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ON THE LENGTH OF THE POWERS OF SYSTEMS OF PARAMETERS IN LOCAL RING

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§ 1. Introduction

Throughout this note, A denotes a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module with $\dim(M) = d$. Let $x_1, \cdots, x_d$ be a system of parameters (s.o.p. for short) for M and I the ideal of A generated by $x_1, \cdots, x_d$. We consider the length $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ over A as a function in the positive integers $n_1, \cdots, n_d$. J-L. Garcia Roig and D. Kirby [5] have shown that this function is generally not a polynomial for $n_1, \cdots, n_d \gg 0$ (sufficiently large) but, if M is a generalized Cohen-Macaulay module, then

$$l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) = n_1 \cdots n_d e(I; M) + \sum_{i=0}^{d-1} \binom{d-1}{i} l(H_{\mathfrak{m}}^i(M))$$

for $n_1, \cdots, n_d \gg 0$, where $e(I; M)$ denotes the multiplicity of M relative to I and $H_{\mathfrak{m}}^i(M)$ is the i-th local cohomology module of M with respect to $\mathfrak{m}$. Therefore, it is natural to ask under which conditions $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ is a polynomial for $n_1, \cdots, n_d \gg 0$? (see [9], Question 1.1).

The purpose of this note is to give an answer to this question. Before stating the main result we need the following definition. Let $x_1, \cdots, x_d$ be a s.o.p. for M. We say that $x_1, \cdots, x_d$ is a p-system of parameters (p-s.o.p. for short) for M if there exists a positive integer n_0 such that

$$(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M \colon x_i^{n_i} = (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M \colon x_i^{n_0}$$

for all $n_1, \cdots, n_d \geq n_0$, $i = 1, \cdots, d$ ($x_0 = 0$).

We say that $x_1, \cdots, x_d$ is an unconditioned p-s.o.p. if for every permutation of the sequence $x_1, \cdots, x_d$, the above condition holds with respect to the same integer n_0.

THEOREM 1. *The function* $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ *is a polynomial for*

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$n_1, \cdots, n_d \gg 0$ *if and only if* $x_1, \cdots, x_d$ *is an unconditioned p-s.o.p. for M.*

We will prove this theorem in Section 2. In Section 3, we will relate p-s.o.p.'s to some special s.o.p.'s in the theory of local ring such as filter regular s.o.p.'s [3] and standard s.o.p.'s [11] and show that a generalized Cohen-Macaulay module can be characterized by p-s.o.p.'s. At the end of this note, in Section 4, we consider the case $\dim(M) = 2$ more closely. In this case, we will see that a p-s.o.p. can be characterized by the finitely generated condition of certain relative Rees ring by using a recent result of P. Schenzel (see [6]).

§ 2. Proof of Theorem 1

For convenience, we will use following notations.

Let $n_1, \cdots, n_d$ be positive integers. Then we put $n(i) = (n_1, \cdots, n_i)$ and $I_{n(i)} = (x_1^{n_1}, \cdots, x_i^{n_i})A$, $I = I_d = (x_1, \cdots, x_d)A$.

If α is an element of the group of permutations S_d, $\alpha = (\alpha(1), \cdots, \alpha(d))$ we set $I_{n(\alpha(i))} = (x_{\alpha(1)}^{n_{\alpha(1)}}, \cdots, x_{\alpha(i)}^{n_{\alpha(i)}})A$.

We first note that if a s.o.p. $x_1, \cdots, x_d$ for M is a regular M-sequence then it is obviously an unconditioned p-s.o.p.. Furthermore, if $x_1^{n_1}, \cdots, x_d^{n_d}$ is a d-sequence which has been introduced by Huneke [13] for all $n_1, \cdots, n_d \gg 0$ then $x_1, \cdots, x_d$ is a p-s.o.p. Therefore, by [3], every s.o.p. for a generalized Cohen-Macaulay module is an unconditioned p-s.o.p. The property of p-s.o.p.'s has been examined more closely by the author in [12]. Here, we only give some characterizations of unconditioned p-s.o.p.'s which we need for the proof of Theorem 1.

LEMMA 2. *The following conditions are equivalent*:

(i) $x_1, \cdots, x_d$ *is an unconditioned p-s.o.p.*

(ii) *There exists a positive integer* n_0 *such that for all* $n_1, \cdots, n_d \geq n_0$ *and any permutation* α *of* S_d, *we have the equality*

$$(I_{n(\alpha(i-1))}M \colon x_{\alpha(i)}^{n_{\alpha(i)}}) \cap I_{n(\alpha(i))}M = I_{n(\alpha(i-1))}M\,,$$

for each $i = 1, \cdots, d$.

(iii) *There exists a positive integer* n_0 *such that for all* $n_1, \cdots, n_d \gg n_0$ *and any permutation* α *of* S_d, *we have the equality*

$$(I_{n(\alpha(d-1))}M \colon x_{\alpha(d)}^{n_{\alpha(d)}}) \cap I_{n(d)}M = I_{n(\alpha(d-1))}M\,.$$

(iv) *There exists a positive integers* n_0 *such that for all* $n_1, \cdots, n_d \gg n_0$

and any permutation α of S_d, we have the equality

$$I_{n(\alpha(d-1))}M: x_{\alpha(d)}^{n_{\alpha(d)}} = I_{n(\alpha(d-1))}M: x_{\alpha(d)}^{n_0}.$$

Proof. (i) $\Rightarrow$ (ii). By renaming the permuted sequence it suffices to show that

$$(I_{n(i-1)}M: x_i^{n_i}) \cap I_{n(i)}M \subseteq I_{n(i-1)}M.$$

Let $a \in (I_{n(i-1)}M: x_i^{n_i}) \cap I_{n(i)}M$. Write $a = \sum_{j=1}^{i} y_j x_j^{n_j}$ for some $y_j \in M$. Since $ax_i^{n_i} \in I_{n(i-1)}M$ therefore $y_i x_i^{2n_i} \in I_{n(i-1)}M$. Hence, for $n_i \geq n_0$, $y_i \in I_{n(i-1)}M: x_i^{2n_i} = I_{n(i-1)}M: x_i^{n_i}$. It follows that $y_i x_i^{n_i} \in I_{n(i-1)}M$ and $a \in I_{n(i-1)}M$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). For $n_1, \cdots, n_d \geq n_0$ we have

$$(I_{n(d-1)}M: x_d^{n_0}) \cap (I_{n(d-1)} + x_d^{n_0})M = I_{n(d-1)}M.$$

Dividing both sides of this relation by $x_d^{n_0}$, we get

$$I_{n(d-1)}M: x_d^{2n_0} = I_{n(d-1)}M: x_d^{n_0}.$$

From this we can deduce that

$$I_{n(d-1)}M: x_d^{n_0} = I_{n(d-1)}M: x_d^{kn_0}$$

for all $n_1, \cdots, n_d \geq n_0$ and $k \geq 1$. Since our proof is independent of the order of the sequence $x_1, \cdots, x_d$ we obtain (iv).

(iv) $\Rightarrow$ (i). By Krull's Intersection Theorem and (iv) we get

$$\begin{aligned} I_{n(i-1)}M: x_i^{n_i} &\subseteq \bigcap_{k=n_0}^{\infty} ((I_{n(i-1)} + x_{i+1}^k + \cdots + x_d^k)M: x_i^{n_i}) \\ &= \bigcap_{k=n_0}^{\infty} ((I_{n(i-1)} + x_{i+1}^k + \cdots + x_d^k)M: x_i^{n_0}) \\ &= I_{n(i-1)}M: x_i^{n_0} \subseteq I_{n(i-1)}M: x_i^{n_i} \end{aligned}$$

for each $i = 1, \cdots, d$ and all $n_1, \cdots, n_d \geq n_0$. Since the proof is independent of the order of the sequence $x_1, \cdots, x_d$ we conclude Lemma 2.

LEMMA 3. *If $l(M/I_{n(d)}M)$ is a polynomial for $n_1, \cdots, n_d \gg 0$ then it is linear in each n_i.*

Proof. Straightforward.

The following formula for the multiplicity, found by Auslander and Buchsbaum (see [1, § 4]), is the starting point for the proof of Theorem 1.

LEMMA 4. *For every s.o.p. $x_1, \cdots, x_d$ for M we have*

$$l(M/(x_1, \cdots, x_d)M) = l(I_{d-1}M\colon x_d/I_{d-1}M) + \sum_{i=0}^{d-1} e(I_d/I_i;\ I_{i-1}M\colon x_i/I_{i-1}M)\,,$$

where $e(I_d/I_i;\ I_{i-1}M\colon x_i/I_{i-1}M) = e(I_d;\ M)$ *for* $i = 0$.

Proof of Theorem 1. ($\Rightarrow$). Using Lemma 4 we have

$$\begin{aligned}
&l(M/I_{n(d-1)}M + x_d^{kn_d}M) - l(M/I_{n(d)}M)\\
&\quad = l(I_{n(d-1)}M\colon x_d^{kn_d}/I_{n(d-1)}M) - l(I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M)\\
&\qquad + \sum_{i=0}^{d-1} e(I_{n(d-1)} + x_d^{(k-1)n_d}A/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_i}/I_{n(i-1)}M)\\
&\quad = l(I_{n(d-1)}M\colon x_d^{kn_d}/I_{n(d-1)}M) - l(I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M)\\
&\qquad + l(M/I_{n(d-1)}M + x_d^{(k-1)n_d}M) - l(I_{n(d-1)}M\colon x_d^{(k-1)n_d}/I_{n(d-1)}M)
\end{aligned}$$

for all positive integers k. For arbitrary $n(d-1) = (n_1, \cdots, n_{d-1})$, we can find k (in general depending on $n_1, \cdots, n_{d-1}$) such that

$$I_{n(d-1)}M\colon x_d^{kn_d} = I_{n(d-1)}M\colon x_d^{(k-1)n_d}\,.$$

Thus,

$$\begin{aligned}
&l(I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M)\\
&\quad = l(M/I_{n(d)}M) + l(M/I_{n(d-1)}M + x_d^{(k-1)n_d}M) - l(M/I_{n(d-1)}M + x_d^{kn_d}M)\,.
\end{aligned}$$

By Lemma 3 each summand of the above right term is a polynomial, linear in each n_i, for all $n_1, \cdots, n_d \gg 0$. Therefore $l(I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M)$ is a polynomial. For fixed $n^0(d-1) = (n_1^0, \cdots, n_{d-1}^0)$, there exists a positive integer t such that

$$I_{n^0(d-1)}M\colon x_d^{n_d} = I_{n^0(d-1)}M\colon x_d^t$$

for $n_d \geq t$. This implies that the polynomial $l(I_{n(d-1)}(M\colon x_d^{n_d}/I_{n(d-1)}M)$ is independent of n_d. Hence, there exists a positive integer n_0 such that $I_{n(d-1)}M\colon x_d^{n_d} = I_{n(d-1)}M\colon x_d^{n_0}$ for $n_1, \cdots, n_d \gg n_0$. As our proof is independent of the order of the sequence $x_1, \cdots, x_d$, it follows by Lemma 2 (iv) that $x_1, \cdots, x_d$ is an unconditioned p-s.o.p. ($\Leftarrow$). For convenience, we set

$$l(I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M) = e(I_{n(d)}/I_{n(d)};\ I_{n(d-1)}M\colon x_d^{n_d}/I_{n(d-1)}M)\,.$$

Then, by Lemma 4, it suffices to show that for each $i = 0, 1, \cdots, d$, $e(I_{n(d)}/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_i}/I_{n(i-1)}M)$ is a polynomial for $n_1, \cdots, n_d \gg 0$. We will argue by induction on d and i.

If $d = 1$ or $i = 0$ and d arbitrary, the statement is trivial.

It $d > 1$ and $i \geq 1$, we suppose that the result is true for $d - 1$ or

$i-1$, and it suffices to show that

$$e(I_{n(d)}/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_i}/I_{n(i-1)}M)$$

is a polynomial. Consider a permutation $\alpha=(\alpha(1),\cdots,\alpha(d))$ of S_d such that $\alpha(i-1)=i$, $\alpha(i)=i-1$ and $\alpha(j)=j$ for all $j\neq i-1$, i. Then by Lemma 3 and the assumption, there exists a positive integer n_0 such that

$$\begin{aligned}0&=l(M/I_{n(d)}M)-l(M/I_{n(\alpha(d))}M)\\&=e(I_{n(d)}/I_{n(i-1)})\colon I_{n(i-2)}M\colon x_{i-1}^{n_0}/I_{n(i-2)}M)\\&\quad+e(I_{n(d)}/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_0}/I_{n(i-1)}M)\\&\quad-e(I_{n(d)}/I_{n(\alpha(i-1))};\ I_{n(i-2)}M\colon x_i^{n_0}/I_{n(i-2)}M)\\&\quad-e(I_{n(d)}/I_{n(\alpha(i))};\ I_{n(\alpha(i-1))}M\colon x_{i-1}^{n_0}/I_{n(\alpha(i-1))}M)\end{aligned}$$

for all $n_1,\cdots,n_d\gg n_0$. It follows that

$$\begin{aligned}&e(I_{n(d)}/I_{n(i-1)};\ I_{n(i-2)}M\colon x_{i-1}^{n_0}/I_{n(i-2)}M)\\&\qquad-e(I_{n(d)}/I_{n(\alpha(i))};\ I_{n(\alpha(i-1))}M\colon x_{i-1}^{n_0}/I_{n(\alpha(i-1))}M)\\&\quad=e(I_{n(d)}/I_{n(\alpha(i-1))};\ I_{n(i-2)}M\colon x_i^{n_0}/I_{n(i-2)}M)\\&\qquad-e(I_{n(d)}/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_0}/I_{n(i-1)}M)\,.\end{aligned}$$

Denote the function on the rihgt of the above formula by F. Since the above left term is a function independent of n_{i-1}, so is F. For $n_1,\cdots,n_d\geq n_0$. we have

$$\begin{aligned}F&=e(x_{i-1}^{n_0},x_{i+1}^{n_{i+1}},\cdots,x_d^{n_d};\ I_{n(i-2)}M\colon x_i^{n_0}/I_{n(i-2)}M)\\&\quad-e(I_{n(d)}/I_{n(i)};\ (I_{n(i-2)}+x_{i-1}^{n_0})M\colon x_i^{n_0}/I_{n(i-2)}M+x_{i-1}^{n_0}M)\,.\end{aligned}$$

Set $\overline{M}=M/x_{i-1}^{n_0}M$. As $\dim(\overline{M})=d-1$, by induction on d, it follows that

$$\begin{aligned}&e(I_{n(d)}/I_{n(i)};\ (I_{n(i-2)}M+x_{i-1}^{n_0}M)\colon x_i^{n_0}/(I_{n(i-2)}+x_{i-1}^{n_0})M\\&\qquad=e(I_{n(d)}/I_{n(i)};\ I_{n(i-2)}\overline{M}\colon x_i^{n_0}/I_{n(i-2)}\overline{M})\end{aligned}$$

is a polynomial for $n_1,\cdots,n_d\geq n_0$. On the other hand, by induction on i, $e(x_{i-1}^{n_0},x_{i+1}^{n_{i+1}},\cdots,x_d^{n_d};\ I_{n(i-2)}M\colon x_i^{n_0}/I_{n(i-2)}M)$ is a polynomial. Thus, F is a polynomial because F is the difference of two polynomials. Consequently, by induction on i,

$$\begin{aligned}&e(I_{n(d)}/I_{n(i)};\ I_{n(i-1)}M\colon x_i^{n_0}/I_{n(i-1)}M)\\&\qquad=e(I_{n(d)}/I_{n(\alpha(i-1))};\ I_{n(i-2)}M\colon x_i^{n_0}/I_{n(i-2)}M)+F\end{aligned}$$

is a polynomial for all $n_1,\cdots,n_d\gg n_0$. The proof of Theorem 1 is now complete.

Remark. For the case $n_1 = \cdots = n_d = n$ as treated in [5] Theorem 1 leads to the following questions:

1. Let $x_1, \cdots, x_d$ be a s.o.p. for M. Then $l(M/(x_1^n, \cdots, x_d^n)M)$ is a polynomial for $n \gg 0$ if and only if $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ is a polynomial for $n_1, \cdots, n_d \gg 0$?

2. Is Theorem 1 still true for this case? That is, $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ is a polynomial for $n_i \gg 0$ if and only if there exists a positive integer n_0 such that, for all $n \geq n_0$ and any permutation α of S_d.

$$(*) \qquad (x_{\alpha(1)}^n, \cdots, x_{\alpha(i-1)}^n)M\colon x_{\alpha(i)}^n = (x_{\alpha(1)}^n, \cdots, x_{\alpha(i-1)}^n)M\colon x_{\alpha(i)}^{n_0}\,,$$

for each $i = 1, \cdots, d$?

Unfortunately, these questions do not always have an affirmative answer as the following example shows: For $d \geq 2$, let $B_d = k[\![Y_1, \cdots, Y_{d+1}]\!]/(Y_1Y_{d+1}, \cdots, Y_dY_{d+1})$, where k is a field and $Y_1, \cdots, Y_{d+1}$ are indeterminates. We denote by x_i the natural image of $Y_i + Y_{d+1}$ in B_d, $i = 1, \cdots, d$, then $x_1, \cdots, x_d$ form an s.o.p. for B_d. It can be verified that

$$l(B_d/(x_1^{n_1}, \cdots, x_d^{n_d})B_d) = n_1 \cdots n_d + \min\{n_1, \cdots, n_d\}\,,$$

for all $n_1, \cdots, n_d \geq 1$ and

$$(x_1^n, \cdots, x_{i-1}^n)B_d\colon x_i^k = (y_1^n, \cdots, y_{i-1}^n, y_{d+1})B_d\,, \qquad i = 1, \cdots, d\,,$$

for all $k \geq 1$, where y_i is the natural image of Y_i in B_d. Therefore $x_1, \cdots, x_d$ satisfy the condition $(*)$ but $l(B_d/(x_1^{n_1}, \cdots, x_d^{n_d})B_d$ is not a polynomial.

§ 3. Generalized Cohen-Macaulay modules

In this section we will see that p-s.o.p.'s are closely related to some specified s.o.p.'s like filter regular s.o.p.'s [3] or standard s.o.p.'s [11] and that one can use the notation of p-s.o.p.'s to characterize the generalized Cohen-Macaulay module which has been first introduced in [3].

Recall that an s.o.p. $x_1, \cdots, x_d$ for M is called a filter regular s.o.p. if $x_i \notin P$ for all $P \in \operatorname{Ass}(M/(x_1, \cdots, x_{i-1})M - \{\mathfrak{m}\}$, $i = 1, \cdots, d$. It is called an unconditioned filter regular s.o.p. if for any order of the sequence $x_1, \cdots, x_d$ it is always a filter regular sequence. This notion was introduced in [3] and has led to some interesting results. For instance, M_P is a Cohen-Macaulay module and $\dim(M_P) + \dim(A/P) = \dim(M)$ for all $P \in \operatorname{Supp}(M) - \{\mathfrak{m}\}$ if and only if every s.o.p. for M is a filter regular s.o.p. [3, Satz 2.5]. In this case, M is called an f-module. M is called a generalized Cohen-Macaulay module if $l(H^i_{\mathfrak{m}}(M)) < +\infty$ for $i = 0, \cdots, d-1$. It is well-known that every generalized Cohen-Macaulay module is an f-module and that the converse holds if A is a factor ring of a Cohen-Macaulay ring [3]. But in general, an f-module is not a generalized Cohen-Macaulay module. Ferrand and Raynaud [4] have constructed a two-dimensional local integral domain R such that the $\mathfrak{m}$-adic completion $\hat{R}$ has a one-dimensional associated prime ideal. Thus, R is an f-ring but it is not a generalized Cohen-Macaulay ring.

An s.o.p. $x_1, \cdots, x_d$ for M is called a standard s.o.p. if $l(M/I_dM) - e(I_d; M) = l(M/(x_1^2, \cdots, x_d^2)M) - e(x_1^2, \cdots, x_d^2; M)$. Trung [11] has shown that M is a generalized Cohen-Macaulay module if and only if there exists a standard s.o.p. for M and that if $x_1, \cdots, x_d$ is a standard s.o.p., then for all $n_1, \cdots, n_d \geq 1$, $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M) - e(x_1^{n_1}, \cdots, x_d^{n_d}; M)$ is a constant. Therefore $x_1, \cdots, x_d$ is an unconditioned p-s.o.p. for M with respect to the integer $n_0 = 1$. As for the converse, we have the following

PROPOSITION 5. *M is a generalized Cohen-Macaulay module if and only if there exists an unconditioned filter regular s.o.p. $x_1, \cdots, x_d$ such that $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ is a polynomial for all $n_1, \cdots, n_d \geq 1$. And in this case, $x_1, \cdots, x_d$ is a standard s.o.p.*

Proof. By the above remark it suffices to show the "if" part of the proposition. Let $x_1, \cdots, x_d$ be an unconditioned filter regular s.o.p. for M. Using the notations as in Section 2, by Corollary 4.8 of [1] we have for $n_1, \cdots, n_d \geq 1$

$$l(M/I_{n(d)}M) - e(I_{n(d)}; M) = l(I_{n(d-1)}M : x_d^{n_d})/I_{n(d-1)}M)$$

is a polynomial for every permutation of $x_1, \cdots, x_d$. Then it follows that this difference is independent of $n_1, \cdots, n_d$. So $x_1, \cdots, x_d$ is a standard s.o.p. for M and M is a generalized Cohen-Macaulay module.

Remark. The condition that $x_1, \cdots, x_d$ is a filter regular s.o.p. for every permutation of the sequence $x_1, \cdots, x_d$ is necessary as the following example shows. Let $A = k[\![X, Y, Z]\!]/(X^2, XYZ, XZ^2)$ and let y, z be the images of Y, Z^2 in A. Then it is easy to see that y, z is an unconditioned p-s.o.p. of A and a filter regular s.o.p. But A is not a generalized Cohen-Macaulay module since z, y is not a filter regular s.o.p. of A.

COROLLARY 6. *Let M be an f-module. If M is not a generalized Cohen-Macaulay module then, for every s.o.p. $x_1, \cdots, x_d$ for M, $l(M/(x_1^{n_1}, \cdots, x_d^{n_d})M)$ is never a polynomial for $n_1, \cdots, n_d \gg 0$.*

Proof. Note that there always exists an unconditioned filter regular s.o.p. for M (see [2]). Then the proof is immediate from Proposition 5.

COROLLARY 7. *The following conditions are equivalent:*

(i) *M is a generalized Cohen-Macaulay module*

(ii) *Every s.o.p. for M is a p-s.o.p.*

(iii) *For every s.o.p. $x_1, \cdots, x_d$ for M,*
$l(M/x_1^{n_1}, \cdots, x_d^{n_d})M)$ is a polynomial for $n_1, \cdots, n_d \gg 0$.

Proof. Immediate.

§ 4. The case $\dim(M) = 2$

In this section we always assume that $\dim(M) = 2$. We will first show that the property of being a p-s.o.p. is stable under permutations.

LEMMA 8. *Every p-s.o.p. is unconditioned.*

Proof. Let x, y be a p-s.o.p. for M. By Lemma 4 we have

$$\begin{aligned} l(M/(x^n, y^m)M) &= l(x^n M : y^m / x^n M) + me(y; 0 :_M x^n) + nme(x, y; M) \\ &= l(y^m M : x^n / y^m M) + ne(x; 0 :_M y^m) + nme(x, y; M) . \end{aligned}$$

Thus

$$l(y^m M : x^n / y^m M) - me(y; 0 :_M x^n) = l(x^n M : y^m / x^n M) - ne(x; 0 :_M y^m) ,$$

since x, y is a p-s.o.p., it follows that the above difference is a constant, say k for $n, m \gg 0$. Then

$$\begin{aligned} l(x^n M : y^m / x^n M) &= ne(x; 0 :_M y^m) + k , \\ l(y^m M : x^n / y^m M) &= me(y; 0 :_M x^n) + k \end{aligned}$$

are polynomials for $n, m \gg 0$ and we get the result by Theorem 1.

THEOREM 9. *The following conditions are equivalent:*

(i) *x, y is a p-s.o.p. for M.*

(ii) *$l(M/x^n, y^m)M)$ is a polynomial for $n, m \gg 0$.*

(iii) *$l(H_1(x^n, y^m; M))$ is a polynomial for $n, m \gg 0$,*

where $H_*(x, y; M)$ is the homology of the Koszul-Complex $K_*(x, y; A) \otimes_A M$ with respect to the elements x, y of A.

Proof. (i) $\Leftrightarrow$ (ii) by Theorem 1 and Lemma 8. (ii) $\Leftrightarrow$ (iii) arises most directly from

$$l(M/(x^n, y^m)M) - l(H_1(x^n, y^m, M)) = nme(x, y; M) - l(0:_M(x^n, y^m))$$

which is a polynomial for $n, m \gg 0$.

Let $N \subseteq M$ a submodule and $J \subseteq A$ an ideal. We set

$$N: \langle J \rangle = \{a \in M;\ aJ^k \subseteq N \text{ for some } k \geq 1\}.$$

We will see that for a filter regular s.o.p., the property of being a p-s.o.p. can be expressed in terms of only one element,

PROPOSITION 10. *Let x, y be a filter regular s.o.p. for M. Then the following conditions are equivalent*:

(i) *$l(M/(x^n, y^m)M)$ is a polynomial for $n, m \gg 0$.*

(ii) *There exists a positive integer k such that*

$$y^m(y^kM: \langle \mathfrak{m} \rangle) + 0: \langle \mathfrak{m} \rangle = y^{k+m}M: \langle \mathfrak{m} \rangle$$

for all $m \geq 0$.

Proof. (i) $\Rightarrow$ (ii). Note that if x, y is a filter regular s.o.p. then x^n, y^m is also a filter regular s.o.p. for all $n, m \geq 1$. By Lemma 4 and Corollary 4.8 of [1] we get

$$\begin{aligned} l(M/(x^n, y^m)M) - nme(x, y; M) &= l(x^nM: y^m/x^nM) \\ &= l(y^mM: x^n/y^mM) + ne(x; 0:_M y^m). \end{aligned}$$

This shows that the above difference is a polynomial depending only on n. Thus there exists a positive integer k such that $l(y^mM: x^n/y^mM)$ is a constant for all $n, m \geq k$. As x, y is a p-s.o.p. we can choose a sufficiently large k such that $y^mM: x^n = y^mM: \mathfrak{m}^k = y^mM: \langle \mathfrak{m} \rangle$ and $0: \langle y \rangle = 0:_M y^k$ for $n, m \geq k$. We have

$$\begin{aligned} l(y^mM: x^n/y^mM) &= l(y^mM: \langle \mathfrak{m} \rangle/y^mM) \\ &= l(y^mM: \langle \mathfrak{m} \rangle/y^mM + 0: \langle \mathfrak{m} \rangle) + l(y^mM + 0: \langle \mathfrak{m} \rangle/y^mM) \\ &= l(y^mM: \langle \mathfrak{m} \rangle/y^mM + 0: \langle \mathfrak{m} \rangle) + l(0: \langle \mathfrak{m} \rangle), \end{aligned}$$

since, by Lemma 2 (ii)

$$y^mM + 0: \langle \mathfrak{m} \rangle/y^mM \simeq 0: \langle \mathfrak{m} \rangle/y^mM \cap 0: \langle \mathfrak{m} \rangle \simeq 0: \langle \mathfrak{m} \rangle.$$

Thus $l(y^mM: \langle \mathfrak{m} \rangle/y^mM + 0: \langle \mathfrak{m} \rangle)$ is a constant for $m \geq k$. Now we consider the mapping

$$f_m\colon y^kM\colon \langle\mathfrak{m}\rangle/y^kM + 0\colon \langle\mathfrak{m}\rangle \longrightarrow y^{k+m}M\colon \langle\mathfrak{m}\rangle/y^{k+m}M + 0\colon \langle\mathfrak{m}\rangle$$

defined by $f_m(a) = a\cdot y^m$ for $a \in M$. We will show now that f_m is injective for all $m \geq 0$. In fact, since

$$\ker(f_m) = ((y^kM\colon \langle\mathfrak{m}\rangle) \cap (y^kM + 0\colon {}_My^m))/y^kM + 0\colon \langle\mathfrak{m}\rangle$$

and

$$(y^kM\colon \langle\mathfrak{m}\rangle) \cap (y^kM + 0\colon {}_My^m) = y^kM + (y^kM\colon \langle\mathfrak{m}\rangle) \cap (0\colon {}_My^m),$$

we only need to show that $(0\colon {}_My^m) \cap (y^kM\colon \langle\mathfrak{m}\rangle) \subseteq 0\colon \langle\mathfrak{m}\rangle$. Let $a \in (0\colon {}_My^m) \cap (y^kM\colon \langle\mathfrak{m}\rangle)$, for arbitrary $b \in \mathfrak{m}^k$, $ab = y^kc$ for some $c \in M$. As $ay^k = 0$, $0 = ay^kb = y^{2k}c$. Thus $c \in 0\colon {}_My^{2k} = 0\colon {}_My^k = 0\colon \langle y\rangle$ and $ab = y^kc = 0$ for all $b \in \mathfrak{m}^k$. So it follows that $a \in 0\colon {}_M\mathfrak{m}^k = 0\colon \langle\mathfrak{m}\rangle$. Since $l(y^{k+m}M\colon \langle\mathfrak{m}\rangle/y^{k+m}M + 0\colon \langle\mathfrak{m}\rangle)$ is a constant and f_m is injective for $m \geq 0$, it follows that f_m is surjective for all $m \geq 0$ and this proves that $y^m(y^kM\colon \langle\mathfrak{m}\rangle) + 0\colon \langle\mathfrak{m}\rangle = y^{k+m}M\colon \langle\mathfrak{m}\rangle$ for all $m \geq 0$.

(ii) $\Rightarrow$ (i). By Theorem 1 and Lemma 8 it is enough to show that y, x is a p-s.o.p. There exists integers t, s such that $y^kM\colon \langle\mathfrak{m}\rangle = y^kM\colon x^t$ and $0\colon \langle x\rangle = 0\colon {}_Mx^s$. Let $n_0 = \max\{k, t, s\}$. Then, for all $m \geq n_0$,

$$\begin{aligned} y^mM\colon \langle\mathfrak{m}\rangle &= y^{m-k}(y^kM\colon \langle\mathfrak{m}\rangle) + 0\colon \langle\mathfrak{m}\rangle \\ &= y^{m-k}(y^kM\colon x^{n_0}) + 0\colon \langle\mathfrak{m}\rangle \subseteq y^mM\colon x^{n_0}. \end{aligned}$$

This completes the proof of Proposition 10.

The Proposition 10 has the following consequences.

COROLLARY 11. *Let x, y be a filter regular s.o.p. for M. If $l(M/(x^n, y^n)M)$ is a polynomial for $n, m \gg 0$ then, for every $z \in A$ such that z, y is a s.o.p. for M, z, y is a filter regular s.o.p. for M and $l(M/(z^n, y^m)M)$ is a polynomial for $n, m \gg 0$.*

Proof. By Proposition 10 we need only to show that if z, y is a s.o.p. for M, then z, y is a filter regular s.o.p. for M. In fact, we have

$$0\colon \langle z\rangle \subseteq y^{m+k}M\colon \langle z\rangle = y^{m+k}M\colon \langle\mathfrak{m}\rangle = y^m(y^kM\colon \langle\mathfrak{m}\rangle) + 0\colon \langle\mathfrak{m}\rangle$$

for all $m \geq 0$ and k as in Proposition 10. It follows that

$$0\colon \langle z\rangle \subseteq \bigcap_{m=0}^{\infty} (y^m(y^kM\colon \langle\mathfrak{m}\rangle) + 0\colon \langle\mathfrak{m}\rangle) = 0\colon \langle\mathfrak{m}\rangle$$

by Krull's Intersection Theorem. Hence we can conclude that $0\colon \langle z\rangle = 0\colon \langle\mathfrak{m}\rangle$.

As for the next corollary, we recall a notation from [6]. Let x, y be an s.o.p. of A and t an indeterminate over A. Then one call the graded algebra $R_{\mathfrak{m}}(x) = \bigoplus_{n=-\infty}^{+\infty} (x^n A: \langle \mathfrak{m} \rangle) t^n$ the $\mathfrak{m}$-relative Rees ring with respect to the ideal xA. Let $R(x) = \bigoplus_{n=-\infty}^{+\infty} (x^n A) t^n$ be the ordinary Rees ring of A with respect to xA.

COROLLARY 12. *Let $M = A$ and x, y form an s.o.p. of A. Then the following conditions are equivalent*:

(i) $R_{\mathfrak{m}}(x)$ *is finitely generated over* $R(x)$.

(ii) $\operatorname{depth}(A) > 0$ *and* $l(A/(x^n, y)A)$ *is a polynomial for* $n, m \gg 0$.

Proof. It is well-known [6] that (i) is equivalent to the following condition

(i′) There exists a positive integer k such that for all $n \geq 0$ $x^n(x^k A: \langle \mathfrak{m} \rangle) = x^{n+k} A: \langle \mathfrak{m} \rangle$.

Thus, (ii) $\Rightarrow$ (i′) by Proposition 10. (i) $\Rightarrow$ (ii) follows from the Proposition 10 and

$$0:_M y \subseteq \bigcap_{n=0}^{\infty} (x^{n+k} A: \langle y \rangle) = \bigcap_{n=0}^{\infty} (x^{n+k} A: \langle \mathfrak{m} \rangle) = \bigcap_{n=0}^{\infty} (x^n (x^k A: \langle \mathfrak{m} \rangle)) = 0 .$$

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