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EVERY ALGEBRAIC KUMMER SURFACE IS THE K3-COVER OF AN ENRIQUES SURFACE

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Introduction

A *Kummer surface* is the minimal desingularization of the surface T/i, where T is a complex torus of dimension 2 and i the involution automorphism on T. T is an abelian surface if and only if its associated Kummer surface is algebraic. Kummer surfaces are among classical examples of K3-surfaces (which are simply-connected smooth surfaces with a nowhere-vanishing holomorphic 2-form), and play a crucial role in the theory of K3-surfaces. In a sense, all Kummer surfaces (resp. algebraic Kummer surfaces) form a 4 (resp. 3)-dimensional subset in the 20 (resp. 19)-dimensional family of K3-surfaces (resp. algebraic K3 surfaces).

An *Enriques surface* is a smooth projective surface Y with $2K_Y = 0$, $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$. The unramified double cover of Y defined by the torsion class K_Y is an algebraic K3-surface. Conversely, if an algebraic K3-surface X admits a fixed-point-free involution τ, then the quotient surface X/τ is an Enriques surface. It is known that all Enriques surfaces form a 10-dimensional moduli space.

Let X be a surface. The *Picard number* of X, denoted by $\rho(X)$, is the rank of the *Néron-Severi group* $\mathrm{NS}(X)$, the sublattice of $H^2(X, \boldsymbol{Z})$ generated by algebraic cycles. The *transcendental lattice* T_X of X is the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \boldsymbol{Z})$. If X is a K3-surface, then $0 \leq \rho(X) \leq 20$. If X is the K3-cover of an Enriques surface, then $\rho(X) \geq 10$.

Let L be a lattice, i.e. a free $\boldsymbol{Z}$-module of finite rank together with a $\boldsymbol{Z}$-valued symmetric bilinear form. For every integer m we denote by $L(m)$ the lattice obtained from L by multiplying the values of its bilinear form by m. The *length* of L, denoted by $l(L)$, is the minimum number of generators of $L^*/j(L)$, where $j: L \to L^* = \mathrm{Hom}(L, \boldsymbol{Z})$ is the natural

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homomorphism.

Let U and E_8 denote the even unimodular lattices of signature (1, 1) and (0, 8) respectively.

THEOREM 1. (Criterion for a K3-surface to cover an Enriques surface). *Let X be an algebraic K3-surface.*
Assume that $l(T_X)+2\leq\rho(X)$. (This is true if $\rho(X)\geq 12$.)
Then the following are equivalent.

(i) *X admits a fixed-point-free involution.*

(ii) *There exists a primitive embedding of T_X into $\Lambda^-=U\oplus U(2)\oplus E_8(2)$ such that the orthogonal complement of T_X in Λ^- contains no vectors of self-intersection -2.*

Remark. The assumption that $l(T_X)+2\leq\rho(X)$ is needed only for the part (ii) $\Rightarrow$ (i).

Theorem 1 is a consequence of the uniqueness theorem on embeddings of even lattices [Nik 2] and the "surjectivity of the period map for Enriques surfaces" [Ho 2]. Making use of Theorem 1 and the criterion, due to Nikulin, for a K3-surface to be Kummer ([Nik 1] & [Mor]) we can prove the following result:

THEOREM 2. *Every algebraic Kummer surface is the K3-cover of some Enriques surface.*

§ 0. Preliminaries

(0.1) DEFINITION. A $\boldsymbol{Z}$-module isomorphism of lattices preserving the bilinear form is called an *isometry.* The group of self-isometries of a lattice L, denoted by $O(L)$, is called the orthogonal group of L (or the group of units).

A lattice is *even* if the associated quadratic form takes on only even integer values, and is *odd* if the quadratic form takes on some odd value.

The *discriminant* of a lattice L, written discr(L), is the determinant of the matrix of its bilinear form. A lattice is *non-degenerate* if its discriminant is non-zero, and *unimodular* if its discriminant is ± 1. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-), where $t_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L\otimes\boldsymbol{R}$. A lattice is *indefinite* if the associated quadratic form takes on both positive and negative values. An indefinite lattice L of signature $(1, t_-)$ or $(t_+, 1)$ is called a *hyperbolic lattice.*

An embedding $L \to M$ is *primitive* if M/L is torsion free.

(0.2) EXAMPLES. (i) By A_n, D_n, E_n we denote the even negative definite lattices defined by the matrix equal to the Cartan matrix of an irreducible root system of type A_n, D_n, E_n respectively [Bour; Chap. VI]. For example, the bilinear form on E_8 is given by the matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

E_8 is the only unimodular lattice among A_n, D_n, E_n's.

(ii) U denotes the hyperbolic lattice of rank 2 defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is an even unimodular lattice; note that $U(-m) \cong U(m)$ for any integer m.

(iii) For any integer n we denote by $\langle n \rangle$ the lattice $\boldsymbol{Z}e$ with $\langle e, e \rangle = n$.

(0.3) THEOREM [Nik 2; Corollary 1.12.3 & Theorem 1.14.4]. *A primitive embedding of an even non-degenerate lattice L of signature (s_+, s_-) into an even unimodular lattice M of signature (t_+, t_-) exists provided that*

$$s_+ \leq t_+, \quad s_- \leq t_-, \quad \textit{and} \quad l(L) + 1 \leq \operatorname{rank}(M) - \operatorname{rank}(L).$$

Furthermore, if the three inequalities are all strict, then the primitive embedding is unique.

The following corollaries will be used later.

(0.3.1) COROLLARY. *There is a primitive embedding of $\langle -2 \rangle \oplus \langle -2m \rangle$ into the even unimodular lattice E_8 for any positive integer m.*

Proof. Note that $\operatorname{sign}(E_8) = (0, 8)$, $\operatorname{sign}(\langle -2 \rangle \oplus \langle -2m \rangle) = (0, 2)$, and $l(\langle -2 \rangle \oplus \langle -2m \rangle) = 2$. The corollary follows from (0.3). Q.E.D.

(0.3.2) Corollary. *There is a primitive embedding of* $\langle -2m \rangle$ *into* E_8 *for any positive integer m.*

(0.4) Let X be a K3-surface. It is known (see [B-P-V; Chap. VIII]) that $H^2(X, \mathbf{Z})$ (with its intersection pairing) is isomorphic as an abstract lattice to the lattice, called the *K3-lattice*,

$$\Lambda = U \oplus U \oplus U \oplus E_8 \oplus E_8 .$$

(0.5) Definition. Let T be a complex torus of dimension two. The involution automorphism i: $T \to T$, $i(x) = -x$, has sixteen fixed points, namely the points of order 2 on T. The quotient surface T/i has sixteen ordinary double points (i.e. singular points of type A_1). Resolving the double points we obtain a smooth surface X, called the *Kummer surface* of T. X has sixteen exceptional curves of self-intersection -2 arising from the resolution of singularities. Note that T is an abelian surface if and only if its associated Kummer surface is an algebraic K3-surface.

(0.6) Theorem [Nik 1]. *There is an even, negative definite lattice* Π *of rank* 16, *called the Kummer lattice, with the following properties*:

(i) Π *admits a unique primitive embedding into the* $K3$ *lattice* Λ *and its orthogonal complement in* Λ *is isomorphic to* $U(2)^3$.

(ii) *If X is a Kummer surface, then the minimal primitive sublattice of* $H^2(X, \mathbf{Z})$ *containing the classes of the sixteen exceptional curves on X is isomorphic to* Π.

(iii) *A K3-surface X is Kummer if and only if there is a primitive embedding of* Π *into the Néron-Severi group* NS(X).

The following criterion is an alternative to (0.6) (iii).

(0.6.1) Corollary [Mor; Cor. 4.4]. *Let X be an algebraic K3-surface.*

(i) *If* $\rho(X) = 19$ *or* 20, *then X is a Kummer surface if and only if there is an even lattice* T' *with* $T_X \cong T'(2)$.

(ii) *If* $\rho(X) = 18$, *then X is a Kummer surface if and only if there is an even lattice* T' *with* $T_X \cong U(2) \oplus T'(2)$.

(iii) *If* $\rho(X) = 17$, *then X is a Kummer surface if and only if there is an even lattice* T' *with* $T_X \cong U(2)^2 \oplus T'(2)$.

(iv) *If* $\rho(X) \leq 16$, *then X is not a Kummer surface.*

§ 1. Proof of Theorem 1

(1.1) Let X be a K3 surface. Since $p_g(X) = 1$, the choice of an isometry ϕ: $H^2(X, \boldsymbol{Z}) \to \Lambda$ determines a line in $\Lambda_{\boldsymbol{C}} = \Lambda \otimes \boldsymbol{C}$ spanned by the $\phi_{\boldsymbol{C}}$-image of a nowhere vanishing holomorphic 2-form ω_X. The point $[\omega_X] \in \boldsymbol{P}(\Lambda_{\boldsymbol{C}})$ is called *the period point* of the *marked K3-surface* (X, ϕ).

The following theorem goes by the name "(weak) Torelli theorem for K3 surfaces".

(1.2) THEOREM ([P–S], [B–R], [B–P–V]). *Two K3 surfaces are isomorphic if and only if there are markings for them, such that the corresponding period points are the same.*

(1.3) Let Y be an Enriques surface. It is known (see [B–P–V; Chap. VIII]) that

$$\mathrm{Pic}(Y) \cong \mathrm{NS}(Y) \cong H^2(Y, \boldsymbol{Z}) \cong \boldsymbol{Z}^{10} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$$

and that the lattice $H^2(Y, \boldsymbol{Z})_f$, the torsion-free-part of $H^2(Y, \boldsymbol{Z})$, is isomorphic to the even unimodular lattice $U \oplus E_8$ of signature (1,9). If X is the K3-cover of Y, then $\mathrm{Pic}(X)$ contains $p^*(\mathrm{Pic}(Y)) \cong U(2) \oplus E_8(2)$ as a primitive sublattice, where p: $X \to Y$ is the covering projection. In particular, $\rho(X) \geq 10$.

(1.4) Let Λ be the K3-lattice, that is,

$$\Lambda = U \oplus U \oplus U \oplus E_8 \oplus E_8 .$$

We fix a basis of Λ of the form $v_1, v_2, v_1', v_2', v_1'', v_2'', e_1', \cdots, e_8', e_1'', \cdots, e_8''$, where the first three pairs are the standard bases of U and the remaining two octuples are the standard bases of E_8.

Let θ: $\Lambda \to \Lambda$ be the involution given by the formula

$$\begin{aligned} &\theta(v_i) = -v_i, \quad \theta(v_i') = v_i'', \quad \theta(v_i'') = v_i', \quad i = 1, 2, \\ &\theta(e_i') = e_i'', \quad \theta(e_i'') = e_i', \quad i = 1, \cdots, 8. \end{aligned}$$

Then the θ-invariant sublattice, denoted by Λ^+, is

$$\Lambda^+ = \boldsymbol{Z}y_1 \oplus \boldsymbol{Z}y_2 \oplus \boldsymbol{Z}e_1 \oplus \cdots \oplus \boldsymbol{Z}e_8,$$

where $y_i = v_i' + v_i''$, $i = 1, 2$ and $e_i = e_i' + e_i''$, $i = 1, \cdots, 8$.
The θ-anti-invariant sublattice, denoted by Λ^-, is

$$\Lambda^- = \boldsymbol{Z}\bar{y}_1 \oplus \boldsymbol{Z}\bar{y}_2 \oplus \boldsymbol{Z}\bar{e}_1 \oplus \cdots \oplus \boldsymbol{Z}\bar{e}_8 \oplus \boldsymbol{Z}v_1 \oplus \boldsymbol{Z}v_2,$$

where $\bar{y}_i = v_i' - v_i''$, $i = 1, 2$, $\bar{e}_i = e_i' - e_i''$, $i = 1, \cdots, 8$.
It is easy to see that

$$\Lambda^+ \cong U(2) \oplus E_8(2), \quad \Lambda^- \cong U(2) \oplus E_8(2) \oplus U, \quad (\Lambda^+)^\perp = \Lambda^-,$$

and that both Λ^- and Λ^- are primitive sublattices of Λ.

(1.5) LEMMA [Ho 1; Theorem 5.1]. *Let X be the $K3$-cover of an Enriques surface Y, and let τ: $X \to X$ be the covering involution.*
Then there exists an isometry

$$\phi\colon H^2(X, \boldsymbol{Z}) \longrightarrow \Lambda$$

such that the following diagram

$$\begin{array}{ccc} H^2(X, \boldsymbol{Z}) & \xrightarrow{\tau^*} & H^2(X, \boldsymbol{Z}) \\ \phi\downarrow & & \phi\downarrow \\ \Lambda & \xrightarrow{\theta} & \Lambda \end{array}$$

commutes.
In particular, ϕ induces an isomorphism

$$\bar{\phi}\colon H^2(X, \boldsymbol{Z})^{\tau^*} = p^*H^2(Y, \boldsymbol{Z}) = p^* \operatorname{Pic}(Y) \longrightarrow \Lambda^+,$$

where p: $X \to Y$ is the covering projection.

(1.6) *Remark.* The choice of ϕ as in Lemma (1.5) is unique up to

$$\Gamma = \{g \in O(\Lambda)\colon g \circ \theta = \theta \circ g\}.$$

(1.7) *A marked Enriques surface* is a pair (Y, ϕ) with Y an Enriques surface and ϕ: $H^2(X, \boldsymbol{Z}) \to \Lambda$ an isometry satisfying $\phi \circ \tau^* = \theta \circ \phi$, as in Lemma (1.5). Since $\tau^*\omega_X = -\omega_X$ (there is no holomorphic 2-form on Y), the period point $[\omega_X]$, called the *period point* of (Y, ϕ), of the marked K3 surface (X, ϕ) belongs to the set

$$\Omega^- = \{[\omega] \in \boldsymbol{P}(\Lambda^- \otimes \boldsymbol{C})\colon \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}.$$

By Remark (1.6), the assignment

$$Y\colon \longrightarrow [\omega_X] \in \Omega^-/\Gamma_r\,,$$

where $\Gamma_r = \{g|_{\Lambda^-}\colon g \in \Gamma\}$, is well defined and called the *period map for Enriques surfaces.*

Global Torelli theorem for Enriques surfaces [Ho 1] says that this period map is injective.

(1.8) NOTATION.

$$\Omega_0^- = \{[\omega] \in \Omega^- : \langle \omega, \delta \rangle \neq 0 \text{ for any } \delta \in \Lambda^-, \langle \delta, \delta \rangle = -2\}$$
$$D_0 = \Omega_0^- / \Gamma_r .$$

The next theorem, which is due to Horikawa, goes by the name "the surjectivity of the period map for Enriques surfaces".

(1.9) THEOREM ([Ho 2] & [B–P–V]). *Every point of D_0 is the period point of an Enriques surface. In other words, every point of Ω_0^- is the period point of some marked Enriques surface.*

(1.10) LEMMA. *Let X be a $K3$ surface, and let d be a divisor on X of self-intersection ≥ -2. Then either d or $-d$ is effective.*

Proof. Riemann-Roch yields the inequality

$$h^0(\mathcal{O}_X(d)) + h^0(\mathcal{O}_X(-d)) = \tfrac{1}{2}\langle d, d\rangle + 2 + h^1(\mathcal{O}_X(-d)) \geq 1 ,$$

and hence $\mathcal{O}_X(d)$ or $\mathcal{O}_X(-d)$ has a non-trivial section. Q.E.D.

(1.11) *Proof of Theorem* 1.

(i) $\Rightarrow$ (ii): Let $p: X \to Y$ be the unramified covering of an Enriques surface Y, and let τ, ϕ be the same as in (1.5).

Then, $\phi(\mathrm{Pic}(X)) \supseteq \phi(p^*(\mathrm{Pic}(Y))) = \Lambda^+$, so we have $\phi(T_X) \subseteq \Lambda^-$. This embedding is primitive, for an isometry preserves primitivity. Now suppose that Λ^- contains a vector v with $v^2 = -2$, $v \perp \phi(T)$. Then the class $d = \phi^{-1}(v)$ belongs to $\mathrm{Pic}(X)$ and, by (1.10), d or $-d$ is effective. But no effective class can be τ^*-anti-invariant as is $\pm d$.

(ii) $\Rightarrow$ (i): Let $\psi_1: T_X \to \Lambda^-$ be a primitive embedding such that no (-2)-vector in Λ^- is orthogonal to $\psi_1(T_X)$.

Claim. ψ_1 extends to an isometry $\psi: H^2(X, \mathbf{Z}) \to \Lambda$.

Proof of the claim. Fix an isometry $\psi_2: H^2(X, \mathbf{Z}) \to \Lambda$.

Then we have two embeddings of $T = T_X$ into Λ, namely,

$$\psi_1: T \longrightarrow \Lambda^- \subseteq \Lambda \quad \text{and} \quad \psi_2|_T: T \longrightarrow \Lambda .$$

Since T has signature $(2, 20-\rho)$ and since $l(T) + 2 \leq \rho(X) = \mathrm{rank}(\Lambda) - \mathrm{rank}(T)$, by (0.3), there exists $\nu \in O(\Lambda)$ such that $\psi_1 = \nu \circ \psi_2|_T$. We take $\psi = \nu \circ \psi_2$ and the claim is proved. Now, since $\omega_X \in T \otimes \mathbf{C} \subseteq H^2(X, \mathbf{C})$, the period point $[\omega_X]$ of the marked K3-surface (X, ψ) belongs to Ω_0^-. By (1.9), there exists a marked Enriques surface (Y, ϕ) whose period point is equal to $[\omega_X]$. But then, by (1.2), the K3-cover X' of Y is isomorphic to X, and the covering involution on X' lifts to a fixed-point-free involution on X. Q.E.D.

§ 2. Proof of Theorem 2

By Theorem 1, it suffices to prove that there exists a primitive embedding of T_X into Λ^- with no (-2)-vectors in the orthogonal complement. We split the proof into four cases.

Case 1. $\rho(X) = 17$.

By (0.6.1) (iii), $T_X = U(2) \oplus U(2) \oplus \langle -4m \rangle$ for some positive integer m. There exists such a primitive embedding $T_X \to \Lambda^- = U(2) \oplus U \oplus E_8(2)$ if there exists a primitive embedding $U(2) \oplus \langle -4m \rangle \to U \oplus E_8(2)$ with no (-2)-vectors in the orthogonal complement.

Let $\{x, y, t\}$ be a standard basis of $U(2) \oplus \langle -4m \rangle$, that is

$$x^2 = y^2 = \langle x, t \rangle = \langle y, t \rangle = 0, \quad \langle x, y \rangle = 2, \quad t^2 = -4m,$$

and let $\{e, f\}$ be a standard basis of U,

$$e^2 = f^2 = 0, \quad \langle e, f \rangle = 1,$$

By (0.3.1), we can pick up two elements w_1 and w_2 of $E_8(2)$ which generate a primitive sublattice of $E_8(2)$ isomorphic to $\langle -4 \rangle \oplus \langle -4m \rangle$. Define a map ϕ: $U(2) \oplus \langle -4m \rangle \to U \oplus E_8(2)$ by the formula

$$\phi(x) = e,$$
$$\phi(y) = e + 2f + w_1,$$
$$\phi(t) = w_2.$$

Then, $\phi(x)^2 = \phi(y)^2 = \langle \phi(x), \phi(t) \rangle = \langle \phi(y), \phi(t) \rangle = 0,$

$$\langle \phi(x), \phi(y) \rangle = 2 \quad \text{and} \quad \phi(t)^2 = -4m.$$

So, ϕ is an embedding.

If kd, $d \in U \oplus E_8(2)$, $k \in Z$, belongs to $\operatorname{im} \phi$, then
$kd = he + i(e + 2f + w_1) + jw_2$ for some $h, i, j \in Z$.

Write $d = u + w$, $u \in U$, $w \in E_8(2)$, then

$$ku = (h + i)e + 2if,$$
$$kw = iw_1 + jw_2.$$

Since $\{w_1, w_2\}$ generates a primitive sublattice of $E_8(2)$, k divides both i and j and hence h. Thus $d \in \operatorname{im} \phi$. This proves that ϕ is primitive.

If $\boldsymbol{d} \in U \oplus E_8(2)$ is orthogonal to $\operatorname{im} \phi$, then $\boldsymbol{d}$ has to be of the form

$$\boldsymbol{d} = k\boldsymbol{e} + \boldsymbol{w}, \quad k \in \boldsymbol{Z}, \quad \boldsymbol{w} \in E_8(2).$$

But then $\boldsymbol{d}^2 = \boldsymbol{w}^2 \neq -2$, for E_8 is even. Therefore, the orthogonal complement of $\operatorname{im} \phi$ contains no (-2)-vectors.

Case 2. $\rho(X) = 18$.

By (0.6.1) (ii), $T_X \cong U(2) \oplus T'(2)$ for some even lattice T' of signature $(1, 1)$. Since $T'(2)$ is indefinite, there is a primitive element $\boldsymbol{y}$ of $T'(2)$, $\langle \boldsymbol{y}, \boldsymbol{y} \rangle < 0$. Extend $\{\boldsymbol{y}\}$ to a basis $\{\boldsymbol{x}, \boldsymbol{y}\}$ of $T'(2)$. Then $\boldsymbol{x}^2 = 4a$, $\boldsymbol{y}^2 = 4c$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 2b$ for some integers a, b and c, $c < 0$.

By (0.3.2), one can pick up a primitive element $\boldsymbol{w}$, $\boldsymbol{w}^2 = 4c$, of $E_8(2)$. Define a map ψ: $T'(2) \to U \oplus E_8(2)$ by the formula

$$\psi(\boldsymbol{x}) = \boldsymbol{e} + 2a\boldsymbol{f}$$
$$\psi(\boldsymbol{y}) = 2b\boldsymbol{f} + \boldsymbol{w},$$

where $\{\boldsymbol{e}, \boldsymbol{f}\}$ is the standard basis of U. Then ψ is an embedding.

If $k\boldsymbol{d}$, $\boldsymbol{d} = \boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} \in U$, $\boldsymbol{v} \in E_8(2)$, $k \in \boldsymbol{Z}$, belongs to $\operatorname{im} \psi$, then

$$k\boldsymbol{d} = k(\boldsymbol{u} + \boldsymbol{v}) = i(\boldsymbol{e} + 2a\boldsymbol{f}) + j(2b\boldsymbol{f} + \boldsymbol{w}).$$

Comparing both sides, we get

$$k\boldsymbol{u} = i\boldsymbol{e} + (2ai + 2bj)\boldsymbol{f},$$
$$k\boldsymbol{v} = j\boldsymbol{w}.$$

The primivitity of $\boldsymbol{w}$ implies that k divides j, and hence i. But then $\boldsymbol{d} \in \operatorname{im} \psi$. So, ψ is primitive.

If $\boldsymbol{d} = m\boldsymbol{e} + n\boldsymbol{f} + \boldsymbol{v}$, $m, n \in \boldsymbol{Z}$, $\boldsymbol{v} \in E_8(2)$, is orthogonal to $\operatorname{im} \psi$, then $0 = \langle \boldsymbol{d}, \psi(\boldsymbol{x}) \rangle = 2am + n$, so n is an even integer. Since E_8 is even, $\boldsymbol{d}^2 = 2mn + \boldsymbol{v}^2 \neq -2$. This proves that no (-2)-vector lies in the orthogonal complement of $\operatorname{im} \psi$.

Case 3. $\rho(X) = 19$.

By (0.6.1) (i),

$$T_X \cong \begin{pmatrix} 4a & 2d & 2e \\ 2d & 4b & 2f \\ 2e & 2f & 4c \end{pmatrix},$$

i.e., there exists a basis $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ of T_X such that

$$x^2 = 4a, \quad y^2 = 4b, \quad z^2 = 4c, \quad \langle x, y\rangle = 2d, \cdots, \text{etc.}$$

Since T_X is indefinite, we may assume $c < 0$.
Let $\{\boldsymbol{e}, \boldsymbol{f}\}$ and $\{\boldsymbol{h}, \boldsymbol{k}\}$ be the standard bases of U and $U(2)$, respectively. Let $\boldsymbol{w}$ be the same as in Case 2, that is, $\boldsymbol{w}$ is a primitive element of $E_8(2)$ with $\boldsymbol{w}^2 = 4c$.
Define η: $T_X \to U \oplus U(2) \oplus E_8(2)$ by the formula

$$\begin{aligned}\eta(\boldsymbol{x}) &= \boldsymbol{e} + 2a\boldsymbol{f}\\ \eta(\boldsymbol{y}) &= 2d\boldsymbol{f} + \boldsymbol{h} + b\boldsymbol{k}\\ \eta(\boldsymbol{z}) &= 2e\boldsymbol{f} + f\boldsymbol{k} + \boldsymbol{w}.\end{aligned}$$

Then η is an embedding.

By the same argument as in Case 1 & 2, the primitivities of $\boldsymbol{w}$, $\boldsymbol{h}$, $\boldsymbol{e}$ imply the primitivity of η.

If $\boldsymbol{d} = i\boldsymbol{e} + j\boldsymbol{k} + k\boldsymbol{h} + m\boldsymbol{k} + \boldsymbol{v}$, $i, j, k, m \in \boldsymbol{Z}$, $\boldsymbol{v} \in E_8(2)$, belongs to the orthogonal complement of $\eta(T_X)$, then $0 = \langle \boldsymbol{d}, \eta(\boldsymbol{x})\rangle = 2ai + j$, so j is an even integer. But then $\boldsymbol{d}^2 = 2ij + 4km + \boldsymbol{v}^2 \neq -2$. This proves the case 3.

Case 4. $\rho(X) = 20$.

Again, by (0.6.1) (i), T_X has a basis $\{\boldsymbol{x}, \boldsymbol{y}\}$ with

$$x^2 = 4a, \quad y^2 = 4c, \quad \langle x, y\rangle = 2b.$$

Define ν: $T_X \to U \oplus U(2) \oplus E_8(2)$ by the formula

$$\begin{aligned}\nu(\boldsymbol{x}) &= \boldsymbol{e} + 2a\boldsymbol{f}\\ \nu(\boldsymbol{y}) &= 2b\boldsymbol{f} + \boldsymbol{h} + c\boldsymbol{k},\end{aligned}$$

where, again, $\{\boldsymbol{e}, \boldsymbol{f}\}$ and $\{\boldsymbol{h}, \boldsymbol{k}\}$ are the standard bases of U and $U(2)$, respectively. Then it is easy to see that ν is a primitive embedding. It is also easy, by the same argument as in Case 3, to see that there are no (-2)-vectors in the orthogonal complement. Q.E.D.

§ 3. Examples

(3.1) (Lieberman). Let A be the product of two elliptic curves E_1 and E_2 and let (e_1, e_2), $e_i \in E_i$, $e_i \neq 0$, $i = 1, 2$, be a 2-torsion point of A. Then the endomorphism σ: $A \to A$ given by the formula

$$\sigma(z_1, z_2) = (-z_1 + e_1, z_2 + e_2), \quad (z_1, z_2) \in A = E_1 \times E_2,$$

induces a fixed-point-free involution on the Kummer surface $\mathrm{Km}(A)$.

(3.2) *Remark.* Let A be an abelian surface.
If A splits, i.e., $A = E_1 \times E_2$, a product of two elliptic curves, then $\rho(\mathrm{Km}(A)) \geq 18$. Indeed,

$\rho(A) = 4$ if E_1 is isogeneous to E_2 and has a complex multiplication,
$= 3$ if E_1 is isogeneous to E_2 but does not have a complex multiplication,
$= 2$ if E_1 is not isogeneous to E_2 (cf. [Mum]).

If $\rho(A) = 4$ (i.e. $\rho(\mathrm{Km}(A)) = 20$), then A always splits [S–M].
If $\rho(A) \leq 3$, then A may not.

(3.3) Let A be a principally polarized abelian surface which does not split. Then A is the Jacobian $J(C)$ of some curve C of genus 2 and $\mathrm{Km}(A)$ is isomorphic to the resolution of a quartic surface F in $\boldsymbol{P}^3$ with sixteen nodes. The equation of F referred to a Göpel tetrad of nodes has the form

$$\begin{aligned} A(x^2t^2 + y^2z^2) + B(y^2t^2 + z^2x^2) + C(z^2t^2 + x^2y^2) \\ = Dxyzt + F(yt + zx)(zt + xy) + G(zt + xy)(xt + yz) \\ + H(xt + yz)(yt + zx) = 0 \quad \text{[Hut]}. \end{aligned}$$

For a generic choice of coefficients, the standard Cremona transformation acts freely.

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