O_{\mathbb{A}}(R) = \{\tau \in O_{\mathbb{A}}(D, n) \mid \tau R = R\}.$$

Then $\Psi \neq 0$ if and only if $\varepsilon = \varepsilon'$.

§ 6. Computation of Euler factors: Good places

Let $\varphi, \psi \in \mathscr{A}^{\varepsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ for some character ε on the group of involutions w_d $(d \mid N)$ as in § 5, assume further that φ and ψ are eigenfunctions of all the Hecke-operators $\tilde{T}(p)$ with eigenvalues λ_p, μ_p for $p \nmid N$ (equivalently, the modular forms in $S_1^2(N)$ corresponding to φ, ψ (if these are cuspidal) under the correspondence of Eichler, Shimizu, Jacquet-Langlands are eigenfunctions of the Hecke-operators $T(p)$ with the same eigenvalues). We can then compute the Satake parameters of $Y^{(n)}(\varphi, \psi)$ in terms of those of φ, ψ.

THEOERM 6.1. *Let $\varphi, \psi \in \mathscr{A}^{\varepsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ as above, let $p \nmid N$, denote by β_p, β_p^{-1} resp. $\tilde{\beta}_p, \tilde{\beta}_p^{-1}$ the Satake parameters of φ, ψ with respect to the Hecke-algebra of $GL_2(\mathbb{Q}_p) \cong D_p^{\times}$ (so that $\lambda_p = p^{1/2}(\beta_p + \beta_p^{-1})$, $\mu_p = p^{1/2}(\tilde{\beta}_p + \tilde{\beta}_p^{-1})$).*

Then $Y^{(n)}(\varphi, \psi)$ is an eigenfunction of the p-component of the Hecke-algebra of GSp_n with Satake-parameters

$\alpha_0(p) = p^{-((n-2)(n-1)/4)}\tilde{\beta}_p^{-1}$ (*normalized to* $\alpha_0^2(p)\alpha_1(p) \cdots \alpha_n(p) = 1$).

$\alpha_1(p) = \beta_p^{-1}\tilde{\beta}_p$

$\alpha_2(p) = \beta_p\tilde{\beta}_p$

$\alpha_{2+j}(p) = p^j \qquad (j \geq 1)$.

Proof. Let $GO^+(D_p)$ denote the group of proper similitudes of (D, n), $PGO^+(D_p) = GO^+(D_p)/\mathbb{Q}_p^{\times}$. Then $(x_1\mathbb{Q}_p^{\times}, x_2\mathbb{Q}_p^{\times}) \mapsto \sigma_{x_1, x_2}\mathbb{Q}_p^{\times}$ (with $\sigma_{x_1, x_2}(y) = x_1 y \bar{x}_2$,) maps $D_p^{\times}/\mathbb{Q}_p^{\times} \times D_p^{\times}/\mathbb{Q}_p^{\times}$ bijectively onto $PGO^+(D_p)$.

Identifying $D_p^{\times}$ with $GL_2(\mathbb{Q}_p)$ we see that the product of the Borel subgroups of $PGL_2(\mathbb{Q}_p)$ is mapped onto a Borel subgroup of $PGO^+(D_p)$ (i.e., the stabilizer of a flag $W_1 \subseteq W_2$ of isotropic subspaces of dimensions 1, 2 respectively of (D_p, n)). Inverting the map, the torus

$$\left\{\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix}\right\}$$

of $PSO(D_p) = SO(D_p)/\{\pm 1\}$ is mapped onto

$$\left\{\begin{pmatrix} t_2 t_1^{-1} & \\ & 1\end{pmatrix}\mathbb{Q}_p^\times \cdot \begin{pmatrix} t_1 t_2 & \\ & 1\end{pmatrix}\mathbb{Q}_p^\times\right\} \subseteq PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_p)\,.$$

A homomorphism ρ: $\mathscr{H}(PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_p),\ PGL_2(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_p)) \to \mathbb{C}^\times$ with Satake parameters (β_p, β_p^{-1}), $(\tilde{\beta}_p, \tilde{\beta}_p^{-1})$ is thus mapped onto ρ^*: $\mathscr{H}(PSO(D_p),\ PSO(R_p)) \to \mathbb{C}^\times$ with parameters $\tilde{\beta}_p\beta_p^{-1}$, $\tilde{\beta}_p\beta_p$.

We restrict this to $\mathscr{H}(PO(D_p),\ PO(R_p))$ (where $PO(D_p) = O(D_p)/\{\pm 1\}$) and apply Rallis' result [Ra2, Ku] generalizing the Eichler commutation relation between Anzahlmatrices and the action of Hecke operators on theta series.

For $n \geq 2$ this asserts the existence of a homomorphism κ_n: $\mathscr{H}(Sp_n(\mathbb{Q}_p),\ Sp_n(\mathbb{Z}_p)) \to \mathscr{H}(O(D_p),\ O(R_p))$ such that a homomorphism ρ^*: $\mathscr{H}(O(D_p),\ O(R_p)) \to \mathbb{C}^\times$ with Satake parameters γ_1, γ_2 is mapped to $\rho^* \circ \kappa_n$ with Satake parameters

$$\alpha_1 = \gamma_1\,, \quad \alpha_2 = \gamma_2\,, \quad \alpha_{2+j} = p^j \quad (j = 1, \cdots, n-2)\,,$$

and such that κ_n commutes with the theta lifting. In particular, by applying the theta lifting to an automorphic form G on $O_{\mathbb{A}}(D)$ which is an eigenfunction of $\mathscr{H}(O(D_p), O(R_p))$ with eigenvalues given by ρ_G^*: $\mathscr{H}(O(D_p),\ O(R_p)) \to \mathbb{C}^\times$ one obtains an eigenfunction F of $\mathscr{H}(Sp_n(\mathbb{Q}_p),\ Sp_n(\mathbb{Z}_p))$ with eigenvalues $\rho_F^* = \rho_G^* \circ \kappa_n$: $\mathscr{H}(Sp_n(\mathbb{Q}_p),\ Sp_n(\mathbb{Z}_p)) \to \mathbb{C}^\times$. In our situation this shows that $F = Y^{(n)}(\varphi, \psi)$ is an eigenfunction of $\mathscr{H}(Sp_n(\mathbb{Q}_p),\ Sp_n(\mathbb{Z}_p))$ with Satake parameters $\alpha_1(p), \cdots, \alpha_n(p)$ as asserted.

The parameter $\alpha_0(p)$ can be computed by (3.3.70) of [An1]:

$$\alpha_0(p) \prod_{i=1}^{n} (1 + \alpha_i(p)) = \lambda_F(p) p^{-n((n+1)/4)}\,,$$

where $\lambda_F(p)$ is the eigenvalue of F under the Hecke-operator $T(p) = T(1, \cdots, 1, p, \cdots, p)$. Using Yoshida's [Y3] computation of the action of $T(p)$ on theta series we obtain

$$Y^{(n)}(\varphi, \psi) \mid T(p) = p^{n-1}(\lambda_p + \mu_p) \prod_{j=1}^{n-2} (1 + p^j) Y^{(n)}(\varphi, \psi)$$

(see also [Y1] for $n = 2$), and thus

$$\begin{aligned}\lambda_F(p) &= p^{n-1/2}(\beta_p + \beta_p^{-1} + \tilde{\beta}_p + \tilde{\beta}_p^{-1}) \prod_{j=1}^{n-2} (1 + p^j) \\ &= p^{n-1/2}\tilde{\beta}_p^{-1} \prod_{i=1}^{n} (1 + \alpha_i(p))\,, \\ \alpha_0(p) &= p^{-((n-2)(n-1)/4)}\tilde{\beta}_p^{-1}\,.\end{aligned}$$

COROLLARY 6.1. *The standard L-function of* $F^{(n)}=Y^{(n)}(\varphi,\psi)$ *is for* $n\geq 2$

$$\begin{aligned}D^{(N)}_{F^{(n)}}(s) &= \prod_{p\nmid N}(1-p^{-s})^{-1}(1-\beta_p\tilde{\beta}_p p^{-s})^{-1}(1-\beta_p\tilde{\beta}_p^{-1}p^{-s})^{-1}(1-\beta_p^{-1}\tilde{\beta}_p p^{-s})^{-1}\\ &\quad\cdot(1-\beta_p^{-1}\tilde{\beta}_p^{-1}p^{-s})^{-1}\prod_{j=1}^{n-2}(1-p^{-s+j})^{-1}(1-p^{-s-j})^{-1}\\ &= \zeta^{(N)}(s)L^{(N)}_{\varphi,\psi}(s)\cdot\prod_{j=1}^{n-2}\zeta^{(N)}(s-j)\zeta^{(N)}(s+j)\,.\end{aligned}$$

The spinor L-function of $F^{(N)}$ *is for* $n\geq 2$

$$\begin{aligned}Z^{(N)}_{F^{(2)}}(s) &= \prod_{p\nmid N}(1-\beta_p p^{-s})^{-1}(1-\beta_p^{-1}p^{-s})^{-1}(1-\tilde{\beta}_p p^{-s})^{-1}(1-\tilde{\beta}_p^{-1}p^{-s})^{-1}\\ &= L^{(N)}(\varphi, s+1/2)L^{(N)}(\psi, s+1/2)\,,\\ Z^{(N)}_{F^{(n)}}(s) &= Z^{(N)}_{F^{(2)}}\left(s+\frac{(n-1)(n-2)}{4}\right)\\ &\quad\prod_{i=1}^{n-2}\prod_{1\leq j_1<\cdots<j_i\leq n-2} Z^{(N)}_{F^{(2)}}\left(s+\frac{(n-1)(n-2)}{4}-j_1-\cdots-j_i\right)\end{aligned}$$

where

$$L^{(N)}_{\varphi,\psi}(s) = (1-\beta_p\tilde{\beta}_p p^{-s})(1-\beta_p\tilde{\beta}_p^{-1}p^{-s})^{-1}(1-\beta_p^{-1}\tilde{\beta}_p p^{-s})^{-1}(1-\beta_p^{-1}\tilde{\beta}_p^{-1}p^{-s})^{-1})\,.$$

Remark. The last formula for $n=3$ has been obtained by Tanigawa [Tan]. We will see that $Y^3(\varphi,\psi)$ can be cuspidal for $\varphi=1$. This gives then some examples of cusp forms of degree 3 whose spinor L-function has a functional equation under $s\mapsto 1-s$ and can be analytically continued.

§7. Computation of Euler factors: p dividing N

Let $\varphi, \psi\in\mathcal{A}^{\epsilon}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ be as in section 6, and assume in addition φ and ψ to have the following properties:

(i) φ and ψ are in the essential part [Hi-Sa] of $\mathcal{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$, i.e. $\int_{D^{\times}\backslash D_{\mathbb{A}}^{\times}}\varphi(x)\overline{\rho(x)}dx=0$ for all $\rho\in\mathcal{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ which are right invariant under $(R'_{\mathbb{A}})^{\times}$ for some order $R'\subseteq D$ strictly containing R.

(ii) For all primes $q\,|\,N_2$, φ and ψ are eigenfunctions of the Hecke-operators $\tilde{T}(1,q)$ given by $\tilde{T}(1,q)\varphi(x)=\int_{D_q^{\times}}\varphi(xy^{-1})\tau_{1,q}(y)\ dy$ where $\tau_{1,q}$ is the characteristic function of $R_q^{\times}\begin{pmatrix}1&0\\0&q\end{pmatrix}R_q^{\times}$.

That is, φ and ψ are cuspidal (unless R is maximal, in which case they may be identically 1 on $D_{\mathbb{A}}^{\times}$) and correspond to newforms of weight

2 and level N, eigenfunctions of all Hecke operators, under the correspondence of Eichler, Shimizu, Jacquet-Langlands.

The functions φ and ψ generate then (if they are cuspidal) irreducible subspaces of the regular representation of $D_{\mathbb{A}}^{\times}$ on $\mathscr{A}(D_{\mathbb{A}}^{\times})$ ([Ge], § 5) whose p-component for $p\,|\,N_2$ is isomorphic to the unique irreducible subspace of the representation of $GL_2(\mathbb{Q}_p)$ induced by the character $\xi_p|\,|_p^{1/2}$, $\xi_p|\,|_p^{-1/2}$ (see [Ge], [Ca1]) where ξ_p is the unramified character on $\mathbb{Q}_p^{\times}$ with $\xi_p(p) = -\varepsilon(w_p)$.

By the results of section 6 we have to compute the action on $F = Y^{(n)}(\varphi, \psi)$ of $T_N(N\cdot 1_n)$ and of the double cosets

$$\Gamma_0^{(n)}(N)\begin{pmatrix} \Pi_t(p)^{-1} & 0 \\ 0 & \Pi_t(p)\end{pmatrix}\Gamma_0^{(n)}(N) \quad \text{with} \quad \Pi_t(p) = \begin{pmatrix} 1_{n-t} & 0 \\ 0 & p1_t\end{pmatrix} \quad \text{for all } p\,|\,N.$$

We notice first that by the methods of [Y3] the action of the Hecke operators can be localized as usual.

LEMMA 7.1. *Let $\gamma \in Sp_n(\mathbb{Q})$, $F \in M_n^k(N)$, let Ψ_F be the automorphic form on $(Sp_n)_{\mathbb{A}}$ associated to it (see preliminaries). Let*

$$K_p^{(n)} = \left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in Sp_n(\mathbb{Z}_p)\,|\, C \equiv 0 \bmod N\mathbb{Z}_p\right\},$$

let $\sigma_{\gamma,p}$ be the characteristic function of $K_p^{(n)}\gamma K_p^{(n)}$ on $Sp_n(\mathbb{Q}_p)$, and let $p_1, \cdots, p_s$ be the primes for which $\gamma \notin K_p^{(n)}$. Put

$$\Psi_F\,|\,K_p^{(n)}\gamma K_p^{(n)} = \int_{Sp_n(\mathbb{Q}_p)} \Psi_F(g\tilde{g}^{-1})\sigma_{\gamma,p}(\tilde{g})d\tilde{g}\,,$$

let $F|K_p^{(n)}\gamma K_p^{(n)}$ be the modular form in $M_n^k(N)$ associated to $\Psi_F|K_p^{(n)}\gamma K_p^{(n)}$. Then

$$F\,|\,\Gamma_0^{(n)}(N)\gamma\Gamma_0^{(n)}(N) = F\,|\,K_{p_1}^{(n)}\gamma K_{p_1}^{(n)}\,|\cdots|\,K_{p_s}^{(n)}\gamma K_{p_s}^{(n)}\,.$$

In particular, if $\Psi_F\,|\,K_{p_i}^{(n)}\gamma K_{p_i}^{(n)} = \lambda_{p_i}\Psi_F$, then

$$F\,|\,\Gamma_0^{(n)}\gamma\Gamma_0^{(n)}(N) = \prod_{i=1}^{s}\lambda_{p_i}\cdot F\,.$$

LEMMA 7.2. *Let $\Psi_F = \tilde{Y}^{(n)}(\varphi, \psi, f)$ as in section 5, let $K_p^{(n)}\gamma K_p^{(n)} = \bigcup_i K_p^{(n)}\gamma_i$. Then*

$$\Psi_F\,|\,K_p^{(n)}\gamma K_p^{(n)} = \tilde{Y}^{(n)}(\varphi, \psi, f')$$

where $f_p'(z) = \sum_i \omega_p(\gamma_i^{-1})f_p$ (ω_p denoting the oscillator representation as in *section* 5) *and* $f'_l(\mathrm{z}) = f_l(\mathrm{z})$ *for all places* $l \neq p$. *If further* $x_k, \tilde{x}_k \in D_p^\times$ *are such that*

$$f'_p(\mathrm{z}) = \sum_k \int_{D_p^\times \times D_p^\times} f_p(x_k^{-1}\mathrm{z}\tilde{x}_k)\tau_{x_k}(x)\tau_{\tilde{x}_k}(\tilde{x})dx\,d\tilde{x}$$

(*with* τ_{x_k} *the characteristic function of* $R_p^\times x_p R_p^\times$), *then*

$$\tilde{Y}^{(n)}(\varphi, \psi, f') = \sum_k \tilde{Y}^{(n)}(\varphi'_k, \psi'_k, f)$$

where

$$\varphi'_k(x) = \int_{D_p^\times} \varphi(xy^{-1})\tau_{x_k}(y)dy \quad and \quad \psi'_k(x) = \int_{D_p^\times} \psi(xy^{-1})\tau_{\tilde{x}_k}(y)dy\,.$$

Proof. The proofs are completely analogous to those given in [Y3], section 1.

The local computations left to be done become particularly easy for $p\,|\,N_1$ (i.e., D_p is a skew field and R_p is its maximal order).

LEMMA 7.3. *Let* $p\,|\,N_1$, $F = Y^{(n)}(\varphi, \psi)$. *Then*

a) $F\,|\,K_p\begin{pmatrix} 0 & -p^{-1}\cdot 1_n \\ p\cdot 1_n & 0 \end{pmatrix} K_p = p^{n(n-1)/2}\cdot s_p(D)^n \gamma_p^n F$ *where* $s_p(D)$ *is the Hasse invariant of the quadratic space* (D, n) ([OM], § 63) *and* γ_p *is an absolute constant.*

b) *F is an eigenfunction of the* $K_p^{(n)}\begin{pmatrix} \prod_i^{-1}(p) & 0 \\ 0 & \prod_i(p) \end{pmatrix} K_p^{(n)}$ *and the Satake parameters* $\beta_{i,p}^{(n)}$ *from section 2 of F are given by* $\beta_{i,p}^{(n)} = p^{n-i-1}$.

Proof. By section 2 (proof of Lemma 2.2) we have

$$K_p^{(n)}\begin{pmatrix} 0 & (\mathrm{M}')^{-1} \\ -\mathrm{M} & 0 \end{pmatrix} K_p^{(n)} = \bigcup_{w, \mathbf{A}} \begin{pmatrix} 0_n & 1_n \\ -1_n & 0 \end{pmatrix} \begin{pmatrix} 1_n & -\mathbf{A} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & (w')^{-1} \end{pmatrix} K_p^{(n)}$$

where $\mathbf{A} = \mathbf{A}(w)$ runs through $p\mathbb{Z}_{p,\mathrm{sym}}^{(n,n)}$ modulo $w\mathbb{Z}_{p,\mathrm{sym}}^{(n,n)}w'$ and

$$GL(n, \mathbb{Z}_p)\mathrm{M}'GL(n, \mathbb{Z}_p) = \bigcup_w GL(n, \mathbb{Z}_p)w'\,.$$

Now, for fixed w we have:

$$\sum_{\mathbf{A}(w)} \omega_p\left(\begin{pmatrix} 1_n & -\mathbf{A} \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w'^{-1} \end{pmatrix}\right) f(\mathbf{z}) = \sum_{\mathbf{A}(w)} \chi_v(\mathrm{tr}\,(-\mathbf{A}Q(\mathbf{z})))f(\mathbf{z}w)\,|\det\, w|_p^2\,.$$

Since $\pi_p^{-1}R_p^n = \{\mathbf{z} \in D_p^n\,|\,Q(\mathbf{z}) \in p^{-1}\mathbb{Z}_{p,\mathrm{sym}}^{(n,n)}\}$ (π_p a prime element in D_p), this is equal to

$$\begin{cases} 0 & \text{if } \mathbf{z} \notin \pi_p^{-1}R_p^n \\ p^{-n(n+1)/2}\,|\det w|_p^{-n+1} & \text{if } \mathbf{z} \in \pi_p^{-1}R_p^n\,. \end{cases}$$

Observing that $\pi_p^{-1}R_p$ is the dual lattice of R_p with respect to the norm form we obtain:

$$\begin{aligned}&\tilde{Y}^{(n)}(\varphi, \psi, f) \,|\, K_p^{(n)}\begin{pmatrix}0 & M'^{-1}\\ -M & 0\end{pmatrix}K_p^{(n)}\\ &= \sum_w p^{-n(n+1)/2}|\det M|_p^{-n+1}p^n\gamma_p^n s_p(D)^n\tilde{Y}^{(n)}(\varphi, \psi, f)\,.\end{aligned}$$

(Here the action of $\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}$ under the oscillator representation is computed with the help of formulas from [Pe], [Sa], see also Lemma 8.2.a) below. For $M = N\cdot 1_n$ this proves a), for general M we see that (with the notation of section 2) $\tilde{\lambda}(M_0)$ is equal to $|\det M_0|_p^{-(n-1)}$ times the number of w. By (2.2) and our normalization of the $\beta_{i,p}$ this implies b).

LEMMA 7.4. *Let* $p\,|\,N_2$, $F = Y^{(n)}(\varphi, \psi)$. *Then*

$$F\,|\,K_p^{(n)}\begin{pmatrix}0 & -p^{-1}1_n\\ p1_n & 0\end{pmatrix}K_p^{(n)} = -\,p^{n(n-1)/2}s_p(D)^n\gamma_p^n F\,.$$

Proof. This can be proved using Evdokimov's [Ev] computation of

$$\sum_{A\in \mathbf{Z}_{\mathrm{sym}}^{(n,n)} \bmod p\mathbf{Z}_{\mathrm{sym}}^{(n,n)}} F|\begin{pmatrix}1_n & A\\ 0 & p1_n\end{pmatrix}.$$

More precisely, denote by S_{ij}, S representatives of the classes of integral positive definite quadratic forms satisfying the following conditions:

(i) S_{ij} is split over $\mathbf{Q}_p$ and equivalent to S over all $\mathbf{Z}_l$ with $l \neq p$.

(ii) The level of S and S_{ij} is not divisible by p^2.

(iii) The discriminant of S is exactly divisible by p^{m-r}, that of S_{ij} is exactly divisible by p^{2i}.

If S_{ij} corresponds to a lattice K, denote by $p^{-1}S_{ij}^*(p)$ the matrix corresponding to $K^{\#,p} = K^{\#}\cap \mathbf{Z}[1/p]K$.

Then it is easily verified that Evdokimov's $r(S, pS_{ij}, G_rD_{r-a+i}G)$ is equal to $r(S, S_{ij}^*(p))$. Applying $K_p^{(n)}\begin{pmatrix}0 & -1_n\\ p1_n & 0\end{pmatrix}K_p^{(n)}$ we obtain

$$\begin{aligned}&\vartheta^{(n)}(S)\,|\,K_p^{(n)}\begin{pmatrix}0 & -p^{-1}1_n\\ p1_n & 0\end{pmatrix}K_p^{(n)}\\ &= p^{n(n+1)/2}\cdot s_p(S)^n\gamma_p^n\sum_{j=0}^{r/2}(-1)^i p^{i(i-1-n)}\sum_{j=1}^{h(i)}\frac{r(S, S_{m/2-i,j})}{e(S_{m/2-j,j})}\vartheta^{(n)}(S_{m/2-i,j})\,.\end{aligned}$$

In our situation ($m = 4$, $r = 2$), this implies

$$\Psi_F\,|\,K_p^{(n)}\begin{pmatrix}0 & -p^{-1}1_n\\ p1_n & 0\end{pmatrix}K_p^{(n)} = -\,p^{n(n-1)/2}s_p(D)^n\gamma_p^n(\Psi_F - \tilde{Y}^{(n)}(\varphi, \psi, f''))$$

where $f_l'' = f_l$ for $l \neq p$ and $f_p''(z)$ is the characteristic function of

$$\left(\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}M_2(\mathbb{Z}_p)\right)^n \cap \left(M_2(\mathbb{Z}_p)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)^n .$$

Since φ, ψ were chosen to be in the essential part of $\mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$, we see that $\tilde{Y}^{(n)}(\varphi, \psi, f'') = 0$, which proves the assertion.

Remark. i) If D_1 and D_2 are quaternion algebras as above, D_1 is split at p and D_2 is ramified at p, then $s_p(D_1) = - s_p(D_2)$, i.e., the eigenvalues of $K_p^{(n)}\begin{pmatrix}0 & -p^{-1}1_n\\ p1_n & 0\end{pmatrix}K_p^{(n)}$ in the split and in the non-split case are the same for odd n, of opposite sign for even n. ii) The proof of Lemma 7.4 is actually valid if only one of φ, ψ is in the essential part of $\mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$. This will be used later.

LEMMA 7.5. *Let $F \in M_n^k(N)$, $p \mid N$, $\prod_i^{(n)} = \begin{pmatrix}1_{n-i} & 0\\ 0 & p1_i\end{pmatrix}$, let Φ as usual denote Siegel's Φ-operator, write $F \mid T_i^{(n)}$ for $F \mid K_p^{(n)}\begin{pmatrix}(\prod_i^{(n)})^{-1} & 0\\ 0 & \prod_i^{(n)}\end{pmatrix}K_p^{(n)}$. Then*

$$F \mid T_n^{(n)} \mid \Phi = p^{2n-2}F \mid \Phi \mid T_{n-1}^{(n-1)} ,$$

$$F \mid T_i^{(n)} \mid \Phi = p^{2n-2}F \mid \Phi \mid T_{i-1}^{n-1} + p^i F \mid \Phi \mid T_i^{(n-1)} \qquad (i < n) .$$

If F is in the subspace of $M_n^k(N)$ generated by theta series of level dividing N, then $F \mid T_i^{(n)}$ is in the same space.

Proof. The first part of the assertion is easily verified along the lines of [Fre 1], IV, § 4 and [Kri]. The second part is clear in the singular case by [Fre 2] and follows in general from the result in the singular case by the commutation relation of the first part.

LEMMA 7.6. *Let $p \mid N_2$, $F = Y^{(n)}(\varphi, \psi)$. Then F is an eigenfunction of the $T_i = K_p^{(n)}\begin{pmatrix}(\prod_i^{(n)})^{-1} & 0\\ 0 & \prod_i^{(n)}\end{pmatrix}K_p^{(n)}$ and the Satake parameters $\beta_{i,p}^{(n)}$ from section 2 of F are given by $\beta_{i,p}^{(n)} = p^{n-i-1}$.*

Proof. By the previous lemma we can restrict attention to the case $n = 4$ and put $\prod_i^{(4)} = \prod_i$, $K_p^{(4)} = K_p$. We have the coset decomposition

$$K_p\begin{pmatrix}\prod_i & 0\\ 0 & \prod_i^{-1}\end{pmatrix}K_p = \bigcup \begin{pmatrix}1 & -\mathbf{A}\\ 0 & 1\end{pmatrix}\begin{pmatrix}w & 0\\ 0 & w'^{-1}\end{pmatrix}K_p$$

where w' runs through representatives of the cosets $GL(4, \mathbb{Z}_p)w'$ in $GL(4, \mathbb{Z}_p)\prod_i GL(4, \mathbb{Z}_p)$ and $\mathbf{A}$ through $\mathbb{Z}_{p,\mathrm{sym}}^{(4,4)}$ mod $w\mathbb{Z}_{p,\mathrm{sym}}^{(4,4)}w'$. Letting the representatives act on the test function f_p via the oscillator representation we obtain as in the proof of Lemma 3:

$$f'_p(\mathbf{z}) = \begin{cases} \sum_{\{w \mid \mathbf{z} \in R_p^4 w^{-1}\}} |\det w|_p^{-3} & \text{if } \ Q(\mathbf{z}) \in \mathbb{Z}_{p,\mathrm{sym}}^{(4,4)} \\ 0 & \text{otherwise} \end{cases}$$
$$= p^{3i} \begin{cases} \#\{w \mid \mathbf{z} \in R_p^4 w^{-1}\} & \text{if } \ Q(\mathbf{z}) \in \mathbb{Z}_{p,\mathrm{sym}}^{(4,4)} \\ 0 & \text{otherwise}\,. \end{cases}$$

For $\mathbf{z} = (z_1, \cdots, z_4) \in D_p^4$ we let $K(\mathbf{z})$ denote the lattice in D_p generated by $(z_1, \cdots, z_4)$ and restrict attention to $\mathbf{z}$ with $K(\mathbf{z})$ of rank 4. Obviously $\mathbf{z} \in R_p^4 w^{-1}$ for some w if and only if $K(\mathbf{z})$ is in $p^{-1}R_p$ and has no more than i elementary divisors p^{-1} with respect to R_p.

If $\mathbf{z} \in R_p^4 w_0^{-1}$ for some fixed w_0, then the cosets $wGL(4, \mathbb{Z}_p)$ with $\mathbf{z} \in R_p^4 w^{-1}$ are parametrized by the cosets $\bar{u} \in GL(4, \mathbb{Z}_p)/(w_0 GL(4, \mathbb{Z}_p) w_0^{-1} \cap GL(4, \mathbb{Z}_p))$ with $\mathbf{z}u \in R_p^4 w_0^{-1}$. Since their number depends only on $K(\mathbf{z})$ we can put $w_0 = \prod_i$ and have $GL(4, \mathbb{Z})/\prod_i GL(4, \mathbb{Z}_p) \prod_i^{-1} \cap GL(4, \mathbb{Z}_p)$ in bijection to $GL(4, \mathbb{F}_p)/\tilde{P}_i$, where $\tilde{P}_i = \left\{\begin{pmatrix} * & * \\ 0_{i,n-1} & * \end{pmatrix} \in GL(4, \mathbb{F}_p)\right\}$ is the parabolic stabilizing the subspace generated by the first $4 - i$ basis vectors of $\mathbb{F}_p^4$. If $K(\mathbf{z})$ has exactly $j \leq i$ elementary divisors p^{-1} with respect to R_p, the number of cosets $\bar{u}$ with $\mathbf{z}u \in R_p^4 \prod_i^{-1}$ is then equal to

$$\binom{4-j}{i-j}_p\,, \quad \text{where} \quad \binom{n}{k}_p = \frac{(p^n - 1)\cdots(p^n - p^{k-1})}{(p^k - 1)\cdots(p^k - p^{k-1})}$$

is the number of k-dimensional subspaces of $\mathbb{F}_p^n$. To summarize, we have

$$f'_p(\mathbf{z}) = p^{3i} \begin{cases} \binom{4-j}{4-i}_p & \text{if } Q(\mathbf{z}) \in \mathbb{Z}_{p,\mathrm{sym}}^{(4,4)},\ K(\mathbf{z}) \subseteq p^{-1}R_p \text{ has } j \leq i \\ & \quad \text{elementary divisors } p^{-1} \text{ w.r.t. } R_p \\ 0 & \text{otherwise} \end{cases}$$

($K(\mathrm{z})$ of rank 4).

By Lemma 5 we know that with some numbers α_K

$$f'_p = p^{3i} \sum_K \alpha_K h_K$$

where h_K is the characteristic function of K^4 for a lattice $K \subseteq D_p$ and the summation runs over all $K = K(\mathbf{z})$ with $Q(z) \in \mathbb{Z}_{p,\mathrm{sym}}^{(4,4)}$. That is,

$$f'_p(\mathbf{z}) = p^{3i} \sum_{K \supseteq K(\mathbf{z})} \alpha_K\,.$$

We can thus use Theorem 1.7.2 of [Ki1] to compute the α_K. We compute first the α_K for K that are $\mathbb{Z}$-isometric to R_p.

Assume K to have $j \leq i$ elementary divisors p^{-1} w.r.t. R_p, denote by β_1 the number of unimodular lattices $M \subseteq p^{-1}R_p$ with $M \supseteq K$ and one elementary divisor p^{-1}, by β_2 the number of such lattices with two elementary divisors p^{-1} with respect to R_p. We have then

$$\alpha_K = \binom{4-j}{i-j}_p - \beta_1\binom{3}{i-1}_p - \beta_2\binom{2}{i-2}_p$$

(where the last term occurs only for $i \geq 2$).

By elementary computations in the matrix ring $M_2(\mathbf{Q}_p)$ we find that the $K \subseteq p^{-1}R_p$ (isometric to R_p) are in bijective correspondence to the pairs of cosets $(xR_p^\times, \tilde{x}R_p^\times)$ with $\det x = p = \det \tilde{x}$,

$$x, \tilde{x} \in \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} R_p^\times \cup R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times \cup R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$$

or

$$x \in R_p^\times \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} R_p^\times \quad \text{and} \quad \tilde{x} \in R_p^\times \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} R_p^\times$$

or

$$x \in R_p^\times \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} R_p^\times \quad \text{and} \quad \tilde{x} \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$$

or one of these with x, $\tilde{x}$ interchanged, or one of these with x, $\tilde{x}$ replaced by

$$\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}^{-1}, \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \tilde{x} \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}^{-1}$$

(and maybe again interchanged), the correspondence of course being given by $K = xR_p\tilde{x}^{-1}$.

We find further that

a) $j = 0$ $\beta_1 = 2$ $\beta_2 = 1$ if $\quad K = R_p$

b) $j = 1$ $\beta_1 = 1$ $\beta_2 = 1$ if $\quad x \in \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} R_p^\times, \quad \tilde{x} \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$

or $\quad x \in \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} R_p^\times, \quad \tilde{x} \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$

or $\quad x \in R_p^\times \begin{pmatrix} 0 & 1 \\ 0 & p \end{pmatrix} R_p^\times, \quad \tilde{x} \in \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} R_p^\times$

or $\quad x \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times, \quad \tilde{x} \in \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} R_p^\times$

c) $j = 1\ \beta_1 = 0\ \beta_2 = 2$ if $x \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$, $\tilde{x} \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$

or $x \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$, $\tilde{x} \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$

d) $j = 1\ \beta_1 = 1\ \beta_2 = 0$ if $x \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$, $\tilde{x} \in R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$

or $x \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$, $\tilde{x} \in R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$

e) $j = 2\ \beta_1 = 0\ \beta_2 = 1$ in all other cases

Thus

$$(7.1) \qquad \alpha_k = \begin{cases} \binom{4}{i}_p - 2\binom{3}{i-1}_p & \text{in case a)} \\ -\binom{2}{i-2}_p & \text{in case b)} \\ \binom{3}{i-1}_p - 2\binom{2}{i-2}_p & \text{in case c)} \\ 0 & \text{in all other cases} \end{cases}$$

(where the expressions involving $(i-2)$ occur only for $i \geq 2$).

The α_K for unimodular K are irrelevant for us by the same argument as in Lemma 7.4. In fact, the unimodular lattices in D_p are given by ideals having right and left order conjugate to $M_2(\mathbb{Z}_p)$. By direct matrix calculations one checks that all such ideals contained in $p^{-1}R_p$ have either left or right order containing R_p, i.e. have characteristic function invariant (on the lef tor on the right) under $(R'_p)^\times$ for an order R' strictly containing R. The same argument applies to the p-modular sublattices containing $pR_p^\#$ which are just

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} M_2(\mathbb{Z}_p) \quad \text{and} \quad M_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

We contend that $\alpha_K = 0$ for all other p-modular sublattices of $p^{-1}R_p$. To see this notice that

$$f'_p(\mathbf{z}) = f'_p(\mathbf{z}') \text{ if } \mathbf{z}' \in D_p^4 \text{ is such that } K(\mathbf{z}') = K(\mathbf{z}) + pR_p^\#.$$

The same holds (by the computations already done) for $\sum' \alpha_K h_K$, where $\sum'$ extends only over K that are unimodular or $\mathbb{Z}_p$-isometric to R_p.

For $K(\mathbf{z}) \not\supseteq pR_p^\#$, $\mathbf{z}'$ as above we have therefore

$$\begin{aligned} p^{3t} \sum_{K \supseteq K(z)} \alpha_K = f'_p(\mathbf{z}) = f'_p(\mathbf{z}') &= p^{3t} \sum{}' \alpha_K h_K(\mathbf{z}') \\ &= p^{3t} \sum{}' \alpha_K h_K(\mathbf{z}) \\ &= p^{3t} \sum_{K \supseteq K(z)}{}' \alpha_K \,. \end{aligned}$$

Taken together, $Y^{(4)}(\varphi, \psi) \mid T_i$ can be expressed as a linear combination of theta series of lattices in the genus of R with coefficients given by (3.1). We denote by τ_1 the characteristic function of $R_p^\times \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} R_p^\times$, by τ_2 that of $R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$, by τ_3 that of $R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$, write $\varphi \mid \tau_i$ for $\int_{D_p^\times} \varphi(xy^{-1}) \tau_i(y) dy$ and obtain

$$\begin{aligned} Y^{(4)}(\varphi, \psi) \mid T_i = & \, p^{3t}\left(\binom{4}{i}_p - 2\binom{3}{i-1}_p\right) Y^{(4)}(\varphi, \psi) \\ & - p^{3t}\binom{2}{i-2}_p (Y^{(4)}(\varphi \mid \tau_1, \psi \mid \tau_3) + Y^{(4)}(\varphi \mid \tau_1, \psi \mid \tau_2) \\ & \qquad + Y^{(4)}(\varphi \mid \tau_3, \psi \mid \tau_1) + Y^{(4)}(\varphi \mid \tau_2, \psi \mid \tau_1)) \\ & + p^{3t}\left(\binom{3}{i-1}_p - 2\binom{2}{i-2}_p\right)(Y^{(4)}(\varphi \mid \tau_3, \psi \mid \tau_2) + Y^{(4)}(\varphi \mid \tau_2, \psi \mid \tau_3)) \,. \end{aligned}$$

By our assumptions on φ, ψ we can compute the $\varphi \mid \tau_i$, $\psi \mid \tau_i$ by considering the action of these Hecke operators on the unique R_p-invariant function g of the unique irreducible subrepresentation of the representation of $GL_2(\mathbf{Q}_p)$ induced from the character

$$\begin{pmatrix} t_1 & s \\ 0 & t_2 \end{pmatrix} \longmapsto \xi(t_1 t_2) \mid t_1 \mid_p^{1/2} \mid t_2 \mid_p^{-1/2} .$$

on its Borel subgroup.

By § 3 of [Ca2] this can be given by

$$g\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = -p, \quad g\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = 1$$

$\left(\text{together with } g\left(\begin{pmatrix} t_1 & s \\ 0 & t_2 \end{pmatrix} x\right) = \xi(t_1 t_2) \mid t_1 \mid_p \mid t_2 \mid_p^{-1} g(x) \text{ this determines } g \text{ uniquely}\right)$. An elementary computation yields then

$$\varphi \mid \tau_1 = -\xi(p)\varphi$$

$$\varphi \mid \tau_2 = \varphi \mid \tau_3 = \xi(p)\varphi \,.$$

ξ being a quadratic character we obtain on substituting this into the expression for $Y^{(4)}(\varphi, \psi)$:

$$Y^{(4)}(\varphi, \psi) \,|\, T_i = p^{3i}\binom{4}{i}_p Y^{(4)}(\varphi, \psi)\,,$$

which is just the same eigenvalue as the one obtained in the non-split case. Using Lemma 7.5 again we obtain the assertion.

COROLLARY 7.1. *Let* $F = Y^{(n)}(\varphi, \psi)$. *Then the function* $\Lambda_N(s)$ *of section 2 is given by*

$$\Lambda_N(s) = \Lambda_N^{\mathrm{triv}}(s) \prod_{p|N} \prod_{j=1}^{n} (1 - p^{-s-2+j})^{-1}\,.$$

Note added (Nov. 1990). We can now also prove a version of Lemma 7.6 and of Corollary 7.1 in the case that only φ is essential (this weakened requirement on the pair (φ, ψ) basically means that $\varphi \otimes \psi$ gives rise to an "essential" form on the orthogonal group of D (or rather on the spin group) in a sense which has of course to be made more precise). To formulate a statement in this case let $\psi = \psi_1 + \varepsilon_p \psi_2$ where $w_p \varphi = \varepsilon_p \varphi$ and where ψ_1 is right invariant under a maximal order $\tilde{R}_p \supseteq R_p$, $\psi_2 = w_p \psi_1$, and let $\alpha_p, \bar{\alpha}_p$ be the p-Satake parameters of ψ_1 (and ψ_2) (normalized to $\alpha_p \bar{\alpha}_p = p$), $\mu_p = \alpha_p + \bar{\alpha}_p$.

Then with $F = Y^{(n)}(\varphi, \psi)$ we have

$$\langle F \,|\, T_i^{(n)}, F\rangle = p^{(n-1)i}\left(\binom{n}{i}_p - p^{i-1}(p + 1 + \varepsilon_p \mu_p)\binom{n-2}{i-1}_p\right)\langle F, F\rangle\,.$$

The proof is similar to that of Lemma 7.6; one gets that

$$F \,|\, T_i^{(n)} - p^{(n-1)i}\left(\binom{n}{i}_p - p^{i-1}(p + 1 + \varepsilon_p \mu_p)\binom{n-2}{i-1}_p\right)F$$

is a linear combination of theta series of lattices on D of discriminants different from N^2 and hence orthogonal to F (see Lemma 9.1 below).

Furthermore, in the discussion of section 2 assume F only to be an eigenform for all the $T_N(\mathrm{M})$ with $\mathrm{M} = N\mathrm{M}_1$, $(\det(\mathrm{M}_1), N) = 1$ and define $\lambda_F(\mathrm{M})$ for any $\mathrm{M} \in M_n(\mathbb{Z})^*$ with $\mathrm{M} \equiv 0 \bmod N$ by $\langle F, T_N(\mathrm{M})F\rangle = \lambda_F(\mathrm{M})\langle F, F\rangle$. Then (with $F = Y^{(n)}(\varphi, \psi)$ again) the p-factor of the function $\Lambda_N(s)$ from Theorem 2.1 is given by

$$(1 + \varepsilon_p \alpha_p p^{-s-1})^{-1}(1 + \varepsilon_p \bar{\alpha}_p p^{-s-1})^{-1} \prod_{j=3}^{n} (1 - p^{-s-2+j})^{-1}\,.$$

If now $\psi \in \mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times})$ is arbitrary we use the decomposition

$$\mathscr{A}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}) = \mathbb{C}\cdot 1 \oplus \bigoplus_{R' \supseteq R} \mathscr{A}_{\mathrm{new}}(D_{\mathbb{A}}^{\times}, (R'_{\mathbb{A}})^{\times})$$

from [Hi-Sa] (where $\mathscr{A}_{\mathrm{new}}(D_{\mathbf{A}}^{\times}, (R'_{\mathbf{A}})^{\times})$ denotes the set of cusp forms in the essential part of $\mathscr{A}(D_{\mathbf{A}}^{\times}, (R'_{\mathbf{A}})^{\times})$) and assume (as always) that ψ is an eigenfunction of all the w_l $(l|N)$ having the same eigenvalues ε_l as φ.

There are two maximal orders $\tilde{R}_p$, $\bar{R}_p$ in D_p containing R_p, and for each $R' \supseteq R$ in the above decomposition R'_p equals one of $\tilde{R}_p$, $\bar{R}_p$, R_p. Writing

$$\begin{aligned}\psi &= c + \sum_{R' \supseteq R} \psi_{R'} \\ &= c + \sum_{R'_p = \tilde{R}_p} \psi_{R'} + \sum_{R'_p = \bar{R}_p} \psi_{R'} + \sum_{R'_p = R_p} \psi_{R'} \\ &= c + \psi_1 + \psi_2 + \psi_3\end{aligned}$$

we find that ψ_3 can be treated as in the proof of Lemma 7.6, while c and $\psi_1 + \varepsilon_p \psi_2$ are of the shape discussed above (and ψ_1 can be assumed to be an eigenform of the Hecke operator $\tilde{T}_p$).

§ 8. Cuspidality properties of theta series

We will need some results connecting the behaviour of theta series of degree n in the $\Gamma_0^{(n)}(N)$-inequivalent boundary components of $\mathbb{H}_n$. Although we don't believe these to be new, we sketch the proofs for lack of a reference.

As usual, for a field K we consider the standard maximal parabolic subgroups $P_{n,r}(K)$ of $Sp(n, K)$ where $P_{n,r}(K)$ has Levi factor $Sp(n-r, K) \times GL(r, K)$ embedded by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, T\right) \longrightarrow \begin{bmatrix} A & 0 & B & 0 \\ 0 & T & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & T'^{-1} \end{bmatrix}$$

(for $K = \mathbb{Q}$ we have then $C_{n,n-r} = Sp(n, \mathbb{Z}) \cap P_{n,r}(\mathbb{Q})$).

LEMMA 8.1. *Let $N = p_1 \cdots p_t$, for $0 \le l_i \le r$ $(i = 1, \cdots, t)$ let $R_N^{(n)}(l_1, \cdots, l_t)$ be given by*

$$R_N^{(n)}(l_1, \cdots, l_t) \equiv \begin{bmatrix} 1_{n-l_i} & 0 & 0_{n-l_i} & 0 \\ 0 & 0_{l_i} & 0 & -1_{l_i} \\ 0_{n-l_i} & 0 & 1_{n-l_i} & 0 \\ 0 & 1_{l_i} & 0 & 0_{l_i} \end{bmatrix} \bmod p_i\mathbb{Z} \qquad (i = 1, \cdots, t)\,.$$

Then

$$Sp(n, \mathbb{Z}) = \bigcup_{l_1,\cdots,l_t=0}^{r} \Gamma_0^{(n)}(N) R_N^{(n)}(l_1, \cdots, l_t) C_{n,n-r}\,.$$

Proof. Dividing first by the principal congruence subgroup $\Gamma^{(n)}(N)$ and using

$$Sp(n, \mathbb{Z})/\Gamma^{(n)}(N) = \prod_{p \mid N} Sp(n, \mathbb{F}_p)$$

we see that we have to determine representatives of the double cosets in

$$P_{n,n}(\mathbb{F}_{p_i}) \backslash Sp(n, \mathbb{F}_{p_i}) / P_{n,r}(\mathbb{F}_{p_i})\,.$$

That these are just the images of the $R_N^{(n)}(\cdots, l_i, \cdots)$ under reduction mod p_i is a well known consequence of the Bruhat decomposition (see e.g. [War], p. 49).

Remark. The double coset of $R_p^{(n)}(r)$ is also represented by $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ mod $p\mathbb{Z}$. Especially for $r = 1$ it may sometimes be more convenient to use this representative.

LEMMA 8.2. *Let L be an integral positive definite lattice on $\mathbb{Q}^m$ with quadratic form Q and associated bilinear form $B(x, y) = Q(x + y) - Q(x) - Q(y)$ of level dividing N (i.e. $NQ(L^\#) \subseteq \mathbb{Z}$), discriminant d, $\vartheta^{(n)}(L)$ its theta series of degree n. Denote by d_p the highest power of p dividing d. Let $R_p^{(n)}(l) = R_N^{(n)}(l_1, \cdots, l_t)$ with $l_i = 0$ for $p_i \neq p$, $l_i = l$ for $p_i = p$,*

$$(L^{\#,p}) = L^\# \cap \mathbb{Z}\left[\frac{1}{p}\right]L, \quad \vartheta^{(n-l,l)}(L, L^{\#,p}; Z) = \sum_{\mathbf{x}} \exp(2\pi i tr\, Q(x)Z)$$

where $\mathbf{x} = (x_1, \cdots, x_n)$ runs over $(x_1, \cdots, x_{n-l}) \in L^{n-l}$, $(x_{n-l+1}, \cdots, x_n) \in (L^{\#,p})^l$ and $Q(\mathbf{x})_{ij} = \frac{1}{2}B(x_i, x_j)$ as usual. Then

a) $$\vartheta^{(n)}(L) \mid R_p^{(n)}(l) = \gamma_p(d_p)^l s_p(Q)^l d_p^{-1/2} \vartheta^{(n-l,l)}(L, L^{\#,p})$$

where $\gamma_p(d_p)$ depends only on $d_p(\mathbb{Q}_p^\times)^2$ and $s_p(Q)$ is the Hasse invariant ([OM], §63)

b) $\vartheta^{(n)}(L) \mid R_p^{(n)}(l) \mid \Phi^r = d_p^{-r/2} \gamma_p(d_p)^r s_p(Q)^r \vartheta^{(n-r)}(L) \mid R_p^{(n-r)}(l - r)$ *for* $l \geq r$.

Proof. b) is an immediate consequence of a). a) is most easily proved using the Weil- or oscillator-representation. It is then a consequence of the explicit formulas for the action of elements of $Sp(nm, \mathbb{Q}_p)$ given in [Sa], [Pe], Prop. 2.14 and [Rao], Lemma 3.2.

Remark. Statement a) above was proved for $n=1$ (in a somewhat different formulation) by Kitaoka [Ki3].

A consequence of the Lemma is:

THEOREM 8.1. *Let $F=\sum \alpha_i \vartheta_{S_i}^{(n)}$ be a linear combination of theta series of quadratic forms of square free level belonging to the same genus. Then F is a cusp form if and only if $F|\Phi=0$.*

Remark. Note that Theorem 8.1 is different from the corollary to Theorem I.1.1 of [Ra3]. Rallis assumes that the $(n-1)$-st theta lifting of a form on the orthogonal group vanishes for *all* test functions and obtains cuspidality of the n-th theta lifting for *all* test functions, whereas we consider one specific test function throughout.

We notice that Lemma 2 allows to determine the part in the space of Klingen-Eisenstein series of a linear combination of theta series that is not cuspidal. In order to state the result we need some more notations.

Recall that for any discrete subgroup Γ of $Sp(n, \mathbb{Q})$ commensurable with $Sp(n, \mathbb{Z})$, $R \in Sp(n, \mathbb{Z})$ and $f \in S_r^k(R^{-1}\Gamma R \cap C_{n,r})_0$ one has the Klingen-Eisenstein series

$$E_{n,r}^k(f, \Gamma, R) := \sum_{M \in (R^{-1}\Gamma R \cap C_{n,r} \backslash R^{-1}\Gamma)} f(M\langle Z\rangle^*) j(M, Z)^{-k} .$$

(Here for $G \subseteq C_{n,r}$ we denote by G_0 the intersection of G with the $Sp(r)$-component of $C_{n,r}$ and by Z^* the upper left hand corner of size $r \times r$ of Z).

This series converges for $k > n+r+1$ and satisfies for $R=\gamma R_0 c$ ($\gamma \in \Gamma$, $R_0 \in Sp(n, \mathbb{Z})$, $c \in C_{n,r}$):

$$E_{n,r}^k(f, \Gamma, \gamma R_0 c) = E_{n,r}^k(f|c_0^{-1}, \Gamma, R_0)$$

(where again c_0 is the $Sp(r)$-part of c). That is, the double coset decomposition of $\Gamma \backslash Sp(n, \mathbb{Z})/C_{n,r}$ determines the Eisenstein series to be considered.

Furthermore, they satisfy

$$E_{n,r}^k(f, \Gamma, R)|L|\Phi^{n-r} = \begin{cases} f & \text{if } R \text{ and } L \text{ are in the same double cosets} \\ 0 & \text{otherwise}. \end{cases}$$

These facts, taken together with Lemma 2, yield

THEOREM 8.2. *Let $f=\sum \alpha_i \vartheta_{S_i}^{(r)} \in S_r^k(N)$ be a linear combination of theta series of positive integral quadratic forms in $2k$ variables in the same* *genus of square free level* $N = p_1 \cdots p_t$ *and discriminant* d *and assume* $k > n + r + 1$. *Let*

$$F^{(n)} = \sum_i \sum_{l_1,\cdots,l_t=0}^{n-r} \prod_{j=1}^{t} d_{p_j}^{-l_j/2} \gamma_{p_j}(S_i)^{l_j} s_{p_j}(S_i)^{l_j} E_{n,r}^k(f, \Gamma_0^{(n)}(N), R_N^{(n)}(l_1, \cdots, l_t)^{-1}) .$$

Then $(F^{(n)} - \sum \alpha_i \vartheta_{S_i}^{(n)}) | R | \Phi^{n-r} = 0$ *for all* $R \in Sp(n, \mathbb{Z})$.

Remark. By a result of S. Kudla (that has not yet been published) the following is true:

Let K be a lattice on the $2k$-dimensional quadratic space V over $\mathbb{Q}$, let $O_{\mathbb{A}}(V) = \bigcup O(V) h_i O_{\mathbb{A}}(K)$ be a double coset decomposition such that $h_i K$ corresponds to the quadratic form S_i and let φ be the right $O_{\mathbb{A}}(K)$-invariant automorphic form on $O_{\mathbb{A}}(V)$ with $\varphi(h_i) = \alpha_i$. Assume further that the form φ is orthogonal to all forms on $O_{\mathbb{A}}(V)$ whose lift to $Sp(r, \mathbb{A})$ vanishes. Then $\sum \alpha_i \vartheta_{S_i}^{(r+1)}$ is orthogonal to $S_{r+1}^k(N)$.

The above theorem then implies

$$F^{(r+1)} = \sum \alpha_i \vartheta_{S_i}^{(r+1)} \qquad \text{if } k > 2r + 2 ,$$

which can be viewed as an extension of Siegel's main theorem. We will arrive at a similar result for Yoshida's lifting in section 10.

§ 9. Non-vanishing

In [Y1], [Y2] Yoshida conjectured that for cuspidal $\varphi, \psi \in \mathscr{A}^\varepsilon(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times)$, $\varphi \neq 0 \neq \psi$ and R maximal in D the lift $Y^{(n)}(\varphi, \psi)$ is different from zero. We shall now prove this conjecture, including the case of Eichler orders under our usual condition that φ and ψ are in the essential part of $\mathscr{A}(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times)$. Thus, for the rest of this section we assume $\varphi, \psi \in \mathscr{A}^\varepsilon(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times)$ to be as in section 7. As always $N = N_1 N_2$ is the level of R, D is split at the prime p if and only if $p \nmid N_1$.

Denote by $\mathscr{A}_{\text{ess}}$ the essential part of $\mathscr{A}(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times)$. By $\Theta_{N_1,N_2}^{(n)}$ we denote the subspace of $\Theta^{(n)}(4, N) \cap M_n^2(N)^{\text{triv}}$ generated by the $Y^{(n)}(\varphi, \psi)$ with $\varphi, \psi \in \mathscr{A}_{\text{ess}}$.

Note added (Nov. 1990). The improvements sketched in the note at the end of section 7 allow us to weaken the assumption $\varphi \in \mathscr{A}_{\text{ess}}$ here too.

Since we are considering only theta series of quadratic forms in one genus we have a splitting

$$\Theta_{N_1,N_2}^{(n)} = \bigoplus_{j=0}^{n} \Theta_{N_1,N_2}^{(n,j)} \tag{9.1}$$

defined inductively by

$$\Theta^{(n,n)}_{N_1,N_2} = \{F \in \Theta^{(n)}_{N_1,N_2} \mid \Phi F = 0\} = \Theta^{(n)}_{N_1,N_2} \cap S^2_n(N)^{\mathrm{triv}} \quad (\text{by } \S 8)$$

and

$$(\Theta^{(n,n)}_{N_1,N_2})^{\perp} \cong \Theta^{(n-1)}_{N_1,N_2} \qquad \text{by the } \Phi\text{-operator}$$

where $\perp$ means the orthogonal complement in $\Theta^{(n)}_{N_1,N_2}$.
(N.B.: Forms in $(\Theta^{(n,n)}_{N_1,N_2})^{\perp}$ need not be orthogonal to all cusp forms).

Thanks to a result of Kitaoka [Ki2] we have $\Theta^{(n)}_{N_1,N_2} \cong \Theta^{(3)}_{N_1,N_2}$ for $n \geq 3$, so the case $n = 3$ of the splitting contains all information about such splittings for any n.

We shall prove that this splitting can also be described in terms of properties of automorphic L-functions.

Results of similar type—but only for "large" weights—were obtained by Harris [Ha1], [Ha2]. Finally we shall describe the splitting in terms of the Yoshida lifting.

LEMMA 9.1. a) $Y^{(n)}(\varphi, \psi) \mid K^{(n)}_p \begin{pmatrix} 0 & -1_n \\ p1_n & 0 \end{pmatrix} K^{(n)}_p = \varepsilon(w_p) \gamma^n_p s_p(D)^n Y^n(\varphi, \psi)$

b) *If S is an integral positive quaternary quadratic form with $\vartheta^{(n)}_S \in \Theta^{(n)}(4, N)$ with* $\det S \neq N_1^2 N_2^2$ *then*

$$\vartheta^{(n)}_S \textit{ is orthogonal to } \Theta^{(n,n)}_{N_1,N_2}\,.$$

c) *If $n = 2$,* $\det S = N_1^2 N_2^2$ *but S is not in the genus of (R, n), then $\vartheta^{(2)}_S$ is orthogonal to $\Theta^{(2,2)}_{N_1,N_2}$.*

Proof. a) As in § 8 we have

$$\begin{aligned}
&Y^{(n)}(\varphi, \psi) \mid K^{(n)}_p \begin{pmatrix} 0 & -1_n \\ p1_n & 0 \end{pmatrix} K^{(n)}_p(Z) \\
&= \gamma^n_p s_p(D)^n \sum_{i,j} \frac{\varphi(y_i)\psi(y_j)}{e_i e_j} \vartheta^{(n)}(I^{\#,p}_{ij}, pZ) \\
&= \gamma^n_p s_p(D)^n \sum_{i,j} \frac{\varphi(y_i)\psi(y_j)}{e_i e_j} \vartheta^{(n)}(y_i \pi_p^{-1} R y_j^{-1}, pZ) \\
&= \gamma^n_p s_p(D)^n \sum_{i,j} \frac{\varphi(y_i)\psi(y_j)}{e_i e_j} \vartheta^{(n)}(z_p y_i \pi_p^{-1} R y_j^{-1}, Z) \\
&= \gamma(w_p) \varepsilon^n_p s_p(D)^n Y^{(n)}(\varphi, \psi)
\end{aligned}$$

(where π_p denotes the element of $D^{\times}_{\mathbb{A}}$ with $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ or a prime element of D_p in the p-component, 1 in all others and $z_p \in D^{\times}$ with $n(z_p) = p$).

b) Assume that $p^4 \mid \det S$ for some $p \mid N_2$ (i.e. S is divisible by p). Then using the formula for the action of $K_p^{(n)}\begin{pmatrix} 0 & -p^{-1}1_n \\ p1_n & 0 \end{pmatrix} K_p^{(n)}$ from § 7 we see that $\vartheta_S^{(n)}$ is an eigenfunction of $K_p^{(n)}\begin{pmatrix} 0 & -p^{-1}1_n \\ p1_n & 0 \end{pmatrix} K_p^{(n)}$ with eigenvalue $-s_p(D)^n \gamma_p^n p^{n(n-1)/2}$. Since this operator is hermitian and $\Theta_{N_1,N_2}^{(n)}$ is an eigenspace with eigenvalue $-s_p(D)^n \gamma_p^n p^{n(n-1)2}$, we see that $\vartheta_S^{(n)}$ must be orthogonal to $\Theta_{N_1,N_2}^{(n,n)}$.

If S is primitive there is some $p \mid N_2$ with $p \nmid \det S$. Writing again $p^{-1}S^* = S^{\#,p}$ and using a) we see

$$\langle Y^{(n)}(\varphi, \psi), \vartheta_S^{(n)} \rangle = \varepsilon(w_p)\gamma_p^{-n} s_p(D)^{-n} p \langle Y^{(n)}(\varphi, \psi), \vartheta_{S^*}^{(n)} \rangle = 0 .$$

c) In this case there is some $p \mid N$ for which S is anisotropic over $\mathbf{Q}_p$ and D_p split or vice versa.

In any case $\vartheta_S^{(2)}$ is (modulo theta series of imprimitive forms that are orthogonal to $\Theta_{N_1,N_2}^{(2,2)}$ by b)) an eigenfunction of $K_p^{(2)}\begin{pmatrix} 0 & -p^{-1}1_2 \\ p1_2 & 0 \end{pmatrix} K_p^{(2)}$ with eigenvalue $\pm \gamma_p^2 p^3$ while $\Theta_{N_1,N_2}^{(2)}$ has the negative of this number as eigenvalue. Again the assertion follows since the operator is hermitian.

Remark. We should note that Lemma 1 remains true if only one of φ, ψ is in the essential part of $\mathscr{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times)$ since the results about $T_N(N1_n)$ used for its proof are valid under this weaker condition (see the remark after Lemma 7.4).

THEOREM 9.1. *Suppose $F \in \Theta_{N_1,N_2}^{(3)}$ is an eigenform of all Hecke operators $T_N(M)$, $M \equiv 0 \bmod N$. Let $N = p_1 \cdots p_l$. Then*

(i) *$F \in \Theta_{N_1,N_2}^{(3,3)}$ implies that $D_F^{(N)}$ is of order at least l in $s = 1$ and has a simple pole in $s = 2$*

(ii) *$F \in \Theta_{N_1,N_2}^{(3,2)}$ implies that $D_F^{(N)}$ is of order $l - 1$ in $s = 1$ and has at most a simple pole in $s = 2$*

(iii) *$F \in \Theta_{N_1,N_2}^{(3,1)}$ implies that $D_F^{(N)}$ is of order $l - 2$ in $s = 1$ and has a simple pole in $s = 2$*

(iv) *$F \in \Theta_{N_1,N_2}^{(3,0)}$ implies that $D_F^{(N)}$ is of order $2l - 3$ in $s = 1$ and has a double pole in $s = 2$.*

Evidently the converses of these statements are also true (in the applications we have in mind only the behaviour in $s = 1$ will be relevant).

Proof. To prove ii), iii), iv) we use the Zharkovskaya relations [Zh]

(9.2) $$D_F^{(N)}(s) = \zeta^{(N)}(s-1)\zeta^{(N)}(s+1)D_{\Phi F}^{(N)}(s) \qquad \text{if } \Phi F \neq 0$$

(9.3) $$D_F^{(N)}(s) = \zeta^{(N)}(s-1)\zeta^{(N)}(s+1)\zeta^{(N)}(s)^2 D_{\Phi^2 F}^{(N)}(s) \qquad \text{if } \Phi^2 F \neq 0$$

(9.4) $$D_F^{(N)}(s) = \zeta^{(N)}(s-1)^2\zeta^{(N)}(s+1)^2\zeta^{(N)}(s)^3 \qquad \text{if } \Phi^3 F \neq 0$$

ii) For $F \in \Theta^{(3,2)}_{N_1,N_2}$ we have $\Phi F \in \Theta^{(2,2)}_{N_1,N_2} \subseteq \Theta^{(2)}(4, N)$ and ΦF is a cusp-form. Combining (9.2) with Theorem 4.1 and Lemma 9.1 we get the first part of ii).

iii) Here we have

$$D_{\Phi^2 F}^{(N)}(s) = \zeta^{(N)}(2s) \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{\lambda(n^2)}{n^{s+1}}$$

where $\lambda(n^2)$ is the eigenvalue of the Hecke operator $T(n^2)$ (in the usual notation for elliptic modular forms) for the cusp form $\Phi^2 F$.

By a standard reasoning (Rankin-Selberg-method) the value of this Dirichlet series at $s = 1$ is essentially the square of the Petersson norm of $\Phi^2 F$ and therefore different from zero; the behaviour in $s = 2$ is clear. This proves iii).

iv) follows from (9.4).

The second assertion of i) follows from Theorem 4.1 and Lemma 9.1. To obtain the result about the behaviour in $s = 1$ we recall the integral representation for $D_F^{(N)}(s)$ obtained in section 4:

$$\left\langle F, E_6^2\left(\begin{pmatrix} * & 0 \\ 0 & -\bar{Z}\end{pmatrix}, \bar{s}, N\right)\right\rangle = \mu(3,2,s)\frac{\lambda_F(N\cdot 1_N)\Lambda_N(2s-1)D_F^{(N)}(2s-1)}{N^{3(3+2s)}\zeta^{(N)}(2s+2)\prod_{i=1}^{3}\zeta^{(N)}(4s+4-2i)}F(Z)$$

In $s = 1$ we have a simple pole for $\mu(3, 2, s)$ and a pole of order l for

$$\Lambda_N(2s-1) = \prod_{q\mid N}\prod_{j=1}^{3}\frac{1}{(1-q^{-2s-1+j})}\,.$$

To prove i) we must show the crucial

PROPOSITION 9.1. *$E_6^2(Z, s, N)$ has a pole of order* 1 *at* $s = 1$.

Proof. We prefer to prove the same statement of $F_6^2(Z, s, N)$. From section 3 we see that the constant term in the Fourier expansion of $F_6^2(Z, s, N)$ indeed has a pole of first order. On the other hand we know by the results of Feit ([Fe], Theorem 9.1. b) 3.) that

$$\gamma(s)L(s)F_6^2(Z, s, N)$$

has at most poles of first order. Here $L(s)$ is a product of (shifted) L-series with the property that $s = 1$ is still in the domain of absolute convergence of their Euler product; $\gamma(s)_{s=1}$ is a product of Γ-factors evaluated at positive arguments.

The second part of ii) follows in a similar way from the fact that $F_4^2(Z, s, N)$ is regular in $s = 1$ (which is implied by [Fe], 9.1. b) 2.).

In [Y1], [Y2] Yoshida conjectured that (for any ε) the lift $Y^2(\varphi, \psi)$ is different from zero if $\varphi, \psi \neq 0$ are elements of $\mathscr{A}^{\varepsilon}_{\mathrm{cusp}}(D^\times_{\mathbf{A}}, R^\times_{\mathbf{A}})$.

We shall verify (a strong version of) Yoshida's conjecture by using the splitting (9.1) of $\Theta^{(3)}_{N_1,N_2}$ and solaing the following

Problem. Suppose $\varphi, \psi \in \mathscr{A}^{\varepsilon}(D^\times_{\mathbf{A}}, R^\times_{\mathbf{A}})$ are eigenfunctions of the Hecke algebra, both different from zero and in the essential part. Determine the subspace $\Theta^{(3,i)}_{N_1,N_2} (0 \leq i \leq 3)$ in which $Y^{(3)}(\varphi, \psi)$ lies!

From the results obtained so far it is clear that $Y^{(3)}(\varphi, \psi)$ does not vanish and that $Y^{(3)}(\varphi, \psi)$ must lie in one of the subspaces $\Theta^{(3,i)}_{N_1,N_2}$. Moreover, according to the Theorem of I.4. the automorphic L-function $D_F^{(N)}(s)$ with $F := Y^{(3)}(\varphi, \psi)$ should contain the solution of this problem.

So let $f, g \in M_1^2(N)$ be modular forms corresponding to φ and ψ and let $a(p) = \alpha_p + \bar{\alpha}_p$, $b(p) = \beta_p + \bar{\beta}_p$ $(\alpha_p\bar{\alpha}_p = \beta_p\bar{\beta}_p = p)$ be the corresponding eigenvalues of the Hecke operators $T(p), p\nmid N$. (Note that this normalization is different from the one used in § 6). We need the L-series

$$\begin{aligned}
L^{(N)}_{\mathrm{sym}}(\varphi, \psi, s) &:= \prod_{p\nmid N} \frac{1}{(1-\alpha_p\beta_p p^{-s})(1-\alpha_p\bar{\beta}_p p^{-s})(1-\bar{\alpha}_p\beta_p p^{-s})(1-\bar{\alpha}_p\bar{\beta}_p p^{-s})} \\
&= \zeta^{(N)}(2s-2) \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{a(n)b(n)}{n^s} \\
&= L^{(N)}_{\varphi,\psi}(s-1)\,.
\end{aligned}$$

(see Corollary 6.1 for the definition of $L^{(N)}_{\varphi,\psi}(s)$).

In special cases we have

$$L^{(N)}_{\mathrm{sym}}(1, 1, s) = \zeta^{(N)}(s)\zeta^{(N)}(s-1)^2\zeta^{(N)}(s-2)$$

and

$$L^{(N)}_{\mathrm{sym}}(1, \psi, s) = L^{(N)}(\psi, s)L^{(N)}(\psi, s-1)$$

where

$$L^{(N)}(\psi, s) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{b(n)}{n^s} = \prod_{p \nmid N} \frac{1}{(1-\beta_p p^{-s})(1-\bar{\beta}_p p^{-s})}\,.$$

From the results of § 6 we have

$$D_F^{(N)}(s) = \zeta^{(N)}(s-1)\zeta^{(N)}(s+1)\zeta^{(N)}(s)L_{\mathrm{sym}}^{(N)}(\varphi, \psi, s+1)\,.$$

Several cases have to be considered separately:

(i) φ and ψ both constant. Then $D_F^{(N)}(s)$ is of order $2l-3$ in $s=1$ and has a double pole in $s=2$.

(ii) φ constant, ψ cuspidal (and vice versa). Since $L^{(N)}(\psi, 2)L^{(N)}(\psi, 3)$ is different from zero, we get a simple pole for $D_F^{(N)}(s)$ in $s=2$. Moreover, the order of $D_F^{(N)}(s)$ in $s=1$ is equal to $l-1+\operatorname{order}_{s=1} L^{(N)}(\psi, s)$.

(iii) φ, ψ both cuspidal, φ not proportional to ψ. Then $D_F^{(N)}(s)$ has a simple pole in $s=2$. We claim that the order of $D_F^{(N)}(s)$ in $s=1$ is indeed $l-1$ or equivalently:

$$L_{\mathrm{sym}}^{(N)}(\varphi, \psi, 2) \neq 0\,.$$

This is precisely what Theorem 4 of Ogg [O] says.

(iv) φ and ψ cuspidal, $\varphi = c\psi$ $(c \neq 0)$. Then $D_F^{(N)}(s)$ has a simple pole in $s=2$ and $D_F^{(N)}(s)$ is of order $l-2$ in $s=1$ since in this case $L_{\mathrm{sym}}^{(N)}(\varphi, \psi, s)$ has a first order pole in $s=2$, the residue is essentially the square of the Petersson norm of f (see e.g. [O]).

We have thus completely solved the problem stated above. We summarize our results, giving at the same time a description of the spaces $\Theta_{N_1,N_2}^{(3,i)}$ in terms of the lift $Y^{(3)}$.

THEOREM 9.2. *Denote by* $\mathscr{A}_{\mathrm{new}}^{\varepsilon}$ *the set of cusp forms in the essential part of* $\mathscr{A}^{\varepsilon}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$, *let* $\{\psi_i\} \subset \mathscr{A}_{\mathrm{new}}^{\varepsilon_i}$ *be a basis of* $\mathscr{A}_{\mathrm{new}}$ *consisting of eigenfunctions of the Hecke algebra (if* R *is maximal, this basis has* $h-1$ *elements,* h *the class number of* R*).*

Then the spaces $\Theta_{N_1,N_2}^{(3,i)}$ *have the following bases (which are mapped to orthogonal bases of* $\Theta_{N_1,N_2}^{(i,i)}$ *under* Φ^{3-i}*)*

$\Theta_{N,1}^{(3,0)}$ $\{Y^{(3)}(1,1)\}$ $(\Theta_{N_1,N_2}^{(3,0)} = \{0\}$ *for* $N_2 \neq 1)$

$\Theta_{N_1,N_2}^{(3,1)}$ $\{Y^{(3)}(\psi_i, \psi_i)\}$

$\Theta_{N_1,N_2}^{(3,2)}$ $\{Y^{(3)}(\psi_i, \psi_j) \mid i<j,\ \varepsilon_i = \varepsilon_j\}$

$$\subset \begin{cases} \emptyset \ \textit{if}\ N_2 \neq 1 \\ \{Y^{(3)}(1, \psi_j) \mid \varepsilon_j = \varepsilon_0,\ L^{(N)}(\psi_j, 1) \neq 0\} \ \textit{if}\ N_2 = 1 \end{cases}$$

$$\Theta_{N_1}^{(3,3)} \quad \{Y^{(3)}(1, \psi_j) \mid \varepsilon_j = \varepsilon_0, L^{(N)}(\psi_j, 1) = 0\} \qquad (\Theta_{N_1,N_2}^{(3,3)} = \{0\} \text{ if } N_2 \neq 1).$$

Proof. Everything has already been proved except for the orthogonality of the images under Φ^{3-i} of the basis elements. But this follows from strong multiplicity one for $\mathscr{A}_{\mathrm{new}}$ which implies (by the results of § 6) that $Y^{(n)}(\psi_i, \psi_j)$ and $Y^{(n)}(\psi_k, \psi_l)$ give (if they are nonzero) different homomorphisms $\mathscr{H}(GSp_n(\mathbf{Q}_p), Sp_n(\mathbf{Z}_p)) \to \mathbf{C}^\times$ for infinitely many primes p unless $\{i, j\} = \{k, l\}$.

COROLLARY 9.1. *Yoshida's conjecture is true, moreover if $N_2 = 1$ (i.e. R is a maximal order), $0 \neq \psi \in \mathscr{A}_{\mathrm{cusp}}^{\varepsilon_0}$ an eigenform we have $Y^{(2)}(1, \psi) = 0$ if and only if $L^{(N)}(\psi, 1) = 0$.*

COROLLARY 9.2. *Let $N_2 = 1$. Then all nontrivial linear relations between the theta series $\vartheta^{(2)}(I_{ij})$ are of the type*

$$\sum_{i,j} \frac{\varphi(y_i) + \varphi(y_j)}{e_i e_j (1 + \delta_{ij})} \vartheta^{(2)}(I_{ij}) = 0$$

where φ is in the subspace of $\mathscr{A}^{\varepsilon_0}(D_{\mathrm{A}}^\times, R_{\mathrm{A}}^\times)$ spanned by those Hecke eigenforms φ with $L^{(N)}(\varphi, 1) = 0$. Furthermore, $\vartheta^{(2)}(I_{ij}) = \vartheta^{(2)}(I_{kl})$ implies $(I_{ij}, n) \cong (I_{kl}, n)$.

Proof. Let $\{K_\kappa\}$ be a set of representatives of the nonisometric lattices among the I_{ij}, $\sum \alpha_\kappa \vartheta^{(2)}(K_\kappa) = 0$, not all $\alpha_\kappa = 0$. Then $\sum \alpha_\kappa \vartheta^{(3)}(K_\kappa) \neq 0$ is a cusp form, thus (by Theorem 9.2) equal to $Y^{(3)}(1, \varphi)$ with φ as asserted, which proves the first part of the corollary. If $\vartheta^{(2)}(I_{ij}) = \vartheta^{(2)}(I_{kl})$, then (by the first part) there is $\varphi \in \mathscr{A}^{\varepsilon_0}(D_{\mathrm{A}}^\times, R_{\mathrm{A}}^\times)$ such that

$$\begin{aligned}
1 &= \frac{2^t}{\#(\mathrm{Fix}(i) \cap \mathrm{Fix}(j))} \frac{\varphi(y_i) + \varphi(y_i)}{e_i e_j (1 + \delta_{ij})} \\
-1 &= \frac{2^t}{\#(\mathrm{Fix}(k) \cap \mathrm{Fix}(l))} \frac{\varphi(y_k) + \varphi(y_l)}{e_k e_l (1 + \delta_{kl})} . \\
&\varphi(y_\nu) + \varphi(y_\mu) = 0 \qquad \text{if } I_{ij} \not\cong I_{\nu\mu} \not\cong I_{kl} .
\end{aligned}$$

(in the notation of § 5). Assume that $j \neq i$ and (say) $\varphi(y_j) > 0$. Then $\varphi(y_j) + \varphi(y_j) > 0$ and $I_{jj} \not\cong I_{ij}$, a contradiction.

Thus $j = i$, $k = l$ and (again using the notation of § 5) $j \neq \eta(k)$ for all η, since otherwise $\varphi(y_j) = \varphi(y_k)$. If $1 \leq \mu \leq h$ is such that $j \neq \mu \neq k$, then $\varphi(y_\mu) = 0$ implies that $\alpha_\kappa \neq 0$ for $K_\kappa \cong I_{j\mu}$, a contradiction.

If on the other hand $\varphi(y_\mu) \neq 0$, then $\alpha_\kappa \neq 0$ for $K_\kappa \cong I_{\mu\mu}$, thus $I_{\mu\mu} \cong I_{jj}$ or $I_{\mu\mu} \cong I_{kk}$. But that implies $\mu = \eta(j)$ or $\mu = \eta(k)$ for some η, thus $\alpha_\kappa \neq 0$

for $K_\varepsilon \cong I_{\mu j}$ or for $K_\varepsilon \cong I_{\mu k}$. Since both these lattices are neither isometric to I_{jj} nor to I_{kk}, this is again a contradiction. We are thus reduced to the case that the class number and the type number of D are both equal to 2. From the tables of Pizer [Pi] one sees that this can happen only for $N = 11, 17, 19$, in which cases the theta series of degree one of the nonisometric ideals are known to be distinct.

COROLLARY 9.3. *Let $N_2 = 1$, N odd, let $R_1, \cdots, R_t$ be representatives of the types of maximal orders in D, let*

$$L_i = (\mathbb{Z}1 + 2R_i) \cap \{x \in D \,|\, \mathrm{tr}(x) = 0\}.$$

Then the theta series $\vartheta^{(1)}(L_i)$ are linearly independent if and only if $L(\psi, 1) \neq 0$ for all Hecke-eigenforms $\psi \in \mathscr{A}^{\varepsilon_0}(D_{\mathrm{A}}^\times, R_{\mathrm{A}}^\times)$.

Proof. We notice first that the injective mapping $J_{2,1}(\Gamma_0(N)) \to M_{3/2}(N)$ of Satz 8 in [Kra] carries the Jacobi theta series of index one of the R_i to the $\vartheta(L_i)$ (for N prime this is Satz 1, ch. II in [Kra]). To see this, let $x = s' + 2y \in L_i$ with $s' \in \mathbb{Z}$, $y \in R_i$. Then $n(x) \in 2\mathbb{Z}$ if and only if $s' \in 2\mathbb{Z}$. For $x \in L_i$ and $s \in \mathbb{Z}$ one sees therefore that $x' := (x + s)/2$ is in R_i if and only if $n(x) \equiv s \bmod 2$, and one has $\mathrm{tr}(x') = s$, $n(x') = (n(x) + s^2)/4$. Using Kramer's notation [Kra],

$$\varphi_0(4\tau) = \sum_{\substack{x \in L_i \\ n(x) \in 2\mathbb{Z}}} \exp(2\pi i n(x)\tau), \quad \varphi_1(4\tau) = \sum_{\substack{x \in L_i \\ n(x) \notin 2\mathbb{Z}}} \exp(2\pi i n(x)\tau)$$

are therefore such that $\varphi_0(4\tau)\vartheta_{0,1}(\tau, z) + \varphi_1(4\tau)\vartheta_{1,1}(\tau, z)$ is the Jacobi theta series of index 1 of R_i, hence $\omega_0(4\tau) + \varphi_1(4\tau) = \vartheta^{(1)}(L_i)(\tau)$ is the modular form of weight 3/2 corresponding to this Jacobi form. Thus, the $\vartheta^{(1)}(L_i)$ are linearly independent if and only if the Jacobi theta series of index one of the R_i are linearly independent. These are, by the same reasoning as in [SP] and using Theorem 4.3. of [Y2], linearly dependent if and only if there is $\psi \in \mathscr{A}^{\varepsilon_0}(D_{\mathrm{A}}^\times, R_{\mathrm{A}}^\times)$. with $Y^{(2)}(1, \psi) = 0$.
(this generalizes a result of Gross [Gr]).

Remarks. 1) We should emphasize that the main ingredients of our proof of Yoshida's conjecture were

—the holomorphy and regularity results of Shimura [Shi1], [Shi2] and Feit [Fe]

—pullbacks of Eisenstein seires and their relations to automorphic L-functions

—the theorem of Ogg about L_{sym} which in turn is based on the Ramanujan-Petersson conjecture for weight 2 (proved by Eichler [E2]). In particular, we have made essential use of the fact that φ, ψ correspond to holomorphic modular forms.

2) We have used Kitaoka's [Ki2] result on linear independence of theta series to restrict attention to the splitting of $\Theta^{(3)}$. Instead of this, one could also use the trivial fact that the theta series of degree 4 are linearly independent and consider the splitting of $\Theta^{(4)}$. Essentially the same type of argument as above can then be used, proving in particular the special case $Y^{(3)}(\varphi, \psi) \neq 0$ of Kitaoka's linear independence result.

3) The elements of the space

$$\mathbb{C}\{Y^{(2)}(1, \psi) \mid \psi \in \mathscr{A}^{\epsilon_0}_{\mathrm{cusp}},\ \psi \text{ eigenform},\ L^{(N)}(\psi, 1) \neq 0\}$$

N	$\dim \Theta^{(3,3)}_{N,1}$	N	$\dim \Theta^{(3,3)}_{N,1}$	N	$\dim \Theta^{(3,3)}_{N,1}$
389	1	2027	1	3271	3
433	1	2029	2	3463	2
563	1	2081	2	3583	2
571	1	2089	1	3701	2
643	1	2251	1	3779	1
709	1	2293	2	3911	2
997	2	2333	4	3943	4
1061	2	2381	2	3967	1
1171	1	2593	4	4027	2
1483	1	2609	2	4093	2
1531	1	2617	2	4139	1
1567	3	2677	1	4217	2
1613	1	2797	1	4253	3
1621	1	2837	1	4357	1
1627	1	2843	4	4481	1
1693	3	2861	2	4547	1
1873	1	2953	1	4787	2
1907	1	2963	2	4799	1
1913	3	3019	2	4951	2
1933	3	3089	2	5003	3

satisfy the Maaß-relations [Y2]. It would be very interesting to see whether the elements of $\Theta_{N_1}^{(3,3)}$ satisfy similar relations.

4) The result about $Y^{(2)}(1, \psi)$ has been proved independently by Yoshida (using methods similar to those of [Y2] and a result of Waldspurger). For N prime it follows also from the proof given in [SP] for the special case of $N = 389$.

5) The case $L^{(N)}(\psi, 1) = 0$ really occurs. For N prime the dimension of the space generated by such ψ has been computed in many cases by K. Hashimoto (using that it is equal to t-(dim span$\{\vartheta^{(1)}(L_i)\}$) by [Gr]). With his kind permission we reproduce some of his results in the following Table.

6) In case $N_2 = 1$, Corollary 2 specifies the precise extent to which the linear independence conjecture of [An2] is wrong for these genera of quadratic forms.

§ 10. Scalar product formulas

In this last section we shall compare various scalar productsof the modular forms and the automorphic forms on $D_{\mathbf{A}}^{\times}$ that occurred so far. We keep the notations of the previous sections and notice that

$$\vartheta^{(n)}(I_{ij}) \,|\, T(p) = p^{n-1} \prod_{j=1}^{n-2} (p^j + 1)(\sum_k B_{jk}(p)\vartheta^{(n)}(I_{ik}) + \sum_l B_{il}(p)\vartheta^{(n)}((I_{lj}))$$

for $n \geq 2$,

$$\vartheta^{(1)}(I_{ij}) \,|\, T(p) = \sum_k B_{ik}(p)\vartheta^{(1)}(I_{kj})$$

($p \nmid N$, $B_{ij}(p)$ the entries of the Brandt matrix).

This can be deduced without difficulty from [Y3] and is proved implicitly in [Y1], § 5 (the case $n = 1$ is due to Eichler). Further, all the eigenvalues $\lambda_p(\varphi)$ for a Hecke-eigenform $\varphi \in \mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ are real and the eigenform φ itself can be chosen to be real.

We note finally that the natural notion of scalar product for $\mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ and $\mathscr{A}(D_{\mathbf{A}}^{\times} \times D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times} \times R_{\mathbf{A}}^{\times})$ is

$$\langle \varphi, \psi \rangle = \int_{D^{\times} \backslash D_{\mathbf{A}}^{\times}} \varphi(x)\bar{\psi}(x)dx = \sum_i \frac{\varphi(y_i)\bar{\psi}(y_i)}{e_i}$$

and

$$\langle \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle \langle \psi_1, \psi_2 \rangle$$

respectively.

We have to consider $Y^{(n)}(\varphi, \psi)$ for $n = 1, 2, 3$.

a) $n = 1$

Let $\varphi \in \mathscr{A}_{\mathrm{new}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$, be a Hecke-eigenform, put

$$F = Y^{(1)}(\varphi, \varphi), \vartheta_{ij} = \vartheta^{(1)}(I_{ij}) .$$

Using $\vartheta_{ij} \mid T(p) = \sum_k B_{ik}(p)\vartheta_{kj}$ we get

$$\sum_k B_{ik}(p)\langle F, \vartheta_{kj}\rangle = \langle F, \vartheta_{ij} \mid T(p)\rangle = \langle F \mid T(p), \vartheta_{ij}\rangle = \lambda_p(\varphi)\langle F, \vartheta_{ij}\rangle$$

for all j and $p \nmid N$.

By strong multiplicity one for $\mathscr{A}_{\mathrm{new}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ this implies

$$\langle F, \vartheta_{ji}\rangle = \langle F, \vartheta_{ij}\rangle = c_j\varphi(y_i)$$

for some c_j,

$$\varphi(y_i) \sum_j B_{kj}(p)c_j = \langle F, \vartheta_{ki} \mid T(p)\rangle = \lambda_p(\varphi)\langle F, \vartheta_{ki}\rangle = c_k\varphi(y_i) .$$

Choosing some i with $\varphi(y_i) \neq 0$ we get, again using strong multiplicity one: $c_j = c\varphi(y_j)$ with some constant $c = c(\varphi)$ depending on φ, which shows

$$\langle F, \vartheta_{ij}\rangle = c\varphi(y_i)\varphi(y_j) .$$

To determine the constant write

$$\langle F, F\rangle = \sum_{i,j} \frac{\varphi(y_i)\varphi(y_j)}{e_i e_j} \langle F, \vartheta_{ij}\rangle = c\Big(\sum_i \frac{\varphi(y_i)^2}{e_i}\Big)^2 .$$

Since $\sum_i \varphi(y_i)^2/e_i$ is the first Fourier coefficient of F we find

$$c = \langle F_0, F_0\rangle$$

where F_0 is the normalized eigenform corresponding to φ. It is well known (Rankin-Selberg method) that

$$\langle F_0, F_0\rangle = c_0 D_F^{(N)}(1)$$

with

$$c_0 = \frac{1}{(4\pi)^2\zeta^{(N)}(2) \prod_{p \mid N}(1 + p^{-1})}$$

(see e.g. [Pet], [Ran]). We have therefore proved

Proposition 10.1. *For $n = 1$, $F = Y^{(1)}(\varphi, \varphi)$ we have*

i) $\langle F, \vartheta_{ij}^{(1)}\rangle = c_0 D_F^{(N)}(1)\varphi(y_i)\varphi(y_j)$

ii) $\langle F, F\rangle = c_0 D_F^{(N)}(1)\langle\varphi, \varphi\rangle^2$

where c_0 depends only on N.

b) $n = 2$.

Let again $\varphi, \psi \in \mathscr{A}_{\text{new}}^{\varepsilon}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ for some ε be Hecke-eigenforms, $\varphi \neq c\psi$, put

$$F = Y^{(2)}(\varphi, \psi) \neq 0 \text{ (a cusp form)}, \qquad \vartheta_{ij} = \vartheta^{(2)}(I_{ij})\,.$$

Obviously $\rho\colon (y_i, y_j) \mapsto \langle F, \vartheta_{ij}\rangle$ is a symmetric function on $D_{\mathbf{A}}^{\times} \times D_{\mathbf{A}}^{\times}$, i.e. $\rho(y_i, y_j) = \rho(y_j, y_i)$.

The space of such symmetric $\rho \in \mathscr{A}(D_{\mathbf{A}}^{\times} \times D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times} \times R_{\mathbf{A}}^{\times})$ is spanned by the functions

$$\rho_{\nu\mu}(y_i, y_j) = \varphi_\nu(y_i)\varphi_\mu(y_j) + \varphi_\nu(y_j)\varphi_\mu(y_i)$$

where φ_ν, φ_μ are Hecke eigenforms in $\mathscr{A}_{\text{ess}}(D_{\mathbf{A}}^{\times}, (R'_{\mathbf{A}})^{\times})$ for some order $R' \supseteq R$ of level $N'|N$, since by [Hi-Sa] $\mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ is the direct sum of the $\mathscr{A}_{\text{ess}}(D_{\mathbf{A}}^{\times}, (R'_{\mathbf{A}})^{\times})$ for the orders $R' \supseteq R$. Define the operator $\hat{T}(p)$ by

$$\rho\,|\,\hat{T}(p)(y_i, y_j) = \sum_k B_{jk}(p)\rho(y_j, y_k) + \sum_l B_{il}(p)\rho(y_l, y_j)\,.$$

Then

$$\rho_{\nu\mu}\,|\,\hat{T}(p) = (\lambda_p(\varphi_\nu) + \lambda_p(\varphi_\mu))\rho_{\nu\mu}$$

for all p not dividing N.

Let now $\varphi_\nu, \varphi_\mu \in \mathscr{A}_{\text{ess}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ and assume that $\rho_{\nu'\mu'}$ has the same eigenvalues under $\hat{T}(p)$ as $\rho_{\nu\mu}$ for $p\nmid N$. Let f_κ be the normalized elliptic newform of weight 2 (and level N' dividing N) corresponding to φ_κ, i.e., having the same Hecke eigenvalues for $p\nmid N$ as φ_κ.

Then $f_\nu + f_\mu - f_{\nu'} - f_{\mu'}$ has Fourier coefficients $a_n = 0$ for $(n, N) = 1$, hence is an oldform of level N [Li], thus orthogonal to f_ν, f_μ. This implies $\{f_\nu, f_\mu\} = \{f'_\nu, f'_\mu\}$, hence $\{\varphi_\nu, \varphi_\mu\} = \{\varphi'_\nu, \varphi'_\mu\}$ (since strong multiplicity one holds for $\mathscr{A}_{\text{ess}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ by [Hi-Sa]. (The argument remains true if one of φ, ψ is identically 1).

Now $\rho(y_i, y_j) = \langle F, \vartheta_{ij}^{(2)}\rangle$ is seen to satisfy

$$\rho\,|\,\hat{T}(p) = \lambda_p(\varphi) + \lambda_p(\psi) \qquad (p\nmid N)$$

(using the expression of $\hat{T}(p)$ by Brandt matrices and proceeding as in the case $n = 1$), and φ, ψ are in $\mathscr{A}_{\text{new}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times})$ by assumption.

Thus (with $\varphi = \varphi_\nu$, $\psi = \varphi_\mu$), $\rho = c\rho_{\nu\mu}$, i.e.,

$$\rho(y_i, y_j) = \langle F, \vartheta_{ij}^{(2)} \rangle = c(\varphi(y_i)\psi(y_j) + \varphi(y_j)\psi(y_i))$$

for some constant $c = c(\varphi, \psi)$ depending on φ and ψ. We are now in the range where the results of part I apply and obtain

$$c = c_0 \operatorname*{Res}_{s=1} D_F^{(N)}(s) \neq 0$$

with

$$c_0 = \frac{\mu(2, 2, \frac{1}{2})\lambda_F(N \cdot 1_2)\Lambda_N^{\mathrm{triv}}(1)}{2\alpha^{(4)}(N_1, N_2)N^6\zeta^{(N)}(3)\zeta^{(N)}(4)\zeta^{(N)}(2)},$$

where $\lambda_F(N \cdot 1_n) = \pm \prod_{p|N} p^{n(n-1)/2} s_p(D)^n \gamma_p^n$ (by Lemmas 7.3 and 7.4) depends only on n, N_1, N_2 but not on φ, ψ. We write

$$\mathrm{const} = \mathrm{const}\,(n, N_1, N_2)$$

for any such constant depending only on n, N_1, N_2 in the following.

PROPOSITION 10.2. *For* $n = 2$, $\varphi, \psi \in \mathscr{A}_{\mathrm{new}}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times)$, $F = Y^{(2)}(\varphi, \psi)$ *we have*

i) $$\langle F, \vartheta_{ij}^{(2)} \rangle = \mathrm{const} \cdot \operatorname*{Res}_{s=1} D_F^{(N)}(s)(\varphi(y_i)\psi(y_j) + \varphi(y_j)\psi(y_i))$$

ii) $$\langle F, F \rangle = \mathrm{const} \cdot \operatorname*{Res}_{s=1} D_F^{(N)}(s)\langle \varphi \otimes \psi, \varphi \otimes \psi \rangle$$

and for $n' > 2$

iii) $$\begin{aligned} Y^{(n')}(\varphi, \psi) &= \mathrm{const} \cdot \frac{D_F^{(N)}(n'-1)}{\operatorname*{Res}_{s=1} D_F^{(N)}(s)} \operatorname*{Res}_{s=(n'-1)/2} E_{n',2}^2(Y^{(2)}(\varphi, \psi), s) \\ &= \mathrm{const} \cdot \frac{L_{\mathrm{sym}}^{(N)}(\varphi, \psi, n')}{L_{\mathrm{sym}}^{(N)}(\varphi, \psi, 2)} \operatorname*{Res}_{s=(n'-1)/2} E_{n',2}^2(Y^{(2)}(\varphi, \psi), s) \end{aligned}$$

Proof. The last fact follows from the first and Theorem 4.1.

c) $n = 3$, R is a maximal order, $\psi = 1$, $L(\varphi, 1) = 0$.

$F = Y^3(\varphi, 1)$ is then a cusp form, we put $\vartheta_{ij}^{(3)} = \vartheta^{(3)}(I_{ij})$. By the same argument as above we obtain:

PROPOSITION 10.3. *For* $n = 3$, R, φ *as above*, $F = Y^{(3)}(\varphi, 1)$ *we have*:

i) $$\langle F, \vartheta_{ij}^{(3)} \rangle = \mathrm{const} \cdot \operatorname*{Res}_{s=2} D_F^{(N)}(s)(\varphi(y_i) + \varphi(y_j))$$

ii) $$\langle F, F \rangle = \mathrm{const} \cdot \operatorname*{Res}_{s=2} D_F^{(N)}(s)\langle 1 \otimes \varphi, 1 \otimes \varphi \rangle$$

and for $n' > 3$

iii) $$Y^{(n')}(\varphi, 1) = \mathrm{const} \cdot \frac{D_F^{(N)}(n'-1)}{\operatorname*{Res}_{s=2} D_F^{(N)}(s)} \cdot \operatorname*{Res}_{s=n'/2} E^2_{n',3}(Y^{(3)}(\varphi, 1), s)$$
$$= \mathrm{const} \cdot \frac{L^{(N)}(\varphi, n'-1) L^{(N)}(\varphi, n'-2)}{L^{(N)}(\varphi, 3) L^{(N)}(\varphi, 2)}$$
$$\times \operatorname*{Res}_{s=n'/2} E^2_{n',3}(Y^{(3)}(\varphi, 1), s)\,.$$

Remarks. 1) The scalar product formulas obtained in all cases are similar to those of Rallis [Ral] relating Petersson norms of forms on the orthogonal group and of their theta liftings. The formulas for $Y^{(n')}(\varphi, \psi)$, $n' > n$ are in a sense generalizations of Siegel's theorem: They exhibit a linear combination of theta series as (residue of) an Eisenstein series.

2) In the results described above, there is one point missing: In Proposition 1 there is no statement (iii) like in Propositions 2 and 3. Here we can only obtain a somewhat weaker result for $Y^{n'}(\varphi, \varphi)$, which however will be essential in future applications. First we remark that Theorem 4.1 remains valid under the weaker assumption $(n + n' + 1)/2 - k > 0$, so we can apply it for $n = 1$, $F = Y^1(\varphi, \varphi)$, $\varphi \in \mathscr{A}_{\mathrm{new}}(D_A^\times, R_A^\times)$ and $n' \geq 3$. For $n' = 2$ we can use Corollary 3.1.

We have to observe now that on the right hand side of (4.1) theta series arising from all Eichler orders of level N —not only from the one we are considering— may give nontrivial contributions (note that theta series of Eichler orders of level $\neq N$ are orthogonal to F by Lemma 1 of section 9).

We obtain then:

PROPOSITION 10.4. *For $N_1'|N$ having an odd number of prime factors denote by $D(N_1')$ the definite quaternion algebra over $\mathbb{Q}$ ramified at ∞ and the primes dividing N_1', unramified at all other places, and by $R(N_1')$ an Eichler order of level N in $D(N_1')$. Denote further by $\varphi^{(N_1')} \in \mathscr{A}_{\mathrm{new}}(D(N_1')_A^\times, R(N_1')_A^\times)$ the (unique) function in $\mathscr{A}_{\mathrm{new}}(D(N_1')_A^\times, R(N_1')_A^\times)$ satisfying $Y^{(1)}(\varphi^{(N_1')}, \varphi^{(N_1')}) = F$. Then*

$$\sum_{N_1'|N} Y^{(n')}(\varphi^{(N_1')}, \varphi^{(N_1')}) = \mathrm{const} \cdot \frac{D_F^{(N)}(n'-1)}{D_F^{(N)}(1)} \operatorname*{Res}_{s=(n'-2)/2} E^2_{n',1}(F, s)$$

for $n' \geq 3$,

$$\sum_{N_1'|N} Y^{(2)}(\varphi^{(N_1')}, \varphi^{(N_1')}) = \mathrm{const} \cdot (E^2_{2,1}(F, 0))\,.$$

For future reference, we finally state the proposition above for $n' = 2$ in a slightly different (but equivalent) form with φ, F, N_1, N_2, N_1' as above, $F_0 = (\sum_i (\varphi(y_i)^2)/e_i)^{-1} \cdot F$, $l :=$ number of primes dividing N. Then

$$\left\langle F, \mathscr{F}_3^2\left(\left(\begin{matrix} -\bar{Z} & 0 \\ 0 & * \end{matrix}\right)\right)\right\rangle = c_1 D_F^{(N)}(1) \sum_{N_1'|N} Y^{(2)}(\varphi^{(N_1')}, \varphi^{(N_1')})(Z)$$

with

$$c_1 = \frac{(-1)^l \pi^2}{2} \prod_{p|N} (1 + p^{-1})^{-1} \cdot \zeta^{(N)}(2)^{-3} .$$

In subsequent work we shall show that the arithmetic version of the identity above (i.e. the corresponding identity between the Fourier coefficients of both sides) implies a version of Waldspurger's formula for values of twisted L-series.

References

[An 1] A. N. Andrianov, Quadratic forms and Hecke operators, Grundlehren d. math. Wiss. 286, Berlin-Heidelberg-New York 1987.

[An 2] ——, Degenerations of rings of Hecke operators on spaces of theta series, Proc. of the Steklov Institute of Math., 1983, no. 3, 1–17.

[An 3] ——, Duality in Siegel's theorem on representations by a genus of quadratic forms and the averaging operator, Math. USSR Sbornik, **50** (1985), 1–10.

[Bö 1] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, J. f. d. reine und angew. Math., **362** (1985), 146–168.

[Bö 2] ——, Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe, Abh. Math. Sem. Univ. Hamburg, **56** (1986), 35–47.

[Bö 3] ——, Siegel modular forms and theta series in: Proceedings of the AMS Summer Research Institute "Theta Functions", Bowdoin College 1987, Providence 1989.

[BöRa] S. Böcherer and S. Raghavan, On Fourier coefficients of Siegel modular forms, J. f. d. reine und angew. Math., **384** (1988), 80–101.

[Ca 1] W. Casselmann, On some results of Atkin and Lehner, Math. Ann., **201** (1973), 301–314.

[Ca 2] ——, An assortment of results on representations of $GL_2(k)$, p. 1–54 in Modular functions of one variable II, Lecture Notes Math., **349**, Berlin-Heidelberg-New York 1973.

[E 1] M. Eichler, Quadratische Formen und orthogonale Gruppen, Berlin-Göttingen-Heidelberg, 1952.

[E 2] ——, Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion, Arch. Math., **5** (1954), 355–366.

[E 3] ——, Zur Zahlentheorie der Quaternionen-Algebren, J. f. d. reine und angew. Math., **195** (1956), 127–151.

[E 4] ——,The basis problem for modular forms and the traces of the Hecke operators, p. 76–151 in Modular functions of one variable I, Lecture Notes Math., **320**, Berlin-Heidelberg-New York 1973.

[Ev] S. A. Evdokimov, Action of the irregular Hecke operator of index p on the theta series of a quadratic form, J. Sov. Math., **38** (1987), 2078–2081.

[Fe] P. Feit, Poles and residues of Eisenstein series for symplectic and unitary groups, Mem. AMS, **61** (1986), no. 346.

[Fre 1] E. Freitag, Siegelsche Modulfunktionen, Berlin-Heidelberg-New York 1983.

[Fre 2] ——, Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren, Math. Zeitschrift, **156** (1977), 141–155.

[Ga] P. Garrett, Decomposition of Eisenstein series: Applications, in: Automorphic forms in several variables (Taniguchi symposium), Boston 1984.

[Ge] S. Gelbart, Automorphic forms on adele groups, Princeton 1975.

[G R] J. Goldman and G.-C. Rota, On the foundations of combinatorial theory IV, Studies in applied math., **49** (1970), 239–258.

[Gr] B. Gross, Heights and the special values of L-series, p. 115–187 in: Number Theory (Montreal 1985), CMS Conf. Proc. 7, Providence 1987.

[Ha 1] M. Harris, The rationality of holomorphic Eisenstein series, Invent. math., **63** (1981), 305–310.

[Ha 2] ——, Eisenstein series on Shimura varieties, Ann. of math., **119** (1984), 59–94.

[Hi Sa] H. Hijikata and H. Saito, On the representability of modular forms by theta series, p. 13–21 in: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Tokyo 1973.

[H-PS] R. Howe and I. I. Piatetski-Shapiro, Some examples of automorphic forms on Sp_4, Duke Math. J., **50** (1983), 55–106.

[Ki 1] Y. Kitaoka, Siegel modular forms and representation by quadratic forms, Bombay 1986.

[Ki 2] ——, Representations of quadratic forms and their applications to Selberg's zeta function, Nagoya Math. J., **63** (1976), 153–162.

[Ki 3] ——, A remark on the transformation formula of theta functions associated to positive definite quadratic forms, J. Number Th., **12** (1980), 224–229.

[Ki 4] ——, Dirichlet series in the theory of Siegel modular forms, Nagoya Math. J., **95** (1984), 73–84.

[Kli] H. Klingen, Zum Darstellungssatz für Siegelsche Modulformen, Math. Z., **102** (1967), 30–43.

[Kn] M. Kneser, Witts Satz für quadratische Formen über lokalen Ringen, Nachr. d. Akad. d. Wiss. Göttingen 1972, 195–204.

[Kra] J. Kramer, Jacobiformen und Thetareihen, Manuscripta math., **54** (1986), 279–322.

[Kri] A. Krieg, Das Vertauschungsgesetz zwischen Hecke-Operatoren und dem Siegelschen Φ-Operator, Arch. Math., **46** (1986), 323–329.

[Ku] S. Kudla, On the local theta correspondence, Invent. math., **83** (1986), 229–255.

[KuRa] S. Kudla and S. Rallis, On the Weil-Siegel formula, J. f. d. reine u. angew. Math., **337** (1988), 1–68.

[Li] W. Li, Newforms and functional equations, Math. Ann. **212** (1975), 285–315.

[Ma] H. Maaß, Siegel's modular forms and Dirichlet series, Lect. Notes Math., **216**, Berlin-Heidelberg-New York 1971.

[O] A. Ogg, On a convolution of L-series, Invent. math., **7** (1969), 279–312.

[OM] O. T. O'Meara, Introduction to quadratic forms, Grundlehren d. math. Wiss., **117**, Berlin-Heidelberg-New York 1973.

[Pe] P. Perrin, Representation de Schrödinger, indice de Maslov et groupe metaplectique, p. 370–407 in: Lect. Notes Math., **880**, Berlin-Heidelberg-New York 1981.

[Pet] H. Petersson, Über die Berechnung der Skalarprodukte ganzer Modulformen, Comm. math. Helv., **22** (1949), 168–191.

[Pf] H. Pfeuffer, Einklassige Geschlechter totalpositiver quadratischer Formen in total-reellen algebraischen Zahlkörpern, J. Number Th., **3** (1971), 371–411.

[Pi] A. Pizer, Type numbers of Eichler orders, J. f. d. reine u. angew. Math., **264** (1973), 76–102.

[Ra 1] S. Rallis, *L*-functions and the oscillator representation, Lect. Notes Math., **1245**, Berlin-Heidelberg-New York 1987.

[Ra 2] ——, Langlands functoriality and the Weil representation, Amer. J. of Math., **104** (1982), 469–515.

[Ra 3] ——, On the Howe duality conjecture, Comp. Math., **51** (1984), 333–399.

[Ra 4] ——, Injectivity properties of liftings associated to Weil representations, Comp. Math., **52** (1984), 139–169.

[Rao] R. Rao, On some explicit formulas in the theory of the Weil representation, unpublished manuscript.

[Ran] R. Rankin, The scalar product of modular forms, Proc. of the London Math. Soc., **3** (1952), 198–217.

[Sa] M. Saito, Representations unitaires des groupes symplectiques, J. Math. Soc. Japan, **24** (1972), 232–251.

[Shi 1] G. Shimura, Confluent hypergeometric functions on tube domains. Math. Ann., **260** (1982), 269–302.

[Shi 2] ——, On Eisenstein series of half-integral weight, Duke Math. J., **52** (1985), 281–314.

[Shz] Shimizu, Theta series and automorphic forms on GL_2, J. of the Math. Soc. of Japan, **24** (1972), 638–683.

[SP] R. Schulze-Pillot, A linear dependence of theta series of degree and weight two, in: Proceedings of the Journées Arithmétiques Ulm 1987, Lect. Notes Math. **1380**, Berlin-Heidelberg-New York 1989.

[Tam] T. Tamagawa, On the ζ-functions of a division algebra, Ann. Math., **77** (1963), 387–405.

[Tan] Y. Tanigawa, Construction of Siegel modular forms of degree three and commutation relations of Hecke operators, Nagoya Math. J., **100** (1985), 83–96.

[Vi 1] M.-F. Vigneras, Arithmétique des algèbres de quaternions, Lect. Notes Math., **800**, Berlin-Heidelberg-New York 1980.

[Vi 2] ——, Correspondences entre representations automorphes de $GL(2)$ sur une extension quadratique de $GSp(4)$ sur $\mathbb{Q}$, conjecture locale de Langlands pour $GSp(4)$, p. 463–527 in: Contemporary Mathematics, **53** (1986).

[Wa 1] J. L. Waldspurger, Correspondance de Shimura, J. Math. pures et appl., **59** (1980), 1–133.

[Wa 2] ——, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. pures et appl., **60** (1981), 375–484.

[Wa 3] ——, Correspondance de Shimura et quaternions, unpublished manuscript.

[War] G. Warner, Harmonic analysis on semi-simple Lie groups I, Grundlehren d. math. Wiss. **188**, Berlin-Heidelberg-New York 1972.

[We] R. Weissauer, Stabile Modulformen und Eisensteinreihen, Lect. Notes Math., **1219**, Berlin-Heidelberg-New York 1986.

[Y 1] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, Invent. math., **60** (1980), 193–248.

[Y 2] ——, On Siegel modular forms obtained from theta series, J. f. d. reine u. angew. Math., **352** (1984), 184–219.

[Y 3] ——, The action of Hecke operators on theta series, p. 197–238 in: Algebraic and topological theories—to the memory of Dr. T. Miyake—, 1985.

[Zh] N. A. Zharkovskaya, The Siegel operator and Hecke operators, Functional Anal. Appl., **8** (1974), 113–120.

Siegfried Böcherer
Mathematisches Institut
der Universität Freiburg
Hebelst. 29
D-7800 Freiburg

Rainer Schulze-Pillot
Fakultät für Mathematik, SFB 343
Universität Bielefeld
Postfach 8640
D-4800 Bielefeld