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HOLOMORPHIC MAPPING INTO ALGEBRAIC VARIETIES OF GENERAL TYPE

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§1. Introduction

We will study holomorphic mappings

$$f\colon M \longrightarrow N$$

from a connected complex manifold M of dimension m to a projective algebraic manifold N of dimension n. Assume first that N is of general type, i.e.

$$\varlimsup_{k\to\infty} \frac{\dim H^0(N, K_N^k)}{k^n} > 0\,,$$

where $K_N \to N$ is the canonical bundle of N. If K_N is positive, then N is of general type.

In 1971, Kodaira [6] obtained that

THEOREM A. *Any holomorphic mapping $f\colon C^m \to N$ has every-where rank less than n.*

P. Griffiths & J. King [2], [3] furthermore proved that

THEOREM B. *If M is a smooth affine algebraic variety, then any holomorphic mapping $f\colon M \to N$ whose image contains an open set is necessarily rational.*

In 1977, W. Stoll [6] extended Theorems A, B to parabolic manifolds M. To state it, we let M possess a parabolic exhaustion τ and denote

$$(1) \qquad \nu = dd^c\tau\,, \qquad \sigma = d^c \log\tau \wedge (dd^c \log\tau)^{m-1}\,.$$

For a form φ of bidegree (1, 1) on M, write

$$(2) \qquad A(t, \varphi) = t^{2-2m}\int_{M[t]} \varphi \wedge \nu^{m-1}\,, \qquad T(r, s; \varphi) = \int_s^r \frac{A(t, \varphi)}{t}\,dt$$

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if the integrals exist, where $M[t] = \{x \in M: \tau(x) \leq t^2\}$. Suppose throughout that L is a positive holomorphic line bundle over N with a hermitian metric ρ along the fibers of L such that the Chern form $c(L, \rho) > 0$. The characteristic function of f for L is defined by

$$T(r, s) = T(r, s;\ f^*c(L, \rho)) . \tag{3}$$

THEOREM C. *If M is a parabolic manifold and if F is an effective Jacobian section such that*

(i) F is dominated by τ with Y as dominator, there exist positive constants c_1, c_2, c_3 such that for $\varepsilon > 0$

$$T(r, s) \leq c_1 \log Y(r) + c_2 \operatorname{Ric}_\tau (r, s) + c_3 \varepsilon \log r \tag{4}$$

with the exception of a set of values (r) of finite measure.

The condition (i) implies $m \geq n = \operatorname{rank} f$ ([8], Lemma 18.1). We remove this restriction (see [4]). To state the generalization of the Theorem C which we shall prove, we take a positive form ψ of class C^∞ and bidegree (1, 1) on N and set

$$\psi_f = \begin{cases} f^*(\psi^m) & \text{if } m \leq n \\ f^*(\psi^n) \wedge \chi & \text{if } m > n \end{cases} \tag{5}$$

where χ be a positive $(m - n, m - n)$-form of class C^∞ on M. Then the form

$$\chi_f = f^*(\operatorname{Ric} \psi^n) - \frac{n}{b} \operatorname{Ric} \psi_f \qquad \text{where } b = \min (m, n) , \tag{6}$$

is well-defined. Take a holomorphic form B of bidegree $(m - 1, 0)$ on M. Define

$$\ddot{\psi}_f = \ddot{\psi}_f(B) = m i_{m-1} f^*(\psi) \wedge B \wedge \bar{B} ,$$
$$e_f = e_f(\psi) = f^*(\operatorname{Ric} \psi^n) - n \operatorname{Ric} \ddot{\psi}_f ,$$

where i_{m-1} is defined in Section 3. Then $\chi_f(h\psi) = \chi_f(\psi)$, $e_f(h\psi) = e_f(\psi)$ for positive functions h of class C^2 on N. Define η by $\ddot{\psi}_f = \eta f^*(\psi) \wedge \nu^{m-1}$ and denote

$$B(r, \eta) = \frac{1}{2} \int_{\partial M[r]} \log \eta\sigma , \tag{7}$$

$$E_f(r, s) = T(r, s;\, e_f) + nB(t, \eta)|_s^r , \tag{8}$$

where $B(t)|_s^r$ means $B(r) - B(s)$. For $\psi = c(L, \rho)$, we obtain that

THEOREM 1. *If there exists an effective Jacobian section of f and if rank $f = b = \min(m, n)$, then exist positive constants c_1 and c_2 such that for $\varepsilon > 0$*

$$c_1 T(r, s) \leq n\,\mathrm{Ric}_\tau(r, s) + E_f(r, s) + c_2\varepsilon \log r \tag{9}$$

with the exception of a set of values (r) of finite measure.

COROLLARY 2. *If M is smooth affine algebraic variety, any non-degenerate holomorphic mapping $f\colon M \to N$ with*

$$\varlimsup_{r\to\infty} \frac{E_f(r, s)}{\log r} < \infty \tag{ii}$$

is necessarily rational.

To draw geometrical consequences, here assume that M and N are hermitian manifolds. Relative to the local coordinates z^i let

$$ds_M^2 = \sum_{i,j} h_{ij} dz^i d\bar{z}^j \qquad 1 \leq i, j \leq m \tag{10}$$

be a positive definite hermitian metric on M with the associated 2-form

$$\varphi = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j\,. \tag{11}$$

Similarly, let

$$ds_N^2 = \sum_{k,l} \tilde{h}_{kl} dw^k d\bar{w}^l \qquad 1 \leq k, l \leq n \tag{12}$$

be a positive definite hermitian metric on N, with the local coordinates w^k, and

$$\psi = \frac{\sqrt{-1}}{2\pi} \sum_{k,l} \tilde{h}_{kl} dw^k \wedge d\bar{w}^l \tag{13}$$

be the associated 2-form. Define the function u on M by

$$\psi_f = u\varphi^m\,. \tag{14}$$

Then we have

$$\partial\bar{\partial} \log u = \mathrm{Ric}_M - \frac{b}{n} f^*(\mathrm{Ric}_N) + \frac{2\pi b\sqrt{-1}}{n} \chi_f\,. \tag{15}$$

When $m \leq n$,

(16) $$u = \frac{\det(\hat{h}_{ij})}{\det(h_{ij})}$$

is geometrically the ratio of the volume elements, where

$$\hat{h}_{ij} = \sum_{k,l} \tilde{h}_{kl} \frac{\partial w^k}{\partial z^i} \frac{\partial \overline{w}^l}{\partial \bar{z}^j}$$

under the mapping f. If $m = n$, (15) implies the Chern formula [1]

(17) $$\frac{1}{2} \Delta \log u = R - \mathrm{Tr}\,(f^*(\mathrm{Ric}_N)),$$

where Δ is the Laplacian in M and R denotes the scalar curvature of M.

Let D_f be the zero divisor of ψ_f, which independent of the choices of ψ and χ. Then χ_f determines an element $[\chi_f] \in H^2_{DR}(M - D_f, \boldsymbol{R})$, the de Rham cohomology group of closed C^∞ differential forms modulo exact ones. We extend the Chern Theorems [1] on holomorphic mappings of hermitian manifolds of the same dimension to non-equidimensional cases. This includes a non-equidimensional version of the Schwarz lemma, which says that if M is the unit m-ball and N is almost einsteinian with $\sqrt{-1}\,\mathrm{Tr}\,(\chi_f) \geq 0$, the mapping f does not increase volume.

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§ 2. The Ricci form and proof of the formula (15)

As usual, we let

$$d = \partial + \bar{\partial} \quad \text{and} \quad d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial).$$

Then

$$dd^c = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}.$$

The Chern form of the line bundle L for the hermitian metric ρ is defined by

$$c(L, \rho) = -dd^c \log |s|^2_\rho \qquad \text{on } U$$

for all open subsets U in N and all $s \in H^0(U, L)$. Let Ψ be a volume form on N. This is the same as a metric on the canonical line bundle K_N, which is denoted by ρ_Ψ. In terms of complex coordinates $w^1, \cdots, w^n$, such a form is one which can be written

$$\Psi(w) = \rho(w)\Phi(w) \qquad \text{where } \Phi(w) = \prod_{j=1}^{n} \frac{\sqrt{-1}}{2\pi} dw^j \wedge d\bar{w}^j$$

and ρ is real >0. In practice one often has

$$\rho(w) = \lambda(w)|g(w)|^{2q},$$

where g is holomorphic not identically zero, q is some fixed rational number >0 and λ is C^∞ and >0. We define the Ricci form of Ψ to be the Chern form of this metric ρ_Ψ on K_N, so

$$\operatorname{Ric}\Psi = c(K_N, \rho_\Psi) = dd^c \log \rho = dd^c \log \lambda,$$

which is independent of the choice of complex coordinates, and defines a real (1, 1)-form.

Now we prove the formula (15). It is well known that the Ricci form of M for the metric ds_M^2 is of

$$\operatorname{Ric}_M = -\partial\bar{\partial} \log \det (h_{ij}). \tag{18}$$

Then we have

$$\operatorname{Ric}\varphi^m = dd^c \log \det (h_{ij}) = \frac{1}{2\pi\sqrt{-1}} \operatorname{Ric}_M. \tag{19}$$

It follows that

$$\begin{aligned} \chi_f &= f^*(\operatorname{Ric}\psi^n) - \frac{n}{b} \operatorname{Ric}\psi_f \\ &= f^*\left(\frac{1}{2\pi\sqrt{-1}} \operatorname{Ric}_N\right) - \frac{n}{b}(dd^c \log u + \operatorname{Ric}\varphi^m), \end{aligned}$$

which implies (15) by (19).

For convenience, we let $\chi = 1$ if $m \leq n$, so that

$$\psi_f = f^*(\psi^b) \wedge \chi.$$

Hence when $m \leq n$, u is independent of the choice of χ and of the expression (16). Thus

$$u = \frac{\det (\tilde{h}_{kl})}{\det (h_{ij})} \left|\det\left(\frac{\partial w^k}{\partial z^i}\right)\right|^2$$

if $m = n$. When $m > n$, $u = u_\chi$ depends on the choice of χ with

$$u_{h\chi} = hu_\chi ,$$

where h is a function on M. Locally we may choose an orthonormal co-frame $\theta_1, \cdots, \theta_m$ for M such that

$$ds_M^2 = \sum_{j=1}^{m} \theta_j \bar{\theta}_j .$$

It is well-known that ds_M^2 induces an intrinsic connection on M and we let

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \bar{\theta}_l$$

be the curvature. Then

$$\mathrm{Ric}_M = \sum_{i=1}^{m} \Omega_{ii} = \frac{1}{2} \sum_{k,l} R_{kl} \theta_k \wedge \bar{\theta}_l ,$$

where

$$R_{kl} = \sum_{i=1}^{m} R_{iikl} .$$

From them we form the scalar curvature

$$R = \sum_{k=1}^{m} R_{kk} .$$

Similarly, let $\omega_1, \cdots, \omega_n$ be an orthonormal co-frame for N such that

$$ds_N^2 = \sum_{k=1}^{n} \omega_k \bar{\omega}_k$$

and let S_{ijkl}, S_{ij} and S be the curvature tensor, the Ricci tensor and scalar curvature of N respectively. We put

$$du = \sum_i (u_i \theta_i + \bar{u}_i \bar{\theta}_i) ,$$

$$\partial \bar{\partial} u = - d \partial u = \sum_{i,j} u_{ij} \theta_i \wedge \bar{\theta}_j .$$

Then the Laplacian of u is defined by

$$\Delta u = 4 \sum_i u_{ii} .$$

If $u > 0$, we find

$$\Delta \log u = \frac{1}{u} \Delta u - \frac{4}{u^2} \sum_i u_i \bar{u}_i . \tag{20}$$

Under the mapping f let us set

(21) $$\omega_i = \sum_{j=1}^{m} a_{ij}\theta_j \qquad 1 \leq i \leq n .$$

If $u > 0$, it follows from (15) that

(22) $$\frac{1}{2}\Delta \log u = R - \frac{b}{n}\sum_{k,l,i} S_{kl}a_{ki}\bar{a}_{li} + \frac{2b}{n}\lambda_f ,$$

where

(23) $$\lambda_f = 2\pi\sqrt{-1}\,\mathrm{Tr}\,(\chi_f) .$$

When $m = n$, (22) implies (17).

To draw geometrical conclusions we start with some definitions: f is said to be degenerate at $p \in M$, if u vanishes at p, totally degenerate if u vanishes identically, volume decreasing or volume increasing according as $u \leq 1$ or $u \geq 1$ for a χ. Proceeding in similar manner as Chern [1], we have

PROPOSITION 3. *Let f: $M \to N$ be a holomorphic mapping, where M, N are hermitian manifolds of dimension m and n respectively, with M compact and N einsteinian. Let R and S be their scalar curvature respectively. Then we have*

(1) *If $R > 0$, $S \leq 0$, $\lambda_f \geq 0$, then f is totally degenerate.*

(2) *If $R < 0$, $S \geq 0$, $\lambda_f \leq 0$, then there is a point of M at which f is degenerate.*

To obtain an upper bound for the scalar function u, Chern impose some conditions on the domain manifold M and the image manifold N. The first property is:

(DO_K). M is exhausted by a sequence of open submanifolds

$$M_1 \subset M_2 \subset M_3 \subset \cdots \subset M$$

whose closures $\overline{M}_\alpha$ are compact, such that: (1) to each $\alpha = 1, 2, \cdots$ there is a smooth function $\nu_\alpha \geq 0$ defined in M_α, which satisfies the inequality

(24) $$\frac{1}{2}\Delta\nu_\alpha \leq R + K\exp(\nu_\alpha/m) ,$$

where K is a given positive constant; (2) $\nu_\alpha(p_\beta) \to \infty$, if p_β is a divergent sequence of points in M_α.

For example, the unit ball $M = D_1$ defined by

$$r^2 = z_1\bar{z}_1 + \cdots + z_m\bar{z}_m < 1$$

in the m-dimensional number space C^m with coordinates $(z_1, \cdots, z_m)$ has the property (DO_K), with

$$\nu_\rho = \log\left(\frac{1-r^2}{\rho^2 - r^2}\right)^{2m} \tag{25}$$

in the exhaustion submanifolds D_ρ of D_1, where D_ρ be defined by $r < \rho$ (<1), and $K = 2m(m+1)$. The unit ball is einsteinian with its scalar curvature $R = -2m(m+1)$ under the kählerian metric

$$ds_M^2 = \frac{1}{1-r^2}\sum_k dz_k d\bar{z}_k + \frac{4r^2}{(1-r^2)^2}\partial r \bar{\partial} r\,. \tag{26}$$

(IM_K). N is said to have the property (IM_K) (or almost einsteinian), if

$$\sum_{i,k} S_{ik}\zeta_i\bar{\zeta}_k \leq -\frac{K}{n}\sum_i \zeta_i\bar{\zeta}_i\,, \qquad \text{for all } \zeta_i\,. \tag{27}$$

For the rest of this section we let $m \leq n$. Define

$$A_{jk} = \sum_{i=1}^{n} a_{ij}\bar{a}_{ik}\,.$$

Then we have

$$u = \det(A_{jk})\,. \tag{28}$$

By Hadamard's well-known determinant inequality we have

$$\frac{1}{m}\sum_{j,k}|A_{jk}|^2 \geq |\det(A_{jk})|^{2/m} = u^{2/m}\,.$$

Hence Cauchy-Hölder's inequality implies

$$(m^{1/2}/n)u^{1/m} \leq \frac{1}{n}\Big(\sum_{j,k}|A_{jk}|^2\Big)^{1/2} \leq \frac{1}{n}\sum_{i,j}|a_{ij}|^2\,. \tag{29}$$

It follows from (22) that if N have the property (IM_K) and $u > 0$ we have

$$\frac{1}{2}\Delta \log u \geq R + (m^{3/2}/n^2)Ku^{1/m} + \frac{2m}{n}\lambda_f\,. \tag{30}$$

Now proceeding in similar manner as Chern [1], we have

PROPOSITION 4. *Let $f\colon M \to N$ be a holomorphic mapping, where M and N are hermitian manifolds of dimension m and n having the properties*

(DO_K) and (IM_{K_0}) respectively, with $K_0 = (n^2/m^{3/2})K$ and $m \leq n$. If $\lambda_f \geq 0$, then $u \leq \exp(\nu_a)$.

PROPOSITION 5. *Let $f\colon D_1 \to N$ be a holomorphic mapping, where D_1 is the unit m-ball with the standard kähler metric and where N is an n-dimensional hermitian einsteinian manifold with scalar curvature $\leq -2n^2(m+1)/m^{1/2}$ and $n \geq m$. If $\lambda_f \geq 0$, then f is volume-decreasing.*

§ 3. Notes on parabolic manifolds

From now on, we will study value distribution on the holomorphic mapping $f\colon M \to N$. Let $L_f \to M$ be the pull-back of $L \to N$ and s_f the pull-back of $s \in H^0(N, L)$. Then $K_M \otimes (K^*_{Nf})$ is called the Jacobian bundle, its holomorphic sections over M are called Jacobian sections. A Jacobian section F is called effective if the set $F^{-1}(0)$ of zeroes is thin, its zero divisor D_F is called the ramification divisor of f for F. Let $A^p_k(U)$ be the vector space of forms of class C^k and degree p on $U \subset N$. Define

$$i_p = \left(\frac{\sqrt{-1}}{2\pi}\right)^p (-1)^{p(p-1)/2} p!\,.$$

Then a Jacobian section F operates on forms of degree $2n$ as follows: Take $\Psi \in A^{2n}_k(U)$ with $\tilde{U} = f^{-1}(U) \neq \emptyset$. Relative to the local coordinates z^i and w^k of M and N respectively, write

$$F = g dz^1 \wedge \cdots \wedge dz^m \otimes \left(\frac{\partial}{\partial w^1} \wedge \cdots \wedge \frac{\partial}{\partial w^n}\right)_f \qquad g \in \mathrm{Hol}(\tilde{U}),$$
$$\Psi = i_n h dw^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^n .$$

Then

$$F[\Psi] = i_m(h \circ f)|g|^2 dz^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^m .$$

If M is Stein and if f has strict rank $\min(m, n)$, effective Jacobian sections exist (see [8]).

Assume that τ is a parabolic exhaustion of M, i.e., a proper map $\tau\colon M \to \boldsymbol{R}^+$ of class C^∞ which satisfies

$$\begin{cases} dd^c \log \tau \geq 0\,, \\ (dd^c\tau)^m \not\equiv 0 \text{ but } (dd^c \log \tau)^m \equiv 0\,, \\ M[0] \text{ has measure zero}\,. \end{cases}$$

For any regular value r of τ, then

$$c = \int_{\partial M[r]} \sigma$$

is a constant. Take a positive form Ω of degree $2m$ and class C^2 on M. Define v by $\nu^m = v\Omega$. The Ricci function of τ is defined by

$$\text{(31)} \qquad \operatorname{Ric}_\tau(r, s) = T(r, s; \operatorname{Ric}\Omega) + B(t, v)|_s^r ,$$

which does not depend on the choice of Ω. Let D be a divisor on M and set $D[r] = D \cap M[r]$. We define

$$n(t, D) = t^{2-2m} \int_{D[t]} \nu^{m-1} ,$$

$$N(r, s; D) = \int_s^r n(t, D) \frac{dt}{t} .$$

If we define v by $\nu^m = vF[\Psi]$ for an effective Jacobian section F and a positive volume form Ψ of class C^∞ and degree $2n$ on N, then

$$\text{(32)} \qquad \operatorname{Ric}_\tau(r, s) = T(r, s; f^*(\operatorname{Ric}\Psi)) + B(t, v)|_s^r + N(r, s; D_F)$$

(For a detailed proof see [8] Theorem 15.5).

Take an effective Jacobian section F and a positive form ψ of class C^∞ and bidegree (1, 1) on N. Define u_0 and u_1 by

$$\text{(33)} \qquad \nu^m = u_0 \ddot{\psi}_f , \qquad \nu^m = u_1 F[\psi^n] .$$

By the definitions of η and $\ddot{\psi}_f$, we have

$$\nu^m = u_0 \eta f^*(\psi) \wedge \nu^{m-1} .$$

Let D_f be the zero divisor of $\ddot{\psi}_f$. Then

$$\text{(34)} \qquad S_f(r, s) = N(r, s; D_F) - nN(r, s; D_f) + B\left(t, \frac{u_1}{u_0^n}\right)\Big|_s^r$$

is defined such that

$$\text{(35)} \qquad E_f(r, s) + S_f(r, s) = (1 - n) \operatorname{Ric}_\tau(r, s) + nB(t, \eta)|_s^r .$$

In fact, the form $\ddot{\psi}_f$ determines a section s_f of K_M such that $\ddot{\psi}_f = |s_f|_\rho^2 \Omega$ for a volume form Ω and a hermitian metric ρ along the fibers of K_M. Then by Green Residue Theorem [9]

$$\text{(36)} \qquad T(r, s; dd^c \log |s_f|_\rho^2) + N(r, s; D_f) = B(t, |s_f|_\rho^2)_s^r$$

for all regular values s and r of τ with $0 < s < r$. Since

$$\operatorname{Ric} \ddot{\psi}_f = dd^c \log |s_f|_\rho^2 + \operatorname{Ric} \Omega ,$$

we have

$$(37) \qquad \begin{aligned} \operatorname{Ric}_\tau(r, s) &= T(r, s; \operatorname{Ric} \Omega) + B(t, u_0 \cdot |s_f|_\rho^2)|_s^r && \text{(by (31))} , \\ &= T(r, s; \operatorname{Ric} \ddot{\psi}_f) + N(r, s; D_f) + B(t, u_0)|_s^r && \text{(by (36))} . \end{aligned}$$

It follows from (32) that

$$(38) \qquad \operatorname{Ric}_\tau(r, s) = T(r, s; f^*(\operatorname{Ric} \psi^n)) + B(t, u_1)|_s^r + N(r, s; D_F) .$$

Multiply (37) by n and minus (38) to obtain (35).

Let D be a divisor given by the zeroes of a holomorphic section $\alpha \in H^0(N, L)$. Since α and $\lambda\alpha$ ($\lambda \not\equiv 0$) define the same divisor and N is compact, we shall assume that $|\alpha(x)|_\rho \leq 1$ for $x \in N$, i.e., the metric ρ is distinguished. Assume that $\alpha_f \not\equiv 0$. The proximity form is defined by

$$m(r, D) = B(r, |\alpha_f|^{-2}) \geq 0 .$$

Then we have F. M. T. for any effective divisor (see [3], [8])

$$(39) \qquad N(r, s; D_f^\alpha) + m(t, D)|_s^r = T(r, s) ,$$

where D_f^α be the divisor of $\alpha_f \in H^0(M, L_f)$.

The following Lemma is well-known (see Nevanlinna [7]):

LEMMA 6. *Let $h(r) \geq 0$, $g(r) \geq 0$ and $\alpha(r) > 0$ be increasing continuous functions of r where $g'(r)$ is continuous and $h'(r)$ is piecewise continuous. Suppose moreover that $\int^\infty (dr/\alpha(r)) < \infty$. Then*

$$h'(r) \leq g'(r)\alpha(h(r))$$

except for a union of intervals $I \subset \boldsymbol{R}^+$ such that $\int_I dg < \infty$.

We use the notation

$$\|_\varepsilon \, a(r) \leq b(r)$$

to mean that the stated inequality holds except on an open set $I \subset \boldsymbol{R}^+$ such that $\int_I r^\varepsilon dr < \infty$ for $\varepsilon > 0$.

LEMMA 7. *Let $\varphi \geq 0$ be a form of bidegree $(1, 1)$ on M such that $T(r, s; \varphi)$ exists. Let $u \geq 0$ be a function on M such that*

$$u\nu^m \leq \varphi \wedge \nu^{m-1} .$$

Then

$$\|_\varepsilon\ B(r, u) \le \frac{\mathfrak{c}}{2}\{(1+2\varepsilon)\log T(r, s; \varphi) + 4\varepsilon \log r\}.$$

Proof. Define

$$\hat{B}(r, u) = \frac{1}{\mathfrak{c}}\int_{\partial M[r]} u\sigma .$$

Since

$$0 \le r^{2m-2}A(r, u\nu) = m\int_{M[r]} u\tau^{m-1}d\tau \wedge \sigma = 2m\int_0^r \left\{\int_{\partial M[t]} u\sigma\right\} t^{2m-1}dt$$
$$= 2m\mathfrak{c}\int_0^r \hat{B}(t, u)t^{2m-1}dt \le r^{2m-2}A(r, \varphi),$$

$\hat{B}(t, u)$ exists for almost all $t > 0$. Now

$$\frac{2}{\mathfrak{c}}B(r, u) = \frac{1}{\mathfrak{c}}\int_{\partial M[r]} \log u\sigma \le \log \hat{B}(r, u)$$

implies

$$H(r) = \int_s^r t^{1-2m}dt\int_0^t r^{2m-1}\exp\left(\frac{2}{\mathfrak{c}}B(r, u)\right)dr$$
$$\le \int_s^r t^{1-2m}dt\int_0^t r^{2m-1}\hat{B}(r, u)dr$$
$$= \frac{1}{2m\mathfrak{c}}\int_s^r A(t, u\nu)\frac{dt}{t} = \frac{1}{2m\mathfrak{c}}T(r, s; u\nu) \le \frac{1}{2m\mathfrak{c}}T(r, s; \varphi).$$

Taking $h(r) = H(r)$, $g(r) = r^{1+\varepsilon}/(1+\varepsilon)$, $\alpha(r) = r^\lambda$ with $\varepsilon > 0$ and $\lambda > 1$, we obtain from Lemma 6 that

$$\|_\varepsilon\ H'(r) = r^{1-2m}\int_0^r r^{2m-1}\exp\left(\frac{1}{\mathfrak{c}}B(r, u)\right)dr \le r^\varepsilon(h(r))^\lambda$$
$$\le r^\varepsilon(T(r, s; \varphi)/(2m\mathfrak{c}))^\lambda .$$

Keeping the same α and g and taking $h(r) = r^{2m-1}H'(r)$, we find

$$\|_\varepsilon\ r^{2m-1}\exp\left(\frac{2}{\mathfrak{c}}B(r, u)\right) = \frac{d}{dr}\left(r^{2m-1}\frac{dH}{dr}\right) \le r^\varepsilon\left(r^{2m-1}\frac{dH}{dr}\right)^\lambda$$
$$\le r^\varepsilon\{r^{\varepsilon+2m-1}(T(r, s; \varphi)/(2m\mathfrak{c}))^\lambda\}^\lambda,$$

which implies

$$\text{(40)}\qquad \|_\varepsilon\ B(r, u) \le \frac{\mathfrak{c}}{2}\{\lambda^2\log T(r, s; \varphi) + (\lambda(\varepsilon+2m-1) + (\varepsilon+1-2m))\log r$$
$$- \lambda^2\log(2m\mathfrak{c})\}.$$

Take $0 < \delta < \min(1, \varepsilon)$ such that $\varepsilon(4 + \delta) + \delta(2m - 1) < 6\varepsilon$. Let $\lambda = 1 + \delta/2$. Then $\lambda^2 < 1 + 2\varepsilon$ and

$$\lambda(\varepsilon + 2m - 1) + \varepsilon + 1 - 2m = \frac{1}{2}\{\varepsilon(4 + \delta) + \delta(2m - 1)\} < 3\varepsilon .$$

Hence Lemma 7 follows if r is large enough. q.e.d.

§ 4. Holomorphic maps into algebraic varieties of general type

Proof of Theorem 1. By Kobayashi-Ochiai [5] and Kodaira [6], an integer $p \in N$ exists such that L^p is ample and $k \in N$ exists such that $H^0(N, I)$ has positive dimension with $I = K_N^k \otimes (L^p)^*$. Take $\alpha \in H^0(N, I)$. Let D_f^α be the divisor of $\alpha_f \in H^0(M, I_f)$ and let $\hat{\rho}$ be a distinguished hermitian metric along the fibers of I. Then (39) implies

$$T(r, s; f^*c(I, \hat{\rho})) = N(r, s; D_f^\alpha) + m(t, D)|_s^r .$$

A form $\Psi > 0$ of class C^∞ and degree $2n$ exists such that $\operatorname{Ric} \Psi = c(K_N, \rho_\Psi)$ and $\hat{\rho} = (\rho_\Psi)^k \otimes (\rho^*)^p$. Hence

$$c(I, \hat{\rho}) = k \operatorname{Ric} \Psi - pc(L, \rho) ,$$

which implies

$$kT(r, s; f^*(\operatorname{Ric} \Psi)) - m(t, D)|_s^r = pT(r, s) + N(r, s; D_f^\alpha) .$$

A function $v \geq 0$ of class C^∞ exists on $M - F^{-1}(0)$ such that $\nu^m = vF[\Psi]$ and such that

$$\operatorname{Ric}_\tau (r, s) = N(r, s; D_F) + B(t, v)|_s^r + T(r, s; f^*(\operatorname{Ric} \Psi))$$

from (32), where F is an effective Jacobian section of f. Define $\tilde{\zeta} = |\alpha_f|_{\hat{\rho}}^{2/k} v^{-1}$. Then

$$\begin{aligned} \operatorname{Ric}_\tau (r, s) + B(t, \tilde{\zeta})|_s^r &= N(r, s; D_F) + T(r, s; f^*(\operatorname{Ric} \Psi)) \\ - \frac{1}{k} m(t, D)|_s^r &= N(r, s; D_F) + \frac{1}{k} N(r, s; D_f^\alpha) + \frac{p}{k} T(r, s) . \end{aligned}$$

Therefore

$$\text{(41)} \qquad nN(r, s; D_f) + \frac{p}{k} T(r, s) \leq \operatorname{Ric}_\tau (r, s) - S_f(r, s) + B(t, \zeta)|_s^r ,$$

where $\zeta = u_1 u_0^{-n} \tilde{\zeta}$ and

$$\psi = c(L, \rho) .$$

Define $\hat{\Psi} = |\alpha|_\rho^{2/k}\Psi$. Then

$$F[\hat{\Psi}] = |\alpha_f|_\rho^{2/k} F[\Psi] = \tilde{\zeta}\nu^m .$$

Since $\hat{\Psi}$ is continuous and $c(L, \rho) > 0$, a constant $\gamma_1 > 0$ exists such that $(\gamma_1 c(L, \rho))^n \geq \hat{\Psi}$, which implies

$$u_1\tilde{\zeta} = u_1 \frac{F[\hat{\Psi}]}{\nu^m} \leq u_1 \frac{F[(\gamma_1 c(L, \rho))^n]}{\nu^m} \leq \gamma_1^n .$$

Hence

$$\zeta^{1/n}\nu^m \leq \frac{\gamma_1}{u_0}\nu^m = \eta\gamma_1 f^*(c(L, \rho)) \wedge \nu^{m-1} .$$

It follows from Lemma 7 that

$$\|_\varepsilon\ B\left(t, \frac{\zeta}{\eta^n}\right)\Big|_s^r = nB(r, \zeta^{1/n}(\eta\gamma)^{-1}) + \frac{c}{2}\log \gamma_1^n - B\left(s, \frac{\zeta}{\eta^n}\right)$$

$$\leq \frac{nc}{2}\{(1 + 2\varepsilon)\log T(r, s) + 5\varepsilon \log r\} \leq \frac{p}{2k} T(r, s) + 3nc\varepsilon \log r$$

if r is large enough. Therefore

$$\|_\varepsilon\ nN(r, s; D_f) + \frac{p}{2k} T(r, s) \leq \mathrm{Ric}_\tau (r, s) - S_f(r, s) + nB(t, \eta)|_s^r + 3nc\varepsilon \log r . \tag{42}$$

Now (35) and (42) yield (9). q.e.d.

Remark. If F be dominated by τ with Y as dominator, i.e.

$$n\left(\frac{F[\psi^n]}{\nu^m}\right)^{1/n}\nu^m \leq Y(r) f^*(\psi) \wedge \nu^{m-1} \qquad \text{on } M[r]$$

holds for all continuous form $\psi \geq 0$ of bidegree (1, 1) on M, which implies

$$n\left(\frac{u_0^n}{u_1}\right)^{1/n}\eta \leq Y(r) .$$

Then

$$S_f(r, s) \geq -\ nN(r, s; D_f) - \frac{nc}{2}\log \frac{Y(r)}{n} + nB(t, \eta)|_s^r . \tag{43}$$

Hence (42) and (43) yield

$$\|_\varepsilon\ \frac{p}{2k} T(r, s) \leq \mathrm{Ric}_\tau (r, s) + \frac{nc}{2}\log \frac{Y(r)}{n} + 3nc\varepsilon \log r ,$$

which is the (4) in Theorem C.

Proof of Corollary 2. By Stoll [8], there exist effective Jacobian sections of f and holds the following

$$0 \leq \lim_{r\to\infty} \frac{\mathrm{Ric}_\tau(r, s)}{\log r} < \infty .$$

Then the condition (ii) and Theorem 1 imply

$$A(\infty) = \lim_{r\to\infty} A(r) = \lim_{r\to\infty} \frac{T(r, s)}{\log r} < \infty ,$$

where $A(r) = A(r, f^*c(L, \rho))$. Hence f is rational (see [8]). q.e.d.

Remark. The condition (ii) can be replaced by

(ii)' $$E_f = \overline{\lim_{r\to\infty}} \frac{E_f(r, s) - nN(r, s; D_f)}{\log r} < \infty .$$

If M is smooth affine algebraic variety with $m \geq n$, then there exists an effective Jacobian section of f and dominated by τ with a constant dominator $Y = m$. It follows from (35) and (43) that

$$\overline{\lim_{r\to\infty}} \frac{E_f(r, s) - nN(r, s; D_f)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{(1-n)\,\mathrm{Ric}_\tau(r, s)}{\log r} \leq 0 .$$

Hence (ii)' holds for this case and Theorem B follows from Corollary 2.

Remark. If $M = C^m$, then $\mathrm{Ric}_\tau(r, s) = 0$ where τ is defined by $\tau(z) = |z|^2$. Now (9) yields

$$E_f \geq c_1 A(\infty) > 0 ,$$

because the line bundle L is positive and rank $f = b$. Hence we have

COROLLARY 8. *Let N be a connected, n-dimensional projective algebraic manifold of general type. Then any holomorphic mappings f: $C^m \to N$ with $E_f \leq 0$ has everywhere rank less than* $\min(m, n)$.

Theorem A follows from Corollary 8 and Remark above.

Remark. If ψ satisfies

$$\overline{\lim_{r\to\infty}} \log T(r, s; f^*(\psi))/T(r, s) = 0$$

by the proof of Theorem 1, Theorem 1 holds for such ψ.

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