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QUANTIZATION OF A POISSON ALGEBRA AND POLYNOMIALS ASSOCIATED TO LINKS

TETSUYA OZAWA

Dedicated to Professor Masahisa Adachi on the occasion of his 60th birthday

§ 0. Introduction

A formal quantization of Poisson algebras was discussed by several authors (see for instance Drinfel'd [D]). A formal Lie algebra generated by *homotopy classes of loops* on a Riemann surface Σ was obtained by W. Goldman in [G], and its Poisson algebra was quantized, in the sense of Drinfel'd, by Turaev in [T]. Briefly speaking, *a quantization of a commutative Poisson algebra P* is a noncommutatively extented algebra A such that the noncommutativity is related to the Lie bracket of the Poisson algebra P through a surjective homomorphism $\rho\colon A \to P$ (see Definition 2.1). In Turaev's quantization, the algebra A is a semi-group algebra generated by *links* in the thickened Riemann surface ((Riemann surface Σ) $\times$ (real line $\mathbf{R}$)).

On the other hand, there is an obvious map from Goldman's Poisson algebra to the ring $\mathbf{Z}[H_1]$ of polynomials with integral coefficients on the first homology group H_1 of the Riemann surface which assigns to each loop its homology class. There arises a natural question whether it is possible to construct a map from Turaev's algebra to the ring $\mathbf{Z}[H_1][h]$, the polynomial ring with coefficient in $\mathbf{Z}[H_1]$, such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{Z}[H_1][h] \\ {\scriptstyle \text{quantization}}\downarrow & & \downarrow{\scriptstyle h \to 0} \\ P & \longrightarrow & \mathbf{Z}[H_1] \end{array}$$

commutes. Our main purpose in this paper is to show that the *polynomial invariant of links* introduced in § 3 (Definition 3.1) gives an answer

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to the above question (Theorem 4.2 and Theorem 5.4.2). In order to satisfy the condition on quantization, the algebra A has to be a quotient of the algebra of links divided by a certain ideal which corresponds to a relation so called the *skein relation.* Since our polynomial invariant satisfies the skein relation for certain types of triples of links, but not for all Conway triples (see the condition (SR) in § 2 for the precise statement), we adopt a quantization which is slightly different from Turaev's, but the idea is following Turaev. We also consider quantizations of the Poisson algebra generated by nonoriented loops, and discuss the same question for nonoriented loops and links.

As an invariant of links, our polynomial invariants have different nature from, for example, Alexander's, Jones,... invariants. Actually if a link is contained in a 3-ball in $\Sigma \times \mathbf{R}$, then our polynomial associated to this link turns out to be zero. Roughly speaking, our polynomial measures the global behavior of a link on the handles of the surface.

In § 1 and § 2, we review Goldman's Lie bracket and modified Turaev's quantization. In § 3, we give a definition of our polynomial invariant of oriented links, ane in § 4, discuss some properties of the polynomial invariant and its relation to the quantization. In § 5, we investigate nonoriented cases. In Appendix, we discuss quantizations in a more general setting, and construct a certain morphism which extends our polynomial invariant.

§ 1. Lie algebra generated by oriented loops

In this paper, we denote by Σ an oriented Riemann surface, not necessarily closed. Let $\hat{\pi}$ be the set of homotopy classes of loops on Σ, (i.e. the set of conjugacy classes in the fundamental group of Σ). Let $\mathbf{Z}\hat{\pi}$ denote the $\mathbf{Z}$-module freely generated by $\hat{\pi}$.

For each $\alpha \in \hat{\pi}$ and a simple point p on α (we use the same notation for a representative of a homotopy class), we denote by α_p the corresponding element in the fundamental group based at p.

DEFINITION 1.1. For homotopy classes α and $\beta \in \hat{\pi}$, we define the bracket by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \alpha_p \cdot \beta_p \in \mathbf{Z}\hat{\pi}\,,$$

where $\varepsilon_p = \pm 1$ is the intersection number of α and β at p, and $\alpha_p \cdot \beta_p$ is the product in the fundamental group (see Fig. 1.1).

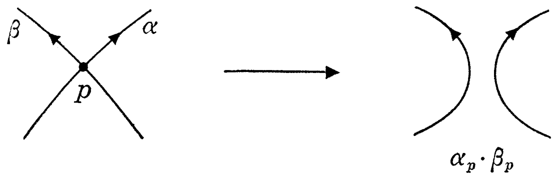

Fig. 1.1.

For the proof of the fact that the above definition of the bracket is independent of the choice of representatives, and for the proof of the following, see [G].

PROPOSITION 1.2. $(\mathbf{Z}\hat{\pi}, [\ ,\ ])$ *is a Lie algebra.*

Let K be a commutative ring (we will consider only the ring $\mathbf{Z}$ of integers for K). Let P be a K-module with a commutative multiplication structure. We call P *a Poisson algebra*, if P carries a Lie algebra structure which satisfies the Leibniz rule with respect to the multiplication of P. Let g be a Lie algebra. The direct sum of symmetric tensor products of g, $S^*(g)$, has the canonical multiplication, and the Lie bracket on g extends to a bracket on $S^*(g)$ using the Leibniz rule. Thus $S^*(g)$ can be considered as a Poisson algebra.

DEFINITION 1.3. We denote by $S(\mathbf{Z}\hat{\pi})$ the Poisson algebra constructed from the Lie algebra $\mathbf{Z}\hat{\pi}$ as above.

§ 2. Quantization of $S(\mathbf{Z}\hat{\pi})$

Let P be a Poisson algebra over a ring K, and A a $K[h]$-algebra. A K-module homomorphism ρ: $A \to P$ is called a $K[h]$-module homomorphism, if for each $f(h) \in K[h]$ and each $a \in A$, the equality $\rho(f(h)a) = f(0)\rho(a)$ holds.

DEFINITION 2.1. If the noncommutativity of A is related with the bracket on P through a surjective $K[h]$-homomorphism ρ: $A \to P$ by

$$ab - ba \in h\rho^{-1}([\rho(a), \rho(b)]),$$

then we call ρ: $A \to P$ *a quantization of* P.

Turaev obtained in [T] a quantization of $S(\mathbf{Z}\hat{\pi})$. We define a quantization of $S(\mathbf{Z}\hat{\pi})$ which is conceptually the same as Turaev's.

Let $\hat{A}_1$ be the $\mathbf{Z}[h]$-module freely generated by oriented links in $\Sigma \times \mathbf{R}$. Define a multiplication $L_1 \cdot L_2$ of two links L_1 and L_2 in $\Sigma \times \mathbf{R}$ as follows; putting L_1 (resp. L_2) in $\Sigma \times \{t > 0\}$ (resp. $\Sigma \times \{t < 0\}$) by parallel translations along $\mathbf{R}$, we define $L_1 \cdot L_2$ as the link obtained by taking the union of L_1 and L_2. Thus $\hat{A}_1$ is considered as a semi-group algebra over $\mathbf{Z}[h]$. Let $\hat{I}_0$ be the ideal of $\hat{A}_1$ generated by the following elements;

(SR) $L_+ - L_- - hL_0$, where $L_\pm$ and L_0 are links such that (i) these three links have the same representatives outside a 3-ball in which they look like as Fig. 2.1 (Conway triple), and (ii) the numbers $|L_\pm|$ and $|L_0|$ of loop components of $L_\pm$ and L_0 satisfy the equality $|L_\pm| = |L_0| + 1$.

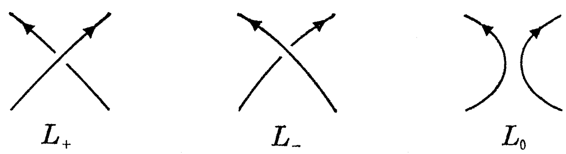

Fig. 2.1.

Let l_i ($i = 1, \cdots, |L|$) be the projection image on Σ of each loop component of a given link L in $\Sigma \times \mathbf{R}$. Here the projection $\Sigma \times \mathbf{R} \to \Sigma$ is the canonical one. We define

$$\hat{\rho}(L) = \prod_{i=1}^{|L|} l_i \in S(\mathbf{Z}\hat{\pi}) .$$

It is easy to see that $\hat{\rho}$ turns out to be a $\mathbf{Z}[h]$-module homomorphism

$$\hat{\rho}\colon \hat{A} = \hat{A}_1/\hat{I}_0 \longrightarrow S(\mathbf{Z}\hat{\pi}) .$$

The proof of the following is the same as that of Theorem 6.3.1. of [T].

PROPOSITION 2.2. *$\hat{\rho}\colon \hat{A} \to S(\mathbf{Z}\hat{\pi})$ is a quantization. In fact $\hat{\rho}$ satisfies*

$$\hat{\rho}\left(\frac{L_1L_2 - L_2L_1}{h}\right) = [\hat{\rho}(L_1), \hat{\rho}(L_2)] .$$

§ 3. Polynomials associated to oriented links

In this section we give a definition of the polynomial invariant. First we construct a polynomial looking at one of the corresponding link-diagrams on the Riemann surface, and then we show the polynomial is independent of the choice of diagrams, i.e. under the Reidemeister moves (see [R]). By *oriented link diagram* on a Riemann surface Σ, we mean an oriented graph D on Σ whose vertices are as in Fig. 3.1, where the vertex in Fig. 3.1. $(+1)$ (resp. Fig. 3.1 (-1)) is called of type $+1$ (resp.

-1). The number $|D|$ of loop components of the corresponding link to D is well defined, since there is associated a unique oriented link in $\Sigma \times \mathbf{R}$ to each oriented link diagram D on Σ,

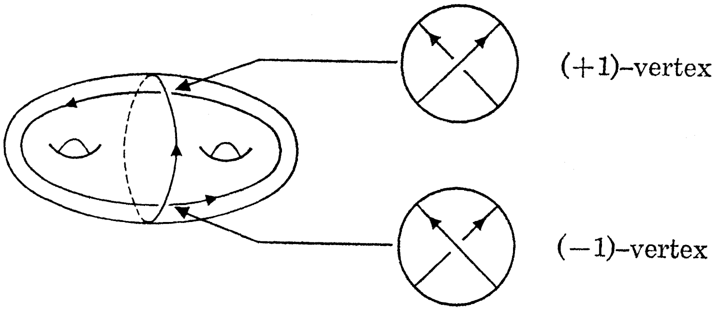

Fig. 3.1.

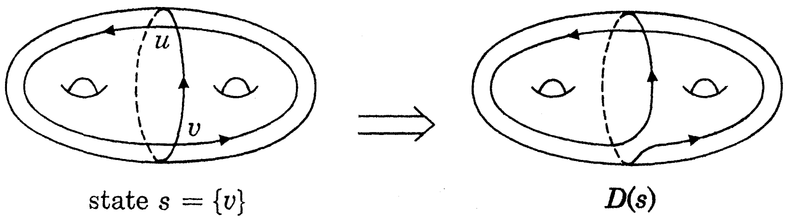

Fig. 3.2.

For a set s of vertices of D, let $D(s)$ be the link diagram obtained from D by changing D near the vertices in s as shown in Fig. 3.2 (for the definition of state, see Definition 3.1).

We fix a basis $\{x_i\}_{i=1}^k$ of the homology group $H_1(\Sigma; \mathbf{Z})$. Let D be an oriented link diagram on Σ, and l_i $(i = 1, \cdots, |D|)$ the loop components on Σ of the corresponding link in $\Sigma \times \mathbf{R}$ (l_i are considered to be projected on Σ). If the component l_i realize the homology class $\sum_{j=1}^k a_j x_j$, then, considering $\sum_{j=1}^k a_j x_j$ as a polynomial in $\mathbf{Z}[x_1, \cdots, x_k]$, we denote this polynomial by $\langle l_i \rangle$. Define

$$\langle D \rangle = \prod_{i=1}^{|D|} \langle l_i \rangle \in \mathbf{Z}[x_1, \cdots, x_k] .$$

DEFINITION 3.1. By a *state* of a link diagram D, we mean a subset s of vertices such that $|D| = |D(s)| + |s|$, where $|s|$ is the number of vertices in s. We define a polynomial $\langle\!\langle D \rangle\!\rangle \in \mathbf{Z}[x_1, \cdots, x_k, h]$ by

$$\langle\!\langle D \rangle\!\rangle = \sum_s (-1)^{|s|-}\langle D(s) \rangle (h/2)^{|s|} ,$$

where the sum is taken over all states s of D, and $|s|_-$ denotes the number of vertices in s of type -1.

THEOREM 3.2. *The polynomial* $\langle\!\langle\ \rangle\!\rangle$ *is an invariant of an oriented link.*

In the following lemma, we prove $\langle\!\langle D\rangle\!\rangle$ is invariant under the Reidemeister moves, I, II and III.

LEMMA 3.3. $\langle\!\langle D\rangle\!\rangle$ *is invariant under the moves* I.

The proof is trivial.

LEMMA 3.4. $\langle\!\langle D\rangle\!\rangle$ *is invariant under the moves* II.

Proof. We denote by u and v the vertices in D' which are not in D (see Fig. 3.3). If a state s contains neither u nor v, then it contributes to $\langle\!\langle D'\rangle\!\rangle$ in the same way as to $\langle\!\langle D\rangle\!\rangle$. A state s of D' cannot contain both u and v. The sum over all states sD' which contain one of u and v is equal to 0, since these vertices have different types $+1$ and -1.

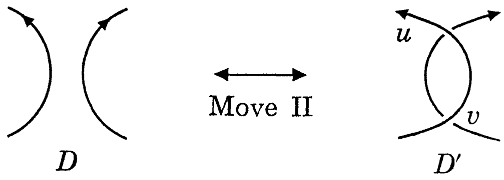

Fig. 3.3.

LEMMA 3.5. $\langle\!\langle D\rangle\!\rangle$ *is invariant under the moves* III.

Proof. We fix a 1:1 correspondence between the sets of vertices of diagrams D and D' as in Fig. 3.4. Note that the corresponding vertices v_i and v_i' have the same type s ± 1, and that the sets of all states of D and D' are also in 1:1 corresponddence. Under this correspondence it is clear that $\langle D(s)\rangle = \langle D'(s)\rangle$ for any state s.

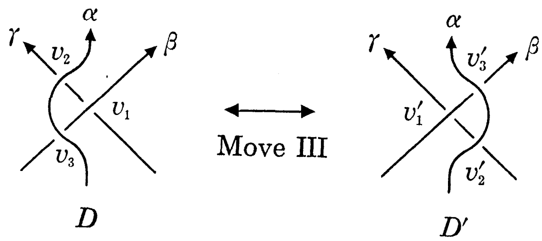

Fig. 3.4.

EXAMPLE 3.6. Let $\Sigma = T^2$ be the 2-torus. We fix generators x and y of $H_1(T^2; Z)$ as in Fig. 3.5. We exhibit examples in Fig. 3.6. In general, we have the equality shown in Fig. 3.7.

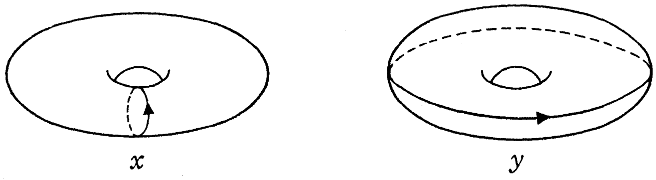

Fig. 3.5.

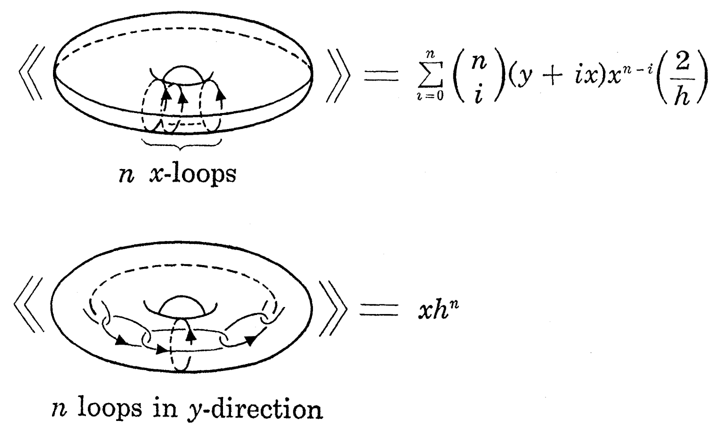

Fig. 3.6.

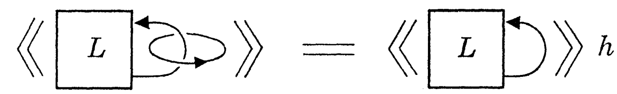

Fig. 3.7.

§ 4. Properties of the polynomial and its relation to the quantization

Given a Riemann surface Σ, we fix a basis $\{x_i\}_{i=1}^k$ of the homology group $H_1(\Sigma; \mathbf{Z})$. To each oriented link L in $\Sigma \times \mathbf{R}$, we associated a polynomial $\langle\langle L \rangle\rangle \in \mathbf{Z}[x_1, \cdots, x_k, h]$. We list some properties of the polynomial in the following;

THEOREM 4.1. *Let L be an oriented link in $\Sigma \times R$, and $\langle\langle L \rangle\rangle$ the polynomial constructed in* § 3.

(1) *If a link consists of d loop components, then* $\langle\langle L\rangle\rangle$ *is homogeneous of degree d.*

(2) $\langle\langle L\rangle\rangle(x_1, \cdots, x_k, 0) = \langle L\rangle(x_1, \cdots, x_k)$.

(3) *If L is semi-local (i.e. there is a nonempty subset of L and a* 3-*ball in* $\Sigma \times R$ *such that the subset is contained in the* 3-*ball and L is disjoint from the boundary of the* 3-*ball), then* $\langle\langle L\rangle\rangle = 0$.

(4) *If* L_+, L_- *and* L_0 *are a Conway triple with* $|L_\pm| = |L_0| + 1$, *then* $\langle\langle L_+\rangle\rangle - \langle\langle L_-\rangle\rangle = h\langle\langle L_0\rangle\rangle$.

(5) *If* L_1 *and* L_2 *have representatives with disjoint projection images on* Σ), *then* $\langle\langle L_1 \cdot L_2\rangle\rangle = \langle\langle L_1\rangle\rangle \cdot \langle\langle L_2\rangle\rangle$.

Proofs. (1) The homogeneity of the polynomial is a consequence of the condition $|D| = |D(s)| + |s|$ on states s.

(2) It is enough to note that $|D(s)| = |D|$ if and only if $|s| = 0$.

(3) If L itself is contained in a 3-ball, then $\langle D(s)\rangle = 0$ for all states s. For general semi-local links, the triviality of the polynomial is a consequence of (5).

(4) Choose link diagrams D_+, D_- and D_0 for L_+, L_- and L_0 such that there is a small 2-disc on Σ where these diagrams look like as Fig. 2.1, and outside the disc they coincide. Let v_0 be the vertex of $D_\pm$ in the disc. Then all states of D_+ are states of D_-, and vise versa. If a state s of D_+ contains the vertex v_0 then $s - \{v_0\}$ is a state of D_0, and $D_+(s) = D_-(s) = D_0(s - \{v_0\})$. This implies the equality.

(5) We can take link diagrams D_1 for L_1 and D_2 for L_2 such that $D_1 \cap D_2 = \emptyset$. Then we have {states of $D_1 \cup D_2$} = {states of D_1} $\times$ {states of D_2}, and $(-1)^{|s_1 \cup s_2|_-} = (-1)^{|s_1|_-} \times (-1)^{|s_2|_-}$ for any states s_1 of D_1 and s_2 of D_2. This completes the proof.

Let $\hat{\rho}: \hat{A} \to S(\mathbf{Z}\hat{\pi})$ be the quantization obtained in the last section. The property (4) in Theorem 4.1 implies the well definedness of the $\mathbf{Z}[h]$-module homomorphism $\langle\langle\ \rangle\rangle: \hat{A} \to \mathbf{Z}[x_1, \cdots, x_k, h]$, and the property (2) implies the commutativity of the following;

THEOREM 4.2. *The diagram*

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\langle\langle\ \rangle\rangle} & \mathbf{Z}[x_1, \cdots, x_k, h] \\ \hat{\rho}\downarrow & & \downarrow h \to 0 \\ S(\mathbf{Z}\hat{\pi}) & \xrightarrow{\langle\ \rangle} & \mathbf{Z}[x_1, \cdots, x_k] \end{array}$$

is well defined and commutative.

§ 5. Nonoriented loops and links

So far we considered oriented loops and oriented links. In this section, we consider the nonoriented case, but again fix the orientation of the Riemann surface. We show the analogous results in parallel with those in the preceding sections, and the indices of Propositions, ... are compatible.

5.1. Lie algebra generated by nonoriented loops

Let $\tilde{\pi}$ be the set of homotopy classes of (nonoriented) loops on a given Riemann surface Σ, and $\mathbf{Z}\tilde{\pi}$ the $\mathbf{Z}$-module generated by $\tilde{\pi}$. For loops α and β given in general position on Σ, and for each point $p \in \alpha \cap \beta$, we denote by $(\alpha\cdot\beta)_p^{\pm}$ the loops obtained by changing α and β near p as shown in Fig. 5.1. We define bracket structure on $\mathbf{Z}\tilde{\pi}$ as follows (see [G]);

DEFINITION 5.1.1.

$$[\alpha, \beta] = \sum_{p\in\alpha\cap\beta} ((\alpha\cdot\beta)_p^{+} - (\alpha\cdot\beta)_p^{-}) .$$

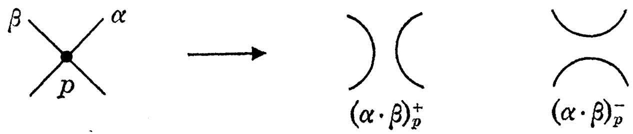

Fig. 5.1.

PROPOSITION 5.1.2. *The* $\mathbf{Z}$*-module* $\mathbf{Z}\tilde{\pi}$ *with the above bracket is a Lie algebra.*

For the proof, see [G].

DEFINITION 5.1.3. Let $S(\mathbf{Z}\tilde{\pi})$ denote the Poisson algebra corresponding to the Lie algebra $\mathbf{Z}\tilde{\pi}$.

5.2. Quantization

We consider nonoriented links in $\Sigma \times \mathbf{R}$, and construct the $\mathbf{Z}[h]$-algebra $\tilde{A}_1$ with a semi-group algebra structure and an ideal $\tilde{I}_0$ in the same way as the oriented case except the condition (SR). In place of (SR), we consider the following;

(GSR) $L_+ - L_- - h(L_0 + L_\infty)$, where $L_\pm$, L_0 and L_∞ are links such that (i) these four links have the same representatives outside a 3-ball in which they are as in Fig. 5.2, and (ii) $|L_\pm| = |L_0| + 1 = |L_\infty| + 1$.

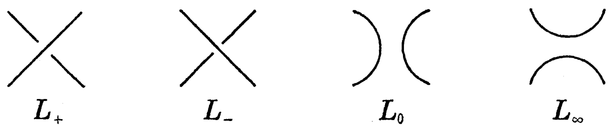

Fig. 5.2.

Multiplications on $\tilde{A}_1$ and on the quotient $\tilde{A} = \tilde{A}_1/\tilde{I}_0$ and a $\mathbf{Z}[h]$-module homomorphism $\tilde{\rho}\colon \hat{A} \to S(\mathbf{Z}\tilde{\pi})$ are all defined in the same way as in the oriented case.

PROPOSITION 5.2.2. *$\tilde{\rho}\colon \hat{A} \to S(\mathbf{Z}\tilde{\pi})$ is a quantization. In fact $\tilde{\rho}$ satisfies*

$$\tilde{\rho}\left(\frac{L_1L_2 - L_2L_1}{h}\right) = [\tilde{\rho}(L_1), \tilde{\rho}(L_2)] .$$

5.3. Polynomials associated to nonoriented links

Let D be a (nonoriented) link diagram on a Riemann surface Σ. For a map s which associated to each vertex of D a number ± 1 or 0, we change D near its vertices as shown in Fig. 5.3 to obtain a new diagram $D(s)$.

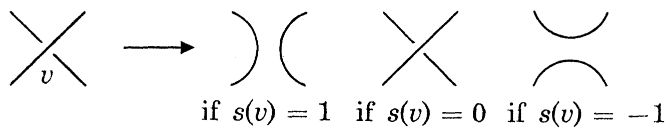

Fig. 5.3.

For the projection image l_i $(i = 1, \cdots, d = |D|)$ of each component of the link corresponding to the diagram D, we choose an orientation on l_i, and write its homology class by $\sum a_i x_i$, where $\{x_i\}_{i=1}^k$ is a fixed basis of $H_1(\Sigma; \mathbf{Z})$. Then we define a polynomial $\langle l_i \rangle$ by $\langle l_i \rangle = \sum |a_i| x_i$, and a polynomial $\langle D \rangle$ by $\langle D \rangle = \prod \langle l_i \rangle$.

DEFINITION 5.3.1. By a *state* of the diagram D, we mean a map s as above such that $|D| = |D(s)| + |s|$, where $|s|$ is the number of vertices which are associated to 1 or -1 by s. Let $|s|_-$ denote the number of vertices which are assigned to -1 by s. We define

$$\langle\langle D \rangle\rangle = \sum (-1)^{|s|_-} \langle D(s) \rangle (h/2)^{|s|} ,$$

where the sum is taken over all states s of D.

THEOREM 5.3.2. *The polynomial* $\langle\!\langle\ \rangle\!\rangle$ *is a nonoriented link invariant.*

Proof. The proof of the invariance under the Reidemeister moves is the same as the oriented case except the following observation about Move III.

We consider two diagrams D and D' as in Fig. 3.4, forgetting their orientations. We fix the canonical 1:1 correspondence between the sets of states of D and D'. Remark that under the correspondence the resulting diagrams $D(s)$ and $D'(s)$ are, in general, not homotopic, if there are two vertices in Fig. 3.4 on which the state s is not zero. In order to see the difference of contributions of such states on the polynomials $\langle\!\langle D'\rangle\!\rangle$ and $\langle\!\langle D'\rangle\!\rangle$, we choose the orientation on each loop component as in Fig. 3.4. Then we reduce the difference in the form

$$\begin{aligned}\langle\!\langle D\rangle\!\rangle - \langle\!\langle D'\rangle\!\rangle &= f\cdot g\\ g &= \langle\alpha\beta\gamma\rangle + \langle\alpha\gamma^{-1}\beta\rangle + \langle\alpha\gamma\beta^{-1}\rangle + \langle\alpha\beta^{-1}\gamma^{-1}\rangle\\ &\quad - \langle\alpha\gamma\beta\rangle - \langle\alpha\beta^{-1}\gamma\rangle - \langle\alpha\beta\gamma^{-1}\rangle - \langle\alpha\gamma^{-1}\beta^{-1}\rangle\,,\end{aligned}$$

where f is some polynomial in $\mathbf{Z}[x_1, \cdots, x_k, h]$, and $\langle\alpha\beta\gamma\rangle, \cdots$ are the polynomials in $\mathbf{Z}[x_1, \cdots, x_k]$ which corresponds to the loops $\alpha\beta\gamma, \cdots$ being forgotten their orientations (the product $\alpha\beta\gamma$ is taken in the fundamental group), in the same way as $\langle D\rangle$. From the definition of the polynomial, $\langle\ \rangle$, we have, for example, $\langle\alpha\beta\gamma\rangle = \langle\alpha\gamma\beta\rangle$, and thus $g = 0$.

5.4. Properties of the polynomial and its relation to the quantization

The proof of the following is similar as Theorem 4.1.

THEOREM 5.4.1. (1) *The polynomial* $\langle\!\langle L\rangle\!\rangle$ *of a nonoriented link* L *with* d *connected components is homogeneous of degree* d.

(2) $\langle\!\langle L\rangle\!\rangle(x_1, \cdots, x_k, 0) = \langle L\rangle(x_1, \cdots, x_k)$.

(3) *If* L *is semi-local (i.e. there is a nonempty subset of* L *and a* 3-*ball in* $\Sigma\times\mathbf{R}$ *such that the subset is contained in the* 3-*ball and* L *is disjoint from the boundary of the* 3-*ball), then* $\langle\!\langle L\rangle\!\rangle = 0$.

(4) *If* L_+, L_-, L_0 *and* L_∞ *satisfy the condition* (*GSR*) *in* (5.2), *then* $\langle\!\langle L_+\rangle\!\rangle - \langle\!\langle L_-\rangle\!\rangle = h(\langle\!\langle L_0\rangle\!\rangle - \langle\!\langle L_\infty\rangle\!\rangle)$.

(5) *If* L_1 *and* L_2 *have representatives with disjoint projection images on* Σ), *then* $\langle\!\langle L_1\cdot L_2\rangle\!\rangle = \langle\!\langle L_1\rangle\!\rangle\cdot\langle\!\langle L_2\rangle\!\rangle$.

Finally we state a relation of the polynomial in (5.3) to the quantization in (5.2) in the following;

THEOREM 5.4.2. *The diagram*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\langle\!\langle\ \rangle\!\rangle} & \mathbf{Z}[x_1, \cdots, x_k, h] \\ \tilde{\rho}\downarrow & & \downarrow h \to 0 \\ S(\mathbf{Z}\tilde{\pi}) & \xrightarrow{\langle\ \rangle} & \mathbf{Z}[x_1, \cdots, x_k] \end{array}$$

is well defined and commutative.

Appendix

In Appendix, we consider an abstract procedure of quantization of Poisson algebras, and construct a homomorphism of a noncommutative algebra to a polynomial ring, in a general setting, and then we relate this homomorphism to our polynomial invariant constructed in § 3.

A.1. Let be given

(1) a commutative ring R,

(2) a module V over R,

(3) a skew symmetric bilinear form B: $V \otimes V \to R$.

From these data, we can construct a Lie algebra structure on the module $g = \bigoplus_{v \in V} Re_v$ with the free generators e_v parametrized by $v \in V$ as follows. The Lie bracket is defined by

$$[e_v, e_w] = B(v, w)e_{v+w} .$$

We extend this bracket to that on the symmetric tensor product space $S(g)$ by the Leibniz rule, obtaining a Poisson algebra (cf. Definition 1.3).

A.2. A quantization of this Poisson algebra is obtained as follows; consider a bracket $[\ ,\ ]'$ on the module $g \otimes R[h]$ defined by

$$[x, y]' = h[x, y] \qquad (x, y \in g) ,$$

and denote by $S_h(g)$ the universal envelopping algebra of $g \otimes R[h]$ with respect to the Lie bracket $[\ ,\ ]'$. The natural map $S_h(g) \to S(g)$ obtained by putting $h = 0$ is a quantization of the Poisson algebra $S(g)$ in the sense of Definition 2.1.

This quantization is functorial in the following sense. If f: $g \to g'$ is a Lie algebra homomorphism, then f canonically extends to a homomorphism $S(g) \to S(g')$ (denoted by the same letter f), and then f is lifted up to a homomorphism F: $S_h(g) \to S_h(g')$, so that the diagram

$$\begin{array}{ccc} S_h(g) & \xrightarrow{F} & S_h(g') \\ \downarrow & & \downarrow \\ S(g) & \xrightarrow[f]{} & S(g') \end{array}$$

commutes.

A.3. Let be given

(4) a commutative R-algebra T

(5) an R-algebra homomorphism λ: $S(g) \to T$.

We are interested in $R[h]$-module homomorphisms Λ such that the following diagram commutes;

$$\begin{array}{ccc} S_h(g) & \xrightarrow{\Lambda} & T[h] \\ {\scriptstyle h \to 0}\downarrow & & \downarrow{\scriptstyle h \to 0} \\ S(g) & \xrightarrow[\lambda]{} & T\,. \end{array}$$

From the Poincare-Birkhoff-Witt theorem, we know that $S_h(g)$ is a free $R[h]$-module, since $S_h(g)$ is a universal enveloping algebra, and also, introducing a linear ordering on V, we can use an R-module basis of $S(g)$ as a basis of $S_h(g)$. So there are prenty of homomorphisms Λ as above. We will see that there exists, among them, a natural homomorphism Λ which is closely related to our polynomial invariant obtained in § 3.

A.4. Fix an element $e_{v_1} \bullet \cdots \bullet e_{v_n} \in S_h(g)$, where $\bullet$ denotes the multiplication in $S_h(g)$ as an enveloping algebra. Let's consider the complete graph $\Gamma(n)$ with n vertices $\{1, \cdots, n\}$ and with $\binom{n}{2}$ edges that are oriented from the smaller number to the larger number. Let Γ be a subgraph of $\Gamma(n)$, and $\{\Gamma_i;\ i = 1, \cdots, |\Gamma|\}$ the set of all connected components of Γ. Let $\langle \Gamma \mid \lambda, B \rangle$ denote the element in T defined by

$$\langle \Gamma \mid \lambda, B \rangle = \left(\prod_{i=1}^{|\Gamma|} \sum_{a \in \mathrm{Vert}\,(\Gamma_i)} \lambda(e_{v_a}) \right) \prod_{u \in \mathrm{Edge}\,(\Gamma)} B(v_{u(0)}, v_{u(1)})\,,$$

where $u(0)$ and $u(1)$ denote the initial and terminal vertices of the edge u. We define

$$\Lambda(e_{v_1} \bullet \cdots \bullet e_{v_n}) = \sum_{\Gamma} \langle \Gamma \mid \lambda, B \rangle (h/2)^{n-|\Gamma|}\,,$$

where the sum is taken over all subgraphs which contain all vertices of

$\Gamma(n)$ and have no 1-cycles. It is a direct calculation to verify that the polynomials $\Lambda(e_{v_1} \bullet \cdots \bullet e_{v_n}) \in T[h]$ satisfy the equality

$$\Lambda(e_{v_1} \bullet e_{v_2} \bullet X) - \Lambda(e_{v_2} \bullet e_{v_1} \bullet X) = h\Lambda([e_{v_1}, e_{v_2}] \bullet X)$$

for any v_1 and $v_2 \in V$ and any $X \in S_h(g)$. Therefore the homomorphism Λ: $S_h(g) \to T[h]$ is well defined, and makes the diagram in A.3 commutative.

A.5. Now we consider the case where

(i) $R = \mathbf{Z}$

(ii) $V = H_1(\Sigma; \mathbf{Z})$

(iii) $B =$ the intersection pairing on $H_1(\Sigma; \mathbf{Z})$.

Let $S_h(g) \to S(g)$ be the quantization associated to the Lie algebra g as is obtained in the previous sections from the above data (i) $\sim$ (iii). If we start with Goldman's Lie algebra $\mathbf{Z}\hat{\pi}$, then we get the quantization $S_h(\mathbf{Z}\hat{\pi}) \to S(\mathbf{Z}\hat{\pi})$. The homomorphism $\mathbf{Z}\hat{\pi} \to g$ induced by the Hurwitz homomorphism $\pi_1 \to H_1(\Sigma; \mathbf{Z})$ is a Lie algebra homomorphism. Therefore we get the following commutative diagram;

$$\begin{array}{ccc} S_h(\mathbf{Z}\hat{\pi}) & \longrightarrow & S_h(g) \\ \downarrow & & \downarrow \\ S(\mathbf{Z}\hat{\pi}) & \longrightarrow & S(g)\,. \end{array}$$

A.6. Let $\hat{A}_0$ be the $\mathbf{Z}[h]$-algebra generated by oriented links in the product space $\Sigma \times \mathbf{R}$ of a Riemann surface Σ with a real line $\mathbf{R}$. We consider the following ideal $\hat{I}_H$ rather than the ideal $\hat{I}_0$ in § 2. The ideal $\hat{I}_H$ in $\hat{A}_0$ is, by definition, generated by the following elements (HSR);

(m) $\quad L_+ - L_- - hL_0\,, \quad$ if $|L_\pm| = |L_0| + 1$ (mixed Conway triple),

(p) $\quad L_+ - L_-\,, \quad$ if $|L_\pm| = |L_0| - 1$ (pure Conway triple),

where $L_\pm$ and L_0 are Conway triples, and the letters p and m are the initials of pure and mixed Conway triples (see [HP]). This is called the *homotopy* skein relation, and the quotient $\hat{A}_H = \hat{A}_0/\hat{I}_H$ is called the homotopy skein module of the 3-manifold. For our 3-manifold $\Sigma \times \mathbf{R}$ the homotopy skein module is of course a $\mathbf{Z}[h]$-algebra. Hoste and Przytycki has shown in [HP] that

$$\hat{A}_H \cong S_h(\mathbf{Z}\hat{\pi})\,,$$

A.7. Let's consider, for the Lie algebra g constructed in §(A.5) from the homology group $V = H_1(\Sigma; \mathbf{Z})$, the following algebra and morphism;

(iv) $T = \mathbf{Z}[x_1, \cdots, x_n]$,

(v) $\lambda: S(g) \rightarrow \mathbf{Z}[x_1, \cdots, x_n]$, the canonical homomorphism,

where $x_1, \cdots$ and x_n are the generators of $H_1(\Sigma; \mathbf{Z})$ as in § 3. Summarizing the constructions from § A.1 to § A.6, we get the following commutative diagram;

$$\begin{array}{ccccc} \hat{A}_H & \longrightarrow & S_h(g) & \longrightarrow & Z[x_1, \cdots, x_n, h] \\ \downarrow & & \downarrow & & \downarrow \\ S(\mathbf{Z}\hat{\pi}) & \longrightarrow & S(g) & \longrightarrow & Z[x_1, \cdots, x_n]\,. \end{array}$$

A direct calculation shows

THEOREM. *The composition of homomorphisms in the upper horizontal line of the above commutative diagram is the same as the polynomial invariant* $\langle\langle \quad \rangle\rangle$ *constructed in* § 3.

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Department of Mathematics
School of Science
Nagoya University
Chikusa-ku, Nagoya, 464–01
Japan