ALGEBRAIC K3 SURFACES
WITH FINITE AUTOMORPHISM GROUPS

SHIGEYUKI KONDÔ

Introduction

The purpose of this paper is to give a proof to the result announced in [3]. Let $X$ be an algebraic surface defined over $\mathbb{C}$. $X$ is called a $K3$ surface if its canonical line bundle $K_X$ is trivial and $\dim H^1(X, \mathcal{O}_X) = 0$. It is known that the automorphism group $\text{Aut}(X)$ of $X$ is isomorphic, up to a finite group, to the factor group $O(S_X)/W_X$, where $O(S_X)$ is the automorphism group of the Picard lattice of $X$ (i.e. $S_X$ is the Picard group of $X$ together with the intersection form) and $W_X$ is its subgroup generated by all reflections associated with elements with square $(-2)$ of $S_X$ ([11]). Recently Nikulin [8], [10] has completely classified the Picard lattices of algebraic $K3$ surfaces with finite automorphism groups.

Our goal is to compute the automorphism groups of such $K3$ surfaces. Let $X$ be an algebraic $K3$ surface with finite automorphism group $\text{Aut}(X)$. By definition, there exists a nowhere vanishing holomorphic 2-form $\omega_X$ on $X$. Since an automorphism $g$ of $X$ preserves $\omega_X$, up to constants, $g^*\omega_X = \alpha_X(g) \cdot \omega_X$ where $\alpha_X(g) \in \mathbb{C}^*$. Therefore we have an exact sequence

$$1 \longrightarrow G_X \longrightarrow \text{Aut}(X) \longrightarrow \mathbb{Z}/m \longrightarrow 1$$

where $\mathbb{Z}/m$ is a cyclic group of $m$-th root of unity in $\mathbb{C}^*$ and $G_X$ is the kernel of $\alpha_X$. Moreover the representation of the cyclic group $\mathbb{Z}/m$ in $T_X \otimes \mathbb{Q}$ is isomorphic to a direct sum of irreducible representations of the cyclic group $\mathbb{Z}/m$ over $\mathbb{Q}$ of maximal rank $\phi(m)$, where $T_X$ is a transcendental lattice of $X$ and $\phi$ is the Euler function. In particular $\phi(m) \leq \text{rank } T_X$ and hence $m \leq 66$ ([6], Theorem 3.1 and Corollary 3.2).

An algebraic $K3$ surface $X$ is called general if the image of $\alpha_X$ is of order at most 2, and $X$ is called special if it is not general. The meaning of this definition is as follows: Let $X$ be an algebraic $K3$ surface with

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a Picard lattice $S_x$. Let $S$ be an abstract lattice which is isomorphic to $S_x$. Denote by $M_s$ the moduli space for algebraic $K3$ surfaces whose Picard lattices are isomorphic to $S$. Then the dimension of $M_s$ is equal to $20 - \text{rank}(S)$. A general $K3$ surface $Y$ with $S_Y = S$ corresponds to a point of the complement of hypersurfaces in $M_s$.

**Theorem.** Let $X$ be an algebraic $K3$ surface with finite automorphism group $\text{Aut}(X)$.

(i) If $X$ is general, then $\text{Aut}(X)$ is as in the following table:

<table>
<thead>
<tr>
<th>$S_X$</th>
<th>$\text{Aut}(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U \oplus E_6 \oplus E_6 \oplus A_1$</td>
<td>$\mathbb{S}_3 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U \oplus E_6 \oplus E_8$, $U \oplus E_8 \oplus E_7$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U \oplus E_6 \oplus D_6$, $U \oplus E_6 \oplus D_4 \oplus A_1$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U \oplus D_8 \oplus D_6$, $U \oplus E_6 \oplus A_1$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U \oplus E_7 \oplus A_1$, $U \oplus D_6 \oplus A_1$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U \oplus D_8 \oplus A_1$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U(2) \oplus D_8 \oplus D_6$, $U \oplus A_1^2$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$U(2) \oplus A_1^2$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
<tr>
<td>otherwise</td>
<td>$\mathbb{Z}/2$ or ${1}$</td>
</tr>
</tbody>
</table>

where $U$ (resp. $U(2)$) is the lattice of rank 2 with the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$), $A_m$, $D_n$ and $E_n$ are negative definite lattices associated with the Dynkin diagrams of type $A_m$, $D_n$ and $E_n$ respectively and $A_1^k$ denotes the direct sum $A_1 \oplus A_1 \oplus \cdots \oplus A_1$ ($k$ times).

(ii) If $X$ is special, then $\text{Aut}(X)$ is a cyclic extension of the group in the above table.

We remark here that there exists a special $K3$ surface $X$ with $\text{Aut}(X) \simeq \mathbb{Z}/66$. This automorphism acts on the Picard group of $X$ as identity. In [4], we studied automorphisms with this property.

Also for Enriques surfaces with finite automorphism groups, we refer the reader to [2], [9].

To prove the above theorem we use the following phenomenon: In
the exact sequence (1), if \( \text{rank}(S_x) \) becomes smaller, then \( G_x \) too becomes smaller, and the group \( \mathbb{Z}/m \) grows bigger.

In Section 1, we recall the Picard lattices of algebraic K3 surfaces with finite automorphism groups. Section 2 is devoted to the results on finite automorphisms of K3 surfaces due to Nikulin [6] and Mukai [5]. In particular from these results we obtain all the possible cases of \( G_x \) (Lemma 2.3). In Sections 4 and 5 we prove the above theorem. In case \( \text{rank}(S_x) \geq 15 \) we have the dual graph of all smooth rational curves on \( X \) ([8], Sect. 4, Part 5, Table 2) and hence we can compute \( \text{Aut}(X) \). In case \( \text{rank}(S_x) \leq 14 \) it follows from the result in Section 2 that \( G_x \) is a subgroup of \( \mathbb{Z}/3 \) or \( \mathbb{Z}/2 \times \mathbb{Z}/2 \). To determine \( \text{Aut}(X) \) we use the theory of symmetric bilinear forms (cf. [7]) and that of elliptic pencils due to Kodaira [1] and Shioda [12] (Sect. 3).

§ 1. Picard lattices of K3 surfaces with finite automorphism groups

In this section we recall the Nikulin’s classification [8], [10] of Picard lattices of algebraic K3 surfaces with finite automorphism groups.

A lattice \( L \) is a free \( \mathbb{Z} \)-module of finite rank endowed with an integral bilinear form \( \langle , \rangle \). By \( L \oplus L^* \) we denote the orthogonal direct sum of lattices \( L \) and \( L^* \). For a lattice \( L \) and an integer \( m \) we denote by \( L(m) \) the lattice whose bilinear form is the one on \( L \) multiplied by \( m \). Also we denote by \( U \) the lattice of rank 2 with the intersection matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and by \( A_n, D_n \) and \( E_8 \) the negative definite lattices associated with the Dynkin diagram of type \( A_n, D_n \) and \( E_8 \) respectively. A lattice \( L \) is called **even** if \( \langle x, x \rangle \in 2\mathbb{Z} \) for all \( x \in L \). Let \( S \) be a non degenerate lattice. We denote by \( S^* = \text{Hom}(S, \mathbb{Z}) \) the dual of \( S \). Put \( A_s = S^*/S \). Then \( A_s \) is a finite abelian group which is called the **discriminant group** of \( S \). We denote by \( l(S) \) the number of minimal generators of \( A_s \). A lattice \( S \) is called a **2-elementary** if \( A_s \) is a 2-elementary abelian group. For a 2-elementary lattice \( S \), we define a **parity** \( \delta(S) \) of \( S \) as follows:

\[
\delta(S) = \begin{cases} 
0 & \text{if } q_s(x) = 0 \text{ for all } x \in A_s \\
1 & \text{otherwise}
\end{cases}
\]

where \( q_s \) is the quadratic form on \( A_s \) induced from the one on \( S \).

**Proposition 1.1** ([8], Theorem 4.3.2). An indefinite 2-elementary even lattice is determined, up to isomorphisms, by the invariants \( (\text{rank}(S), l(S)) \),
The following tables give the description of Picard lattices of rank \( \geq 9 \) of algebraic \( K3 \) surfaces with finite automorphism groups which we need for the proof of our theorem.

**Table 2 (\( S_x \) is 2-elementary, rank \( S_x \geq 9 \).)**

<table>
<thead>
<tr>
<th>rank(( S_x ))</th>
<th>( S_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>( U\oplus E_6\oplus E_6\oplus A_1 )</td>
</tr>
<tr>
<td>18</td>
<td>( U\oplus E_6\oplus E_7 )</td>
</tr>
<tr>
<td>17</td>
<td>( U\oplus E_6\oplus E_7 )</td>
</tr>
<tr>
<td>16</td>
<td>( U\oplus E_6\oplus D_6 )</td>
</tr>
<tr>
<td>15</td>
<td>( U\oplus E_6\oplus D_6\oplus A_1 )</td>
</tr>
<tr>
<td>14</td>
<td>( U\oplus E_6\oplus D_6, U\oplus D_6\oplus D_6, U\oplus E_6\oplus A_1 )</td>
</tr>
<tr>
<td>13</td>
<td>( U\oplus E_6\oplus A_1, U\oplus E_7\oplus A_1 )</td>
</tr>
<tr>
<td>12</td>
<td>( U\oplus E_6\oplus A_2, U\oplus E_7\oplus A_2, U\oplus D_6\oplus A_1 )</td>
</tr>
<tr>
<td>11</td>
<td>( U\oplus E_6\oplus A_1, U\oplus E_7\oplus A_1, U\oplus D_6\oplus A_1, U\oplus D_6\oplus A_2 )</td>
</tr>
<tr>
<td>10</td>
<td>( U\oplus E_8, U\oplus D_8, U\oplus D_6\oplus D_6, U(2)\oplus D_6\oplus D_6, U\oplus E_7\oplus A_1, U\oplus D_6\oplus A_1, U\oplus D_6\oplus A_2, U\oplus A_1 )</td>
</tr>
<tr>
<td>9</td>
<td>( U\oplus E_9, U\oplus D_8\oplus A_1, U\oplus D_6\oplus A_1, U\oplus A_1, U(2)\oplus A_1 )</td>
</tr>
</tbody>
</table>

**Table 3 (\( S_x \) is not 2-elementary and rank(\( S_x \)) \( \geq 9 \).)**

<table>
<thead>
<tr>
<th>rank(( S_x ))</th>
<th>( S_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>( U\oplus E_8\oplus A_3 )</td>
</tr>
<tr>
<td>12</td>
<td>( U\oplus E_8\oplus A_2 )</td>
</tr>
<tr>
<td>11</td>
<td>( U\oplus E_8\oplus A_2 )</td>
</tr>
<tr>
<td>9</td>
<td>( U\oplus A_7, U\oplus D_8\oplus A_3, U\oplus D_6\oplus A_3, U\oplus D_9, U\oplus E_8\oplus A_1 )</td>
</tr>
</tbody>
</table>
§ 2. Finite automorphisms of K3 surfaces

Let $X$ be an algebraic K3 surface. We denote by $\text{Aut}(X)$ the group of automorphisms of $X$. Let $G$ be a finite subgroup of $\text{Aut}(X)$ and let $\omega_X$ be a nowhere vanishing holomorphic 2-form on $X$. Then for $g \in G$, $g^*\omega_X = \alpha_x(g) \cdot \omega_X$ where $\alpha_x(g) \in C^*$. Therefore we have an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow Z/m \longrightarrow 1$$

where $Z/m$ is a cyclic group of $m$-th root of unity in $C^*$ and $K$ is the kernel of $\alpha_x$. Moreover the representation of the cyclic group $Z/m$ in $T_X \otimes \mathbb{Q}$ is isomorphic to a direct sum of irreducible representations of the cyclic group $Z/m$ over $\mathbb{Q}$ of maximal rank $\phi(m)$, where $\phi$ is the Euler function. In particular $\phi(m) \leq \text{rank}(T_X)$ and hence $m \leq 66$ ([6], Theorem 3.1 and Corollary 3.2).

An automorphism $g$ of $X$ is called symplectic if $\alpha_x(g) = 1$. The classification of finite symplectic automorphism groups of K3 surfaces is recently given by S. Mukai [5], based on the study of abelian groups due to Nikulin [6].

**Proposition 2.1 ([6], § 5, [5], (0.1)).** Let $g$ be a symplectic automorphism of finite order $n$ of a K3 surface. Then $n \leq 8$ and the number of fixed points $f(n)$ depends only on $n$ and is as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Let $G$ be a finite symplectic automorphism group of a K3 surface. Put $f(1) = 24$ and $\mu(G) = (1/|G|) \sum_{g \in G} f(|g|)$. By the Lefschetz fixed point formula and an elementary representation theory, we have

**Proposition 2.2 ([5], Proposition 2.4).** $\mu(G) = 2 + \text{rank}(L^o)$ where $L = H^1(X, \mathbb{Z})$ and $L^o = \{ x \in L | g^*x = x \text{ for any } g \in G \}$.

In what follows we assume that $\text{Aut}(X)$ is finite. Then we have an exact sequence

$$1 \longrightarrow G_x \longrightarrow \text{Aut}(X) \longrightarrow Z/m \longrightarrow 1$$

where $G_x$ is the kernel of $\alpha_x$. In Section 5 we shall need the following:
Lemma 2.3. (i) If rank(S_x) ≤ 14, then G_x is a subgroup of Z/3 or Z/2 × Z/2; (ii) If rank(S_x) ≤ 12, then G_x is a subgroup of Z/2; (iii) If rank(S_x) ≤ 8, then G_x = {1}.

Proof. It follows from [6], Theorem 1.1 that \( L^{G_x} \) contains \( T_x \). Since \( G_x \) is finite, the signature of \( S_x^{G_x} \) is equal to \((1, r)\), where \( r \) is a non-negative integer. Hence rank(\( L^{G_x} \)) ≥ rank(\( T_x \)) + 1. Note that rank(\( T_x \)) + rank(S_x) = 22. Now the assertions easily follows from Propositions 2.1 and 2.2.

Proposition 2.4 ([6], § 10). Assume that \( G = G_x \) is a subgroup of Z/3 or Z/2 × Z/2. Then the discriminant group \( A_{L^G} \) of \( L^G \) is as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
G & Z/2 & Z/2 × Z/2 & Z/3 \\
\hline
A_{L^G} & (Z/2)^8 & (Z/2)^8 × (Z/4)^2 & (Z/3)^6 \\
\hline
\end{array}
\]

§ 3. Elliptic pencils on K3 surfaces

Let \( X \) be a K3 surface. An elliptic pencil \( π: X → P^1 \) is a holomorphic map \( π \) from \( X \) to \( P^1 \) whose general fibres are smooth elliptic curves. An effective divisor \( D \) is called a \( m \)-section of \( π \) if \( D F = m \), where \( F \) is a fibre of \( π \) and \( m ∈ N \). A 1-section is simply called a section. All singular fibres of an elliptic pencil were classified by Kodaira [1]. We use the terminology of singular fibres in [1]. The following lemma follows from [11], § 3, Corollary 3, the Riemann-Roch theorem and the classification of singular fibres of elliptic pencils [1].

Lemma 3.1. Let \( X \) be an algebraic K3 surface and let \( S_x \) be the Picard lattice of \( X \). Assume that \( S_x = U ⊕ K \), where \( K \) is a negative definite lattice. Then

(i) there exists an elliptic pencil \( π: X → P^1 \) with a section.

(ii) If \( K = K_1 ⊕ N \), where \( K_1 \) and \( N \) are negative definite lattices and \( N \) is generated by elements with square \((-2)\), then \( π \) has a singular fibre \( F \) as in the following table:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
N & A_1 & A_2 & A_n (n ≥ 3) & D_n (n ≥ 4) & E_6 & E_7 & E_8 \\
\hline
F & I_2 or III & I_3 or IV & I_{n+1} & I_{n-4}^* & IV^* & III^* & II^* \\
\hline
\end{array}
\]
The following will be used in the latter to prove the existence of symplectic automorphisms.

**Proposition 3.2** ([1], Theorem 12.2, [12], Corollaries 1.5, 1.7). Let $X$ be an algebraic $K3$ surface and $S_X$ the Picard lattice of $X$. Let $\pi: X \to P^1$ be an elliptic pencil with a section and let $F_\nu$ ($1 \leq \nu \leq k$) be all singular fibres of $\pi$. We denote respectively by $\varepsilon_\nu$, $m_\nu$ or $\mu_\nu$ the Euler number of $F_\nu$, the number of irreducible components of $F_\nu$ or the number of simple components of $F_\nu$. Then

1. $\sum_{\nu=1}^{k} \varepsilon_\nu = 24$ (the Euler number of $X$),
2. $\text{rank}(S_X) = r + 2 + \sum_{\nu=1}^{k} (m_\nu - 1)$

where $r$ is the rank of the group of sections of $\pi$,

3. when $r = 0$, let $n$ denote the order of the group of sections of $\pi$.

Then we have

$$|\det(S_X)| = \prod_{\nu=1}^{k} \mu_\nu / n^2.$$

§ 4. **Proof of the Theorem—the case when $\text{rank}(S_X) \geq 15$**

In this section and the next we prove our theorem. By our proof in the following, we can see:

**Proposition.** Let $X$ be an algebraic $K3$ surface with finite automorphism group $\text{Aut}(X)$. Then the subgroup $G_X$ of symplectic automorphisms of $\text{Aut}(X)$ is uniquely determined by the isomorphism class of $S_X$.

The assertion (ii) in Theorem follows from this Proposition and the exact sequence (1). For simplicity, in the following, we assume that $X$ is a general algebraic $K3$ surface with finite automorphism group.

Let $X$ be a general algebraic $K3$ surface with finite automorphism group and $\text{rank}(S_X) \geq 15$. Then $S_X$ is a 2-elementary lattice (see Table 2). By [8], Section 4, there exists an automorphism $\sigma$ of order 2 such that $\sigma^*|S_X = 1_{S_X}$ and $\sigma^*|T_X = -1_{T_X}$. Therefore we have an exact sequence:

$$1 \longrightarrow G_X \longrightarrow \text{Aut}(X) \xrightarrow{\alpha_X} \mathbb{Z}/2 \longrightarrow 1$$

where $\mathbb{Z}/2$ is generated by $\sigma$. Since $g^*|T_X = 1_{T_X}$ for all $g \in G_X$, $g^* \circ \sigma^* = \sigma^* \circ g^*$. It follows from the global Torelli theorem [11] that $g \circ \sigma = \sigma \circ g$. Hence the above exact sequence splits: $\text{Aut}(X) \simeq G_X \times \mathbb{Z}/2$.

A **dual graph** of smooth rational curves is the following simplicial complex $\Gamma$: (i) the set of vertices is a set of smooth rational curves on
To determine the group $G_X$ we use the dual graph of all smooth rational curves on $X$. Such graphs were found by Nikulin [8]. However for $S_X = U \oplus E_8 \oplus E_8 \oplus A_1$, his graph is not complete (compare the following graph in Figure 1 with the table 2 in [8], § 4, Part 5). It follows from [13], Proposition 1 and [14], Lemma 2.4 that the following graph represents all smooth rational curves on $X$.

Let $\Gamma$ be the dual graph of all smooth rational curves on $X$ (see Figures 1-5). Consider the natural homomorphism $\rho: \text{Aut}(X) \to \text{Aut}(\Gamma)$, where $\text{Aut}(\Gamma)$ is the symmetry group of $\Gamma$. Since $S_X$ is generated by the classes of smooth rational curves in $\Gamma$, the kernel of $\rho$ acts on $S_X$ as identity. Hence the symplectic group $G_X$ is regarded as a subgroup of $\text{Aut}(\Gamma)$.

(4.1) $S_X = U \oplus E_8 \oplus E_8 \oplus A_1$. The following diagram $\Gamma$ is the dual graph of all smooth rational curves on $X$:

![Diagram](image)

Figure 1

Obviously the symmetry group $\text{Aut}(\Gamma)$ is isomorphic to $\mathbb{Z}_3$.

We now claim that $G_X \cong \mathbb{Z}_3$. First consider the elliptic pencil $|A_1| = |\sum_{i=1}^{18} E_i|$ which has a section and a singular fibre of type $I_{18}$. By the formulas in Proposition 3.2, we can see that $|A_1|$ has only one reducible singular fibre of type $I_{18}$ and the group of sections of $|A_1|$ is isomorphic to $\mathbb{Z}/3$. These sections act on $X$ as a symplectic automorphism of order 3 which is a rotation of $\Gamma$ of order 3. Next consider the elliptic pencil $|A_2| = |E + E_{11-19}|$ which has a section and two singular fibres of type $I_5$ and of type $I_{15}^*$. Again it follows from the formulas in Proposition 3.2 that $|A_2|$ has only two reducible singular fibres of type $I_5$ and of type...
$I^*_{12}$ and the group of sections of $|\Delta|$ is isomorphic to $\mathbb{Z}/2$. Therefore $G_x \simeq \mathbb{Z}/2$.

(4.2) $S_x = U \oplus E_8 \oplus E_6$. The following diagram $\Gamma$ is the dual graph of all smooth rational curves on $X$:

![Figure 2](Image)

We claim that $G_x \simeq \text{Aut}(\Gamma) \simeq \mathbb{Z}/2$. Let $\varphi$ be an isometry of $S_x$ defined by $\varphi((x, y, z)) = (x, z, y)$ where $(x, y, z) \in U \oplus E_8 \oplus E_6$. Note that the second cohomology lattice $L = H^2(X, \mathbb{Z})$ is the direct sum of $S_x$ and $T_x$. Put $\bar{\varphi} = (\varphi, 1_{T_x}) : S_x \oplus T_x \to S_x \oplus T_x$. Then by the global Torelli theorem [11], there exists an automorphism $g$ of $X$ such that $g^* = \bar{\varphi}$ on $L$. By construction, $g$ is symplectic and generates Aut($\Gamma$). Hence $G_x \simeq \mathbb{Z}/2$ and $\text{Aut}(X) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

(4.3) $S_x = U \oplus E_8 \oplus E_6$. The following diagram $\Gamma$ is the dual graph of all smooth rational curves on $X$:

![Figure 3](Image)

Obviously $\text{Aut}(\Gamma) \simeq \mathbb{Z}/2$. By considering the elliptic pencil $|E_1 + E_2|$ with a section, we have a symplectic automorphism of order 2 which acts on $\Gamma$ as a symmetry of order 2. Hence we have $\text{Aut}(X) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

(4.4) $S_x = U \oplus E_8 \oplus D_6$. The following diagram $\Gamma$ is the dual graph of all smooth rational curves on $X$:

![Figure 4](Image)
We can see $\text{Aut}(\Gamma) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. We select a generator $\{\gamma_1, \gamma_2\}$ of $\text{Aut}(\Gamma)$ as follows; $\gamma_1$ is the reflection of $\Gamma$ with $\gamma_1(E_i) = E_i$ and $\gamma_2$ is the reflection with respect to the middle horizontal line. By considering the elliptic pencil $|E_1 + E_2|$ with a section, we have a symplectic automorphism $g$ whose action on $\Gamma$ coincides with $\gamma_1$. On the other hand, if $\gamma_2$ is represented by a symplectic automorphism $g'$, then $g'$ preserves 15 smooth rational curves respectively (see Figure 4). Hence the number of fixed points of $g'$ is greater than 8 which is impossible (Proposition 2.1). Thus we have $G_x \simeq \mathbb{Z}/2$ and $\text{Aut}(X) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

(4.5) $S_x = U \oplus E_1 \oplus D_1 \oplus A_1$. The following diagram $\Gamma$ is the dual graph of all smooth rational curves on $X$:

![Diagram](image)

Figure 5

We can see that $\text{Aut}(\Gamma) \simeq \mathbb{S}_3 \times \mathbb{Z}/2$ where $\mathbb{Z}/2$ is generated by the reflection $\gamma$ with $\gamma(E_i) = E_i$ and $\mathbb{S}_3$ is the permutations of the set $\{E_i, F_i, L_i\}$. By considering the elliptic pencil $|E_1 + E_2|$ with a section, $\gamma$ is represented by a symplectic automorphism of order 2. On the other hand, any element of $\mathbb{S}_3$ is not represented by a symplectic automorphism because a symplectic automorphism of order 2 (resp. of order 3) has exactly 8 (resp. 6) isolated fixed points (Proposition 2.1). Therefore we have $G_x \simeq \mathbb{Z}/2$ and $\text{Aut}(X) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

§ 5. **Proof of the Theorem—the case when $\text{rank}(S_x) \leq 14$**

(5.1) First we remark that $G_x$ is trivial if $\text{rank}(S_x) \leq 8$ (Lemma 2.3, (iii)). Hence it suffices to consider the case that $9 \leq \text{rank}(S_x) \leq 14$. In these cases, $G_x$ is a subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/3$ (Lemma 2.3). Consider a primitive embedding $T_x \subseteq \mathcal{L}_{ax}$ and denote by $T_x^{\perp}$ the orthogonal complement of $T_x$ in $\mathcal{L}_{ax}$. Then $T_x \oplus T_x^{\perp}$ is a sublattice of $\mathcal{L}_{ax}$ of finite index and $A_{\mathcal{L}_{ax}}$ is a quotient group of $A_{T_x \oplus T_x^{\perp}}$, and hence $l(T_x \oplus T_x^{\perp}) \geq l(\mathcal{L}_{ax})$. 


Since \( \text{rank}(T^\perp_x) \geq l(T^\perp_x) \) and \( l(T^\perp_x) = l(S^\perp_x) \), we have \( l(S^\perp_x) + \text{rank}(T^\perp_x) \geq l(L^{\perp_x}) \). Therefore it follows from Proposition 2.4 that:

\[ G^\perp_x = \{1\} \text{ or } \mathbb{Z}/2 \text{ if } S^\perp_x = U \oplus E_8 \oplus D_4, \ U \oplus D_4 \oplus D_4, \ U \oplus E_8 \oplus A^+_8, \ U \oplus E_8 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus E_8 \oplus A^+_4, \ U \oplus E_4 \oplus A^+_4, \ U \oplus E_4 \oplus A^+_4, \ U \oplus A^+_4, \ U \oplus A^+_8, \ or \ U(2) \oplus A^+_4 \text{ and } G^\perp_x = \{1\} \text{ if } S^\perp_x \text{ is otherwise.} \]

Moreover, if \( G^\perp_x = \mathbb{Z}/2 \) and \( S^\perp_x = U \oplus E_8 \oplus D_4, \ U \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus A^+_4, \ U \oplus A^+_8, \ U \oplus D_4 \oplus A^+_4, \ U \oplus A^+_4, \ U(2) \oplus A^+_4 \) or \( U(2) \oplus A^+_4, \) then \( A^\perp_{L^\perp_x} = A_{T^\perp_x} \) and hence \( L^{\perp_x} = T^\perp_x \oplus T^\perp_x \). This is a contradiction because \( L^{\perp_x} \) is a 2-elementary lattice with \( \delta_{T^\perp_x} = 0 \) and, on the other hand, \( T^\perp_x \) is a 2-elementary lattice with \( \delta_{T^\perp_x} = 1 \). Also, if \( S^\perp_x = U \oplus E_8 \oplus D_4 \) and \( G^\perp_x = \mathbb{Z}/2 \), then \( l(L^{\perp_x}) = l(T^\perp_x) + l(T^\perp_x) \) and hence \( L^{\perp_x} = T^\perp_x \oplus T^\perp_x \). Hence \( T^\perp_x \) is a 2-elementary lattice with \( \text{rank}(T^\perp_x) = 6, \ l(T^\perp_x) = 6 \) and \( \delta_{T^\perp_x} = 0 \). However, by [7], Theorem 3.6.2, such lattice does not exist.

Hence \( G^\perp_x = \{1\} \) if \( S^\perp_x = U \oplus E_8 \oplus D_4, \ U \oplus E_8 \oplus A^+_8, \ U \oplus E_8 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus A^+_4, \ U \oplus A^+_8, \) or \( U \oplus A^+_4 \).

In the following we shall see that \( G^\perp_x = \mathbb{Z}/2 \) if \( S^\perp_x = U \oplus D_4 \oplus D_4, \ U \oplus E_8 \oplus A^+_8, \ U \oplus E_8 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus D_4 \oplus A^+_4, \ U \oplus D_4 \oplus D_4, \ U \oplus A^+_4, \) or \( U(2) \oplus A^+_4 \).

(5.2) \( S^\perp_x = U \oplus D_4 \oplus D_4 \). Note that there exists an elliptic pencil with a section whose reducible singular fibres are of type \( I^*_8 \) and of type \( I^*_8 \) (Lemma 3.1). Hence we have the following dual graph of smooth rational curves on \( X \):

[Diagram of smooth rational curves]

where \( E_i \) is a section of this pencil and others are components of singular fibres. Let us consider the elliptic pencil \( |D| = |2E_1 + 4E_2 + 6E_3 + 3E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_9| \). Then \( E_{10}, E_{11}, E_{12} \) and \( E_{13} \) are components of a singular fibre \( F \) of this pencil \( |D| \). By Proposition 3.2, \( F \) is of type \( I^*_8 \) and hence there exists a smooth rational curve \( E_{14} \) with \( E_{10} + E_{11} + E_{12} + E_{13} + 2E_{14} \in |D| \). Since \( E \) is a 2-section of \( |D| \), \( E \cdot E_{14} = 2 \). Then the elliptic pencil \( |E_{14} + E| \) has two sections \( E_{13}, E_9 \) and these two sections define a symplectic automorphism. Therefore \( G^\perp_x \simeq \mathbb{Z}/2 \).

(5.3) \( S^\perp_x = U \oplus E_8 \oplus A^+_4 \). First we remark that \( U \oplus E_8 \oplus A^+_4 \) is isomor-
phic to $U \oplus E_r \oplus D_i \oplus A_i$ (Proposition 1.1). Therefore there exists an elliptic pencil with a section which has three reducible singular fibres of type $\Pi^*_a$, $I^*_b$ and $I_c$ (Lemma 3.1). Hence we have the following dual graph of smooth rational curves on $X$:

![Dual Graph](image)

where $E_g$ is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|\mathcal{A}| = |E_1 + E_2 + E_3 + 2(E_2 + E_3)|$. Then $E_j$, $9 \leq j \leq 14$, are contained in some singular fibre $F$ of $|\mathcal{A}|$. It follows from Proposition 3.2 that $F$ is of type $I^*_c$. Hence there exists a smooth rational curve $E$ with $E + E_9 + E_{10} + 2(E_9 + E_{10}) \in |\mathcal{A}|$. Since $E_9$ is a $2$-section of $|\mathcal{A}|$, $E \cdot E_9 = 2$. Then the elliptic pencil $|E + E_9|$ has two sections $E_3$ and $E_{10}$ which define a symplectic automorphism of order $2$. Therefore we have $G_x \simeq \mathbb{Z}/2$.

(5.4) $S_x = U \oplus E_r \oplus A_i^i$. First note that $U \oplus E_r \oplus A_i^i \simeq U \oplus D_i \oplus D_i \oplus A_i$ (Proposition 1.1). Since there exists an elliptic pencil with a section which has three reducible singular fibres of type $I^*_a$, $I^*_b$ and $I_c$ (Lemma 3.1), we have the following dual graph:

![Dual Graph](image)

where $E_g$ is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|\mathcal{A}| = |E_1 + E_2 + E_3 + 2(E_2 + E_3)|$. Then $E_j$, $8 \leq j \leq 12$, are components of singular fibres of $|\mathcal{A}|$. Since $K$ is a section of $|\mathcal{A}|$ and $K \cdot E_g = K \cdot E_9 = 1$, $E_9$ is not a component of a singular fibre containing $E_g$. It now follows from Proposition 3.2 that the reducible singular fibres of $|\mathcal{A}|$ are of type $I^*_a$, $I^*_b$ and $I_c$. Hence there exists a smooth rational curve $E$ with $E + E_9 + E_{10} + 2E_{10} \in |\mathcal{A}|$. Since $F$ is a $2$-section of $|\mathcal{A}|$, $E \cdot F = 2$. The elliptic pencil $|E + F|$ has two sections $E_3$ and $E_{10}$, and hence $G_x \simeq \mathbb{Z}/2$. 
(5.5) $S_x = U \oplus D_6 \oplus A^4_4$. First note that $U \oplus D_6 \oplus A^4_4 \simeq U \oplus D_5 \oplus D_5 \oplus A^4_4$ (Proposition 1.1). Since there exists an elliptic pencil with a section which has 4 reducible singular fibres of type $I^*_o$, $I^*_o$, $I_2$, and $I_2$ (Lemma 3.1), we have the following dual graph:

![Diagram](https://example.com/diagram)

where $E_5$ is a section of this pencil and others are components of singular fibres. Then the elliptic pencil $|E_1 + E_2 + E_3 + E_4 + 2E_5|$ has two sections. Hence $G_x \simeq \mathbb{Z}/2$.

(5.6) $S_x = U \oplus D_6 \oplus A^6_4$. Since there exists an elliptic pencil with a section which has one singular fibre of type $I^*_o$ and 6 singular fibres of type $I_2$ (Lemma 3.1), we have the following dual graph of smooth rational curves:

![Diagram](https://example.com/diagram)

where $E_5$ is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|A| = |E_1 + E_2 + E_3 + E_4 + 2E_5|$. Then $E_j$, $6 \leq j \leq 11$, are components of singular fibres of $|A|$. By Proposition 3.2, the following two cases occur: (a) $|A|$ has reducible singular fibres of type $I^*_o$, $I^*_o$, $I_2$, and $I_2$; (b) $|A|$ has two reducible singular fibres of type $I^*_o$ and $I^*_o$. In case (a), we may assume that there exists a smooth rational curve $E$ with $E + E_6 \in |A|$. Since $E_{12}$ is a 2-section of $|A|$, we have $E \cdot E_{12} = 2$. Then the elliptic pencil $|E + E_{12}|$ has two sections $E_5$ and $E_9$, and hence $G_x \simeq \mathbb{Z}/2$. In case (b), we may assume that there exists a smooth rational curve $F$ with $E_5 + E_9 + E_{11} + 2E_3 + 2E_{10} + 2F \in |A|$. Then the elliptic pencil $|E_5 + E_{10} + E_{11} + E_{12} + 2E_9|$ has two sections $E_5$ and $F$, and hence $G_x \simeq \mathbb{Z}/2$. 
\[(5.7) \quad S_x = U \oplus D_4 \oplus A_1^i. \text{ In this case, the same argument as in (5.6) shows } G_x \simeq \mathbb{Z}/2.\]

\[(5.8) \quad S_x = U(2) \oplus D_4 \oplus D_v. \text{ First we claim that } S_x \text{ is isomorphic to } U \oplus K, \text{ where } K \text{ is a negative definite lattice of rank 8. Let } \{e, f\} \text{ be a basis of } U(2) \text{ and } \{e_j, f_j\} \text{ the two copies of a basis of } D_i \text{ as in the following dual graphs:}\]

\[\begin{array}{c}
  e_1 & e_2 & e_3 & e_4 \\
  f_1 & f_2 & f_3 & f_4
\end{array}\]

Put \(\delta = e + f + e_i + f_i\). Then \(\delta^2 = 0\) and \(\langle \delta, e_i \rangle = 1\). Hence \(\delta\) and \(e_i\) generate a sublattice of \(S_x\) isomorphic to \(U\). So we have \(S_x \simeq U \oplus K\). Therefore there exists an elliptic pencil \(|\Delta|\) with a section (Lemma 3.1). It follows from Proposition 3.2, (ii) that \(K\) has a sublattice \(K'\) of finite index which is generated by some components of singular fibres of \(|\Delta|\). Since \(K\) is a 2-elementary lattice with rank \(K = 8\), \(\det K = 2^6\) and \(\delta_x = 0\), we can see that \(K \neq K'\). Hence the group of section of \(|\Delta|\) is not trivial (Proposition 3.2, (iii)). Therefore \(G_x \simeq \mathbb{Z}/2\).

\[(5.9) \quad S_x = U(2) \oplus D_4 \oplus A_1^i. \text{ In these cases, to prove } G_x \simeq \mathbb{Z}/2, \text{ we give a lattice theoretic construction of a symplectic automorphism.}\]

In case \(S_x = U \oplus A_1^i\), consider a sublattice \(\langle 2 \rangle \oplus \langle -2 \rangle \oplus A_1^i\) of \(S_x\). Since a 2-elementary lattices \(S\) is determined by \(\text{rank}(S_x), l(S)\) and the parity of \(S\), this sublattice is isomorphic to \(\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)\) (Proposition 1.1). By this isomorphism, we consider \(\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)\) as a sublattice of \(S_x\). Let \(\iota\) be an involution of \(\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)\) such that \(\iota|\langle 2 \rangle \oplus \langle -2 \rangle = 1\) and \(\iota|E_8(2) = -1\). Since \(\langle 2 \rangle \oplus \langle -2 \rangle\) and \(E_8(2)\) are 2-elementary, \(\iota\) extends to an involution \(\iota'\) of \(S_x\). By construction, \(\iota'\) acts on the discriminant group \(A_{s_x}\) as identity. Hence \(\iota'\) extends to an involution \(\iota\) of \(L_x\) with \(\iota|T_x = 1\). \(\iota\) preserves a period of \(X\) and the Kähler cone because \(E_8(2)\) contains no \((-2)\)-elements. Hence by the global Torelli theorem [11], \(\iota\) is represented by a symplectic automorphism of order 2.

In case \(S_x = U(2) \oplus A_1^i\), we define two involutions \(\sigma\) and \(g\) of \(L_x\) as follows: let \(\{\alpha_i, \beta_i\}\) be a copy of a basis of \(U(1 \leq i \leq 3)\) and let \(\{e_j, f_j\}\) be copies of a basis of \(E_8(1 \leq j \leq 8)\). Then \(\{\alpha_i, \beta_i, e_j, f_j\}\) \((1 \leq i \leq 3, 1 \leq j \leq 8)\) is a basis of \(L_x = U \oplus U \oplus U \oplus E_8 \oplus E_8\). Put \(g|U \oplus U \oplus U = 1\) and

\[\begin{array}{c}
  e_i & e_i \\
  f_j & f_j
\end{array}\]

\[\begin{array}{c}
  \alpha_i & \beta_i \\
  e_j & f_j
\end{array}\]
$g(e_j) = f_j, \ 1 \leq j \leq 8, \ \sigma(\alpha_i) = \beta_i, \ \sigma(\beta_i) = -\alpha_i, \ \sigma(\gamma_i) = -\beta_i, \ 2 \leq i \leq 3,$ and $\sigma(e_j) = -f_j, \ 1 \leq j \leq 8$. Then $L^{(\sigma)}$ is isomorphic to $\langle 2 \rangle \oplus E_8(2) \cong U(2) \oplus A_1$ which is generated by $\{\alpha_i + \beta_i, e_j - f_j | j = 1, \ldots, 8\}$. On the other hand $L^{(\sigma)}$ is isomorphic to $U \oplus U \oplus U \oplus E_8(2)$ which is generated by $\{\alpha_i, \beta_i, e_j + f_j | i = 1, 2, 3, j = 1, \ldots, 8\}$. How we consider $L^{(\sigma)}_X$ as a Picard lattices $S_X$. Then we can easily see that $g$ preserves the Kähler cone of $X$ and a period of $X$. Hence by the global Torelli theorem [11], $g$ is represented by a symplectic automorphism. Thus we have $G_x \simeq \mathbb{Z}/2$.

**References**


Department of Mathematical Sciences
Tokyo Denki University
Hatoyama-machi, Hiki-gun,
Saitama-ken, 350-03 Japan