M. Nisio
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STOCHASTIC DIFFERENTIAL GAMES AND VISCOSITY SOLUTIONS OF ISAACS EQUATIONS

MAKIKO NISIO

§ 1. Introduction

Recently P.L. Lions has demonstrated the connection between the value function of stochastic optimal control and a viscosity solution of Hamilton-Jacobi-Bellman equation [cf. 10, 11, 12]. The purpose of this paper is to extend partially his results to stochastic differential games, where two players conflict each other. If the value function of stochatic differential game is smooth enough, then it satisfies a second order partial differential equation with max-min or min-max type nonlinearity, called Isaacs equation [cf. 5]. Since we can write a nonlinear function as min-max of appropriate affine functions, under some mild conditions, the stochastic differential game theory provides some convenient representation formulas for solutions of nonlinear partial differential equations [cf. 1, 2, 3].

Now we will consider stochastic differential games on a finite interval [0, T], for simplicity. Let Γ_i be a compact and convex subset of R^{k_i}. $B(t)$, $t \geq 0$, denotes a standard d-dimensional Brownian motion, defined on a probability space (Ω, F, P). A B-adapted process U is called a control of player i, if $U_i(t) \in \Gamma_i$. We denote the totality of controls of player i by A_i, equipped with $L_2([0, T] \times \Omega)$ − topology.

For $U_i \in A_i$, $i = 1, 2$, the system X is evoluted by the following controlled stochastic differential equation (CSDE in short),

$$(1.1)\qquad \begin{cases} dX(t) = \alpha(X(t), U_1(t), U_2(t))dB(t) + \Upsilon(X(t), U_1(t), U_2(t))dt \\ X(0) = \chi \end{cases}$$

where α and Υ are symmetric matrix and vector valued functions, defined on $R^d \times \Gamma_1 \times \Gamma_2$, respectively. We assume some regularity, see (A1) and (A2). The solution X of (1, 1) is denoted by $X(t, \chi, U_1, U_2)$, if χ, U_1 and U_2

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are stressed. Let $c\,(\geq 0)$ and f be real valued functions on $R^d \times \Gamma_1 \times \Gamma_2$, with conditions (A1) and (A2). Putting

$$
\begin{aligned}
(1.2)\quad & \Phi(s, t, \chi, U_1, U_2, \phi; f) \\
&= \int_s^t f(X(\theta), U_1(\theta), U_2(\theta)) \exp\Big(-\int_s^\theta c(X(z), U_1(z), U_2(z))\,dz\Big) d\theta \\
&\quad + \phi(X(t)) \exp\Big(-\int_s^t c(X(z), U_1(z), U_2(z))dz\Big)
\end{aligned}
$$

where $X(t) = X(t, \chi, U_1, U_2)$, we define the pay-off function J as follows

$$
(1.3)\qquad J(t, \chi, U_1, U_2, \phi; f) = E\Phi(0, t, \chi, U_1, U_2, \phi; f)\,.
$$

Since we fix f in this paper, we can drop f in Φ and J without any confusion, except Section 4. Here player 1 wants to maximize the pay-off by a suitable control of A_1 and player 2 wants to minimize it by a suitable control of A_2. Moreover both players act step-by-step with a small size of step.

Now we introduce the upper and lower values of game in the following way. Put $I(N, j) = [j2^{-N}, (j+1)2^{-N}]$ and denote by $A_i(N, j)$ the totality of Γ_i-valued B-adapted processes on the time interval $I(N, j)$. Then A_i can be identified with $A_i(N, 0) \times A_i(N, 1) \times \cdots$ in the usual way. Put

$$
(1.4)\qquad B_i(N, j) = \{U \in A_i(N, j);\ U(\theta) = U(j2^{-N}) \quad \text{on } I(N, j)\}
$$

Setting $U_i = (U_i^0, U_i^1, \cdots U_i^l)$, where $U_i^k \in A_i(N, k)$, we define the upper valve V^+ and the lower valve V^- as follows, for $l = [2^N t]$,

$$
(1.5)\quad V^-(t, \chi, \phi) = \lim_{N\to\infty} \sup_{B_1(N0)} \inf_{A_2(N0)} \cdots \sup_{B_1(Nl)} \inf_{A_2(Nl)} J(t, \chi, U_1^0 \cdots U_1^l, U_2^0 \cdots U_2^l, \phi)
$$

$$
(1.6)\quad V^+(t, \chi, \phi) = \lim_{N\to\infty} \inf_{B_2(N0)} \sup_{A_1(N0)} \cdots \inf_{B_2(Nl)} \sup_{A_1(Nl)} J(t, \chi, U_1^0 \cdots U_1^l, U_2^0 \cdots U_2^l, \phi)
$$

In this paper we always assume the following two conditions

(A1) $|h(\chi, u_1, u_2) - h(y, v_1, v_2)|^2 \leq K|\chi - y|^2 + m(\sum_{i=1}^2 |u_i - v_i|)$

where K is a positive constant and m is increasing and bounded continuous on $[0, \infty)$ with $m(0) = 0$.

(A2) $\|h\| = \sup_{\chi uv} |h(\chi, u, v)| \leq b$ for $h = \alpha, \Upsilon, c, f$.

For simplicity, we denote by BUC(R^d) a Banach lattice of bounded and uniformly continuous functions on R^d, with supremum norm $\|\ \|$.

THEOREM 1. *For $\phi \in$ BUC(R^d), $V^+(t, \chi, \phi)$ and $V^-(t, \chi, \phi)$ exist. Moreover $V^+(\cdot, \phi)$ and $V^-(\cdot, \phi)$ belong to* BUC$([0, T] \times R^d)$

We define two transformations $V^+(t)$ and $V^-(t)$ on BUC(R^d) by

$$V^+(t)\phi = V^+(t, \cdot, \phi) \quad \text{and} \quad V^-(t)\phi = V(t, \cdot, \phi) \tag{1.7}$$

respectively. Setting $C_b^2 = C_b^2(R^d) = \{\phi \in \mathrm{BUC}(R^d);\ \partial_i\phi,\ \partial_i\partial_j\phi \in \mathrm{BUC}(R^d),\ i, j = 1, \cdots, d\}$, where $\partial_i = \partial/\partial x_i$, we see

THEOREM 2. *$V^+(t)$ and $V^-(t)$ provide nonlinear semigroups on* BUC(R^d) *with the following properties*

(i) *monotone*; $V^+(t)\phi \leq V^+(t)\phi$, $V^-(t)\phi \leq V^-(t)\psi$, *whenever* $\phi \leq \psi$

(ii) *contraction*; $\|V^+(t)\phi - V^+(t)\psi\| \leq \|\phi - \psi\|$,
$\|V^-(t)\phi - V^-(t)\psi\| \leq \|\phi - \psi\|$

(iii) *weak generator*; *Let G^+ and G^- be weak generators of $V^+(t)$ and $V^-(t)$ respectively. Then, under the min-max condition,*

$$\mathscr{D}(G^+) \cap \mathscr{D}(G^-) \supset C_b^2$$

and, for $\phi \in C_b^2$,

$$\begin{aligned} G^+\phi(\chi) = G^-\phi(\chi) &= \sup_{u\in\Gamma_1} \inf_{v\in\Gamma_2} (A(u, v)\phi(x) + f(\chi, u, v)) \\ &= \inf_{v\in\Gamma_2} \sup_{u\in\Gamma_1} (A(u, v)\phi(\chi) + f(\chi, u, v)) \end{aligned} \tag{1.8}$$

where

$$A(u, v) = \sum a_{ij}(\chi, u, v)\partial_i\partial_j + \sum \Upsilon_i(\chi, u, v)\partial_i - c(\chi, u, v) \tag{1.9}$$

with $a = \frac{1}{2}\alpha^2$.

For any $u \in \Gamma_1$ and $v \in \Gamma_2$, two operators $I(t, u)$ and $S(t, v)$ are defined by

$$I(t, u)\phi(\chi) = \inf_{U\in A_2} J(t, \chi, u, U, \phi), \qquad 0 \leq t \leq T \tag{1.10}$$

and

$$S(t, v)\phi(\chi) = \sup_{U\in A_1} J(t, \chi, U, v, \phi), \qquad 0 \leq t \leq T,$$

respectively. These operators turn out semigroups on BUC(R^d), related to stochastic optimal controls [cf. 14, 15]. The following theorem gives connection between "V^- and I" and "V^+ and S"

THEOREM 3. (i) *$V^-(t)$ is the upper envelope of $\{I(t, u),\ u \in \Gamma_1\}$ i.e.*

$$V^-(t)\phi \geq I(t, u)\phi, \quad \text{for any } \phi,\ t \text{ and } u. \tag{1.11}$$

If a semigroup $W(t)$ on BUC(R^d) *satisfies* (1.11), *then*

(1.12) $$W(t)\phi \geq V^-(t)\phi, \qquad \textit{for any } \phi \textit{ and } t\,.$$

(ii) *$V^+(t)$ is the lower envelope of $(S(t, v),\ v \in \Gamma_2\}$, i.e.*

(1.13) $$V^+(t)\phi \leq S(t, v)\phi, \qquad \textit{for any } \phi,\ t \textit{ and } v\,.$$

If a semigroup $W(t)$ on BUC(R^d) *satisfies* (1.13), *then*

(1.14) $$W(t)\phi \leq V^+(t)\phi, \qquad \textit{for any } \phi \textit{ and } t\,.$$

We will prove Theorems $1\sim 3$ in Section 3. In Section 4 we consider the connection between the upper and lower values and viscosity solutions of Isaacs equation, namely we will prove the following two theorems, under the min-max condition.

THEOREM 4. *$V^+(t, \chi, \phi)$ and $V^-(t, \chi, \phi)$ are viscosity solutions of Cauchy problem of Isaacs equations,*

(1.15) $$\begin{cases} \partial_t V + F(\partial^2 V, \partial V, V, \chi) = 0, & \text{in } (0, T) \times R^d \\ V(0) = \phi \end{cases}$$

where $\partial_t = \partial/\partial t$, $\partial = (\partial_1, \cdots, \partial_d)$ and

(1.16) $$F(\xi, p, w, \chi)$$
$$= \inf_{v\in\Gamma_2} \sup_{u\in\Gamma_1} \left(-\sum a_{ij}(\chi, u, v)\xi_{ij} - \sum \Gamma_i(\chi, u, v)p_i + c(\chi, u, v)w - f(\chi, u, v)\right)$$
$$= \sup_{u\in\Gamma_1} \inf_{v\in\Gamma_2} \left(-\sum a_{ij}(\chi, u, v)\xi_{ij} - \sum \Gamma_i(\chi, u, v)p_i + c(\chi, u, v)w - f(\chi, u, v)\right.$$

THEOREM 5. *Assume* (A3)$\sim$(A5), *besides* (A1) *and* (A2),

(A3) $\bar{c} = \inf_{\chi u v} c(\chi, u, v) > 0$

(A4) $\sup_{uv}|f(\chi, u, v)| \to 0$ *and* $|\phi(\chi)| \to 0$, *as* $|\chi| \to \infty$.

(A5) $h(\cdot, u, v) \in C_b^2$ *and* $|||h||| = \sup_{uv} \|h(\cdot, u, v)\|_{C_2} < \infty$, *for* $h = \alpha_{ij}, \Gamma_i, c, f$ *where* $\|\psi\|_{C^2} = \|\psi\| + \max_i \|\partial_i\psi\| + \max_{ij} \|\partial_i\partial_j\psi\|$.
Then

(1.17) $$\sup_{t\leqslant T} |V^+(t, \chi, \phi)|, \quad \sup_{t\leqslant T} |V^-(t, \chi, \phi)| \to 0, \quad \textit{as } |\chi| \to \infty.$$

Moreover V^+ and V^- are the maximum subsolution and minimum supersolution of (1.15) *respectively in the set*

$$C_0 = \{W \in C([0, T] \times R^d);\ \sup_{t\leqslant T} |W(t, \chi)| \to 0, \quad \textit{as } |\chi| \to \infty\}\,.$$

Therefore V^+ and V^- are extremum viscosity solutions in C_0. In Section 5, we will deal with the so-called Verification Theorem in Isaacs equation.

§ 2. Preliminaries

First we summarize some propositions on CSDE, which we need in later sections. Put $X(t) = X(t, \chi, U_1, U_2)$ and $\bar{X}(t) = X(t, \bar{\chi}, \bar{U}_1, \bar{U}_2)$. Then by the routine method, we have the following evaluation.

PROPOSITION 1.

$$\begin{aligned}(2.1)\qquad E|X(t) - \bar{X}(t)|^2 &\leq |\chi - \bar{\chi}|^2 e^{(2b+1)t} \\ &+ 2\int_0^t Em\left(\sum_{i=1}^2 |U_i(s) - \bar{U}_i(s)|\right)[\exp(2b+1)(t-s)]ds\,.\end{aligned}$$

$$(2.2)\qquad E|X(t) - X(s)|^2 \leq 2\|\alpha\|^2|t-s| + 2\|\Upsilon\|^2|t-s|^2$$

PROPOSITION 2. *For any $\phi \in \mathrm{BUC}(R^d)$, $J(\cdot, \phi)$ is a uniformly continuous function on $[0, T] \times R^d \times A_1 \times A_2$.*

Proof. For $\varepsilon > 0$, there exists a positive δ such that $m(\delta) < \varepsilon$. Put $\chi_\Lambda =$ indicator of Λ and $\Lambda = \{(t, \omega) \in [0, T] \times \Omega;\ \sum_{i=1}^2 |U_i(t, \omega) - \bar{U}_i(t, \omega)| > \delta\}$. Then we have

$$\begin{aligned}(2.3)\qquad E\int_0^t m\left(\sum_{i=1}^2 |U_i(t) - \bar{U}_i(t)|\right)dt &\leq \varepsilon T + E\int_0^t m\left(\sum_{i=1}^2 |U_i(t) - \bar{U}_i(t)|\right)\chi_\Lambda dt \\ &\leq \varepsilon T + 2m(\infty)\frac{\sum_{i=1}^2 \|U_i - \bar{U}_i\|^2}{\delta^2}\,.\end{aligned}$$

Hence we can complete the proof, appealing to Proposition 1.

Put $\alpha_\varepsilon = \alpha + \varepsilon I$, where I is a $d \times d$ unit matrix. Replacing α by α_ε, we denote a solution of (1.1) by X_ε. Again by the routine method, we get

PROPOSITION 3. *There is a positive constant K_1, such that*

$$(2.4)\qquad E|X_\varepsilon(t, \chi, U_1, U_2) - X(t, \chi, U_1, U_2)|^2 \leq K_1\varepsilon \qquad \textit{for any } t, \chi, U_1 \textit{ and } U_2.$$

PROPOSITION 4. *Under the condition* (A5), *$X(t, \chi) = X(t, \chi, U_1, U_2)$ has the first and second $L^2(\Omega)$ derivatives w, r, to χ, i.e. setting $e_j =$ unit vector for j th coordinate, $Y_{ij}(t, \chi) = \mathrm{l.i.m.}_{\varepsilon\to 0}(1/\varepsilon)(X_i(t, \chi + \varepsilon e_j) - X_i(t, \chi))$ exists and satisfies CSDE,*

$$\begin{cases} dY_{ij}(t) = \sum_{kp} (\partial_k \alpha_{ip})(X(t), U_1, U_2)Y_{kj}(t)dB_p(t) + \sum (\partial_k \Upsilon_i)(X(t), U_1, U_2)Y_{kj}(t)dt \\ Y_{ij}(0) = \delta_{ij}\ (=\textit{Kronecker delta}) \end{cases}$$

$$Z_{ijl}(t, \chi) = \underset{\varepsilon\to 0}{\mathrm{l.i.m.}}\frac{1}{\varepsilon}(Y_{ij}(t, \chi + \varepsilon e_l) - Y_{ij}(t, \chi))$$

exists and satisfies CSDE

$$\begin{cases} dZ_{ijl}(t) = \sum_{kp} (\partial_k \alpha_{ip})(X(t), U_1, U_2) Z_{kjl}(t) dB_p(t) \\ \qquad + \sum_{k} (\partial_k \Upsilon_i)(X(t), U_1, U_2) Z_{kjl}(t) dt \\ \qquad + \sum_{mkp} (\partial_m \partial_k \alpha_{ip})(X(t), U_1, U_2) Y_{kj}(t) Y_{ml}(t) dB_p(t) \\ \qquad + \sum_{mk} (\partial_m \partial_k \Upsilon_i)(X(t), U_1, U_2) Y_{kj}(t) Y_{ml}(t) dt \\ Z_{ijl}(0) = 0\,. \end{cases}$$

Moreover $J(t, \cdot, U_1, U_2, \phi) \in C_b^2$ and

$$\|\partial_i \partial_j J(t, \cdot, U_1, U_2, \phi)\| \leq K_2(1 + \|\phi\|_{C^2}) \tag{2.5}$$

where K_2 depends only on T and $|||h|||$, $h = \alpha, \Upsilon, c, f$.

As [8, Chap. 2, 6], we can prove solvability of CSDE, i.e.

Proposition 5. *Assume* (A6), *besides* (A1) *and* (A2).

(A6) *There exists a constant* $\mu > 0$, *such that* $\sum_{ij} \alpha_{ij} (\chi, u, v) y_i y_j \geq \mu |y|^2$, *for any* χ, u, v.

Let u; $[0, T] \times R^d \to \Gamma_1$ *be Borel measurable. Then, for any* $U \in A_2$, *the following CSDE.*

$$\begin{cases} d\xi(t) = \alpha(\xi(t), u(t, \xi(t)), U(t)) dB(t) + \Upsilon(\xi(t), u(t, \xi(t)), U(t)) dt \\ \xi(0) = \chi \end{cases} \tag{2.6}$$

has a weak solution.

By a weak solution, we mean $(\bar{\xi}, \bar{U}, \bar{B})$ on a suitable probability space $(\bar{\Omega}, \bar{F}, \bar{P})$, such that

(i) $(\bar{U}, \bar{B})$ has the same law as (U, B)

(ii) $(\bar{\xi}, \bar{U}, \bar{B})$ satisfies (2.6).

From the condition (i), $\bar{B}$ is a d-dimensional Brownian motion and $\bar{U}$ is $\bar{B}$-adapted.

Proof. Put S_n = sphere with center 0 and radius 2^n. Since Γ_1 is compact and convex, we can choose an approximate smooth function u_k; $[0, T] \times R^d \to \Gamma_1$, such that u_k tends to u a.e., and

$$\lim_{k \to \infty} \|u_k - u\|_{L^{d+1}([0, T] \times S_k)} = 0 \tag{2.7}$$

By virtue of smoothness of u_k, the following CSDE

$$\begin{cases} d\xi = \alpha(\xi(t), u_k(t, \xi(t)), U(t)) dB(t) + \Upsilon(\xi(t), u_k(t, \xi(t)), U(t)) dt \\ \xi(0) = \chi \end{cases} \tag{2.8}$$

has a unique B-adapted solution. ξ_k. Moreover, $\{\xi_k, k = 1, 2, \cdots\}$ is totally bounded in Prohorov topology. So $\{(\xi_k, U, B), k = 1, 2, \cdots\}$ is also totally bounded. Hence there exist $(\bar{\xi}_k, \bar{U}_k, \bar{B}_k)$ and $(\bar{\xi}, \bar{U}, \bar{B})$ on a suitable probability space $(\bar{\Omega}, \bar{F}, \bar{P})$, such that

(i) $(\bar{\xi}_k, \bar{U}_k, \bar{B}_k)$ has the the same law as (ξ_k, U, B).

(ii) As $k \to \infty$, $\bar{\xi}_k \to \bar{\xi}$ and $\bar{B}_k \to \bar{B}$ in $C[0, T]$ and $\bar{U}_k \to \bar{U}$ in $L^2[0, T]$, with probability 1.

For simplicity, putting $\alpha_k(s, v) = \alpha(\bar{\xi}_k(s), v(s, \bar{\xi}_k(s)), \bar{U}_k(s))$ and $\alpha(s, v) = \alpha_\infty(s, v) = \alpha(\bar{\xi}(s), v(s, \bar{\xi}(s), \bar{U}(s))$ etc, we see, from (2.8)

$$d\bar{\xi}_k = \alpha_k(t, u_k)d\bar{B}_k + \gamma_k(t, u_k)dt\,. \tag{2.9}$$

For $\varepsilon > 0$, we can choose $D = S_n$, such that

$$\bar{P}(\bar{\xi}_k(t) \in D, \text{ for any } t \leq T) > 1 - \varepsilon,\ k = 1. 2, \cdots \infty\,. \tag{2.10}$$

Now Krylov's inequality derives that there is a positive K, such that

$$E\int_0^T |h(s, \bar{\xi}_k(s))|ds \leq K_3\|h\|_{L^{d+1}([0,T]\times D)} + \varepsilon\|h\|T \tag{2.11}$$

for any bounded continuous function h and $k = 1, 2, \cdots, \infty$. Hence (2.11) holds for any bounded Borel function h.

Next we evaluate

$$\begin{aligned}
&\int_0^t \alpha_k(s, u_k)d\bar{B}_k(s) - \int_0^t \alpha(s, u)d\bar{B}(s) \\
&= \int_0^t (\alpha_k(s, u_k) - \alpha_k(s, u_p))d\bar{B}(s) \\
&\quad + \int_0^t \alpha_k(s, u_p)d\bar{B}_k(s) - \int_0^t \alpha(s, u_p)d\bar{B}(s) \\
&\quad + \int_0^t (\alpha(s, u_p) - \alpha(s, u))d\bar{B}(s) \\
&= I_1 + I_2 + I_3\,.
\end{aligned} \tag{2.12}$$

For $\varepsilon > 0$, we take a positive δ such that $m(\delta) < \varepsilon$, and choose $D = S_m$ such that

$$P(\xi_k(t) \in D, \quad \text{for any } t \leq T) > 1 - \delta\varepsilon,\ k = 1, 2, \cdots. \tag{2.13}$$

Moreover there exists a large k_0 such that

$$\|u_k - u_p\|_{L^{d+1}([0,T]\times D)} < \varepsilon\delta \qquad \text{for } k, p \geq k_0 \tag{2.14}$$

From (2.11) (2.13) and (2.14); we can see

$$(2.15) \qquad E|I_1|^2 = E\int_0^\tau |\alpha_k(s, u) - \alpha_k(s, u_p)|^2 ds$$
$$\leq E\int_0^t m(|u_k(s, \xi_k(s)) - u_p(s, \xi_k(s))|)ds$$
$$\leq \varepsilon T + \frac{m(\infty)}{\delta} E\int_0^T |u_k(s, \xi_k(s)) - u_p(s, \xi_k(s))| ds$$
$$\leq \varepsilon[T + m(\infty)(K_3 + (\operatorname{diam} \Gamma_1)T)] .$$

I_3 has the same evaluation as (2.15). Now we deal with I_2. For $\Delta > 0$, we put $\theta(s) = k\Delta$ on $[k\Delta, (k+1)\Delta)$.

$$(2.16) \qquad E\int_0^t |\alpha_k(s, u_p) - \alpha_k(\theta(s), u_p)|^2 ds \leq KE\int_0^t |\bar{\xi}_k(s) - \bar{\xi}_k(\theta(s))|^2 ds$$
$$+ E\int_0^t m(|u_p(s, \bar{\xi}_k(s)) - u_p(\theta(s), \bar{\xi}_k(\theta(s))| + |\bar{U}_k(s) - \bar{U}_k(\theta(s))|)ds .$$

$$(2.17) \qquad E\int_0^t |u_p(s, \bar{\xi}_k(s)) - u_p(\theta(s), \bar{\xi}_k(\theta(s))| ds$$
$$\leq \|\partial_t u_p\| \Delta T + \|\partial u_p\| E\int_0^t |\bar{\xi}_k(s) - \bar{\xi}_k(\theta(s))| ds$$

$$(2.18) \qquad E\int_0^t |\bar{U}_k(s) - \bar{U}_k(\theta(s))|^2 ds = E\int_0^t |U(s) - U(\theta(s))|^2 ds .$$

Combining (2.17) and (2.18) with (2.16), we have

$$(2.19) \qquad \sup_{k=1,2,\cdots,\infty} E\int_0^t |\alpha_k(s, u_p) - \alpha_k(\theta(s), u_p)|^2 \, ds \to 0, \text{ as } \Delta \to 0 .$$

Since, as $k \to \infty$,

$$(2.20) \qquad \int_0^t \alpha_k(\theta(s), u_p) d\bar{B}_k(s) \to \int_0^t \alpha(\theta(s), u_p) d\bar{B}(s)$$

with probability 1, $(\bar{\xi}, \bar{U}, \bar{B})$ satisfies (2.6). This completes the proof.

§ 3. Proof of Theorems 1~3

We prove these theorems for V^-, because we can apply the same method to V^+.

Proof of Theorem 1. Put $\Delta = 2^{-N}$ and

$$(3.1) \qquad V_N(\chi, \phi) = \sup_{u \in \Gamma_1} \inf_{U \in A_2} J(\Delta, \chi, u, U, \phi)$$
$$= \sup_{u \in \Gamma_1} I(\Delta, \chi, u, \phi) .$$

Then $V_N(\cdot, \phi)\phi \in \mathrm{BUC}(R^d)$ by Proposition 2, and we can define V_N; $\mathrm{BUC}(R^d) \to \mathrm{BUC}(R^d)$ by

$$V_N\phi = V_N(\cdot, \phi) \tag{3.2}$$

Moreover,

$$M(\chi) = \{u \in \Gamma_1;\ V_N(\chi, \phi) = I(\Delta, \chi, u, \phi)\} \tag{3.3}$$

is non-empty and compact. Suppose that χ_n tends to χ and $u_n \in M(\chi_n)$ tends to u. Then $u \in M(\chi)$ by the continuity of V_N and I. Therefore a Borel selector $\bar{u}(\cdot) = \bar{u}(\cdot\,; \Delta, \phi)$ of $M(\chi)$ exists [17], *i.e.* $\bar{u}$ is a Borel function on R^d and $\bar{u}(\chi) \in M(\chi)$.

LEMMA 1. *For* $U_1^j \in \boldsymbol{B}_1(N, j)$ *and* $U_2^j \in A_2(N, j)$, $j = 0, \cdots, k-1$, *we have*

$$\begin{aligned} &\sup_{U_1 \in \boldsymbol{B}_1(N,k)} \inf_{U_2 \in A_2(N,k)} J((k+1)\Delta, \chi, U_1^0 \cdots U_1^{k-1}U_1, U_2^0 \cdots U_2^{k-1}U_2, \phi) \\ &\qquad = J(k\Delta, \chi, U_1^0 \cdots U_1^{k-1}, U_2^0 \cdots U_2^{k-1}, V_N\phi) \end{aligned} \tag{3.4}$$

Proof. Put $W_i = (U_i^0 \cdots U_i^{k-1})$ and $\sigma_t = \sigma_t(B)$. Since $U_1 \in \boldsymbol{B}_1(N, k)$ is $\sigma_{k\Delta}$-measurable, we see

$$\begin{aligned} &E(\Phi(k\Delta, (k+1)\Delta, \chi, W_1U_1, W_2U_2, \phi)/\sigma_{k\Delta}) \\ &\qquad \geq I(\Delta, X(k\Delta, \chi, W_1, W_2), U_1, \phi)\,. \end{aligned} \tag{3.5}$$

Therefore we have

$$J((k+1)\Delta, \chi, W_1U_1, W_2U_2, \phi) \geq J(k\Delta, \chi, W_1, W_2, I(\Delta, \cdot, U_1, \phi))\,.$$

Taking the infinimum with respect to $U_2 \in A_2(N, k)$, we get

$$\inf_{U_2 \in A_2(N,k)} J((k+1)\Delta, \chi, W_1U_1, W_2U_2, \phi) \geq J(k\Delta, \chi, W_1, W_2, I(\Delta, \cdot, U_1, \phi))\,. \tag{3.6}$$

Since the right hand side of (3.5) turns out $V_N\phi(X(k\Delta, \chi, W_1, W_2)$ at $U_1 = \bar{u}(X(k\Delta, \chi, W_1, W_2))$ by (3.3), we see

(3.7) left hand side of (3.4) $\geq$ right hand side of (3.4).

For the converse inequality, we will choose a nearly optimal control in the following way. Using Proposition 2, we can take, for $\varepsilon > 0$, a positive $\delta = \delta(\varepsilon, \phi)$ such that

$$\sup_{U \in A_2} |J(\Delta, \chi, u, U, \phi) - J(\Delta, \chi', u', U, \phi)| < \frac{\varepsilon}{3}\,, \tag{3.8}$$

whenever $|\chi - \chi'| < \delta$ and $|u - u'| < \delta$. Let $\{D_n,\ n = 1, 2, \cdots\}$ be a partition of $R^d \times \Gamma_1$ with diam $(D_n) < \delta$. For any fixed $(\chi_n, u_n) \in D_n$, we can choose $U_n^* \in A_2(N, 0)$ such that

$$J(\varDelta, \chi_n, u_n, U_n^*, \phi) - \frac{\varepsilon}{3} \leq I(\varDelta, \chi, u_n, \phi)\,. \tag{3.9}$$

Since U_n^* is B-adapted, there is a Borel function v_n; $[0, \varDelta] \times C([0, \varDelta] \to R^d) \to \Gamma_2$, which is progressively measurable and $U_n^*(t) = v_n(t, B)$.

Now define $\bar{v} = \bar{v}(\cdot, \varDelta, \phi)$ by

$$\bar{v}(t, B, \chi, u) = \sum_n v_n(t, B)\chi_{D_n}(\chi, u)\,.$$

Then $\bar{v} \in A_2(N, 0)$ and we see, from (3.8) and (3.9),

$$J(\varDelta, \chi, u, \bar{v}, \phi) - \varepsilon \leq I(\varDelta, \chi, u, \phi)\,. \tag{3.10}$$

For $U_1 \in \boldsymbol{B}_1(N, k)$, put

$$\bar{U}_2(t) = \bar{v}(t - k\varDelta, B_{k\varDelta}^+,\ X(k\varDelta, \chi, W_1, W_2), U_1)) \tag{3.11}$$

where $B_s^+ = B(\cdot + s) - B(s)$. Then $\bar{U}_2 \in A_2(N, k)$ and

$$\begin{aligned} E(\Phi(k\varDelta, (k+1)\varDelta, \chi, W_1U_1, W_2\bar{U}_2, \phi/\sigma_{k\varDelta}) &\leq I(\varDelta, X(k\varDelta, \chi, W_1, W_2),\ U_1, \phi) + \varepsilon \\ &\leq V_N\phi(X(k\varDelta, \chi, W_1, W_2) + \varepsilon \end{aligned} \tag{3.12}$$

This yields

$$\inf_{U_2 \in A_2(N,k)} J((k+1)\varDelta, \chi,\ W_1U_1,\ W_2U_2,\ \phi) \leq J(k\varDelta, \chi, W_1, W_2, V_N\phi) + \varepsilon\,. \tag{3.13}$$

Taking the supremum with respect to $U_1 \in \boldsymbol{B}_1(N, k)$, we obtain the required inequality and complete the proof.

Repeating the same argument, we have

$$\sup_{U^0 \in \boldsymbol{B}_1(NO)} \inf_{W^0 \in A_2(NO)} \cdots \sup_{U^k \in \boldsymbol{B}_1(NK)} \inf_{W^k \in A_2(NK)} J((k+1)\varDelta, \chi, U^0 \cdots U^k, W^0 \cdots W^k, \phi) = V_N^{k+1}\phi(\chi)\,. \tag{3.14}$$

LEMMA 2. *V_N has the following properties*

(i) *monotone*; $V_N\phi \leq V_N\psi$, *whenever* $\phi \leq \psi$

(ii) *contraction*; $\|V_N\phi - V_N\psi\| \leq \|\phi - \psi\|$

(iii) $V_{N-1}\phi \leq V_N^2\phi$.

Proof. (i) is clear from the definition of J.

Using $|\sup \chi_\alpha - \sup y_\alpha| \leq \sup |\chi_\alpha - y_\alpha|$ and $|\inf \chi_\alpha - \inf y_\alpha| \leq \sup |\chi_\alpha - y_\alpha|$, (ii) is clear, by the following evaluation,

$$|J(t, \chi, U_1, U_2, \phi) - J(t, \chi, U_1, U_2, \psi)| \leq \|\phi - \psi\| . \tag{3.15}$$

From the semigroup property of $I(t, u)$, we have

$$I(2\Delta, u)\phi = I(\Delta, u)I(\Delta, u)\phi \leq I(\Delta, u)V_N\phi \tag{3.16}$$

Taking the supremum with respect to $u \in \Gamma_1$, we have

$$V_{N-1}\phi(\chi) \leq V_N V_N\phi(\chi) = V_N^2\phi(\chi) .$$

So we conclude (iii).

Define $V_N(t)\phi = V_N^k\phi$, for $t = 2^{-N}k$. Then Lemma 2 (iii) guarantees that, for any binary t, $V_N(t)\phi$ is increasing, as $N \to \infty$. Since Proposition 2 and (3.14) imply that $\{V_N(t)\phi, N \geq j\}$ is a totally bounded subset of $\mathrm{BUC}(R^d)$ for $t = 2^{-j}k$,

$$V(t, \chi, \phi) = \lim_{N\to\infty} V_N(t)\phi(\chi) \tag{3.17}$$

exists. Recalling the definition of V^-, we obtain

$$V(t, \chi, \phi) = V^-(t, \chi, \phi) \qquad \text{for binary } t.$$

Namely $V^-(t, \chi, \phi)$ exists for binary t. Hence appealing to Proposition 2 again, we can easily prove that $V^-(t, \chi, \phi)$ exists for any t. Now we complete the proof of Theorem 1.

Proof of Theorem 2. (i) and (ii) are clear. Since $V^-(t, \chi, \phi) \in \mathrm{BUC}([0, T] \times R^d)$, by Proposition 2, we have

$$\| V^-(t)\phi - V^-(s)\phi\| \longrightarrow 0, \qquad \text{as } t - s \to 0. \tag{3.18}$$

Putting $V = V^-$, we show semigroup property of V. For binary t and s,

$$V(t + s)\phi = \lim_{N\to\infty} V_N(t + s)\phi = \lim_{N\to\infty} V_N(t)V_N(s)\phi \tag{3.19}$$

Using the following calculation,

$$\begin{aligned} &\| V_N(t)V_N(s)\phi - V(t)V(s)\phi\| \\ &\leq \| V_N(t)V_N(s)\phi - V_N(t)V(s)\phi\| + \| V_N(t)V(s)\phi - V(t)V(s)\phi\| \\ &\leq \| V_N(s)\phi - V(s)\phi\| + \| V_N(t)V(s)\phi - V(t)V(s)\phi\| \end{aligned} \tag{3.20}$$

we see $V(t)V(s)\phi = \lim_{N\to\infty} V_N(t)V_N(s)\phi$. Recalling (3.19) we have

(3.21) $$V(t+s)\phi = V(t)V(s)\phi \qquad \text{for binary } t \text{ and } s.$$

Let t_n and s_n be approximate binary of t and s respectively. Then $V(s_n)\phi$ tends to $V(s)\phi$ by (3.18). Again using the similar argument as (3.20), we can show

(3.22) $$V(t+s)\phi = \lim_{n\leftarrow\infty} V(t_n)\phi V(s_n)\phi = V(t)V(s)\phi$$

Next we will calculate its weak generator. For $\phi \in C_b^2$, $(1/t)(I(t, u)\phi(X) - \phi(X))$ tends to $\inf_{\lambda\in\Gamma_2}(A(u, v)\phi(X) + f(X, u, v))$, as $t\to 0$, [cf. 15]. Therefore, for $u \in \Gamma_1$,

(3.23) $$\varliminf_{t\to 0} \frac{1}{t}(V(t)\phi(X) - \phi(X)) \geq \inf_{v\in\Gamma_2}(A(u, x)\phi(X) + f(X, u, v)).$$

This yield

(3.24) $$\varliminf_{t\to 0} \frac{1}{t}(V(t)\phi(X) - \phi(X)) \geq \sup_{u\in\Gamma_1}\inf_{v\in\Gamma_2}(A(u, v)\phi(X) + f(X, u, v)).$$

In the same way we get

(3.25) $$\varlimsup_{t\to 0} \frac{1}{t}(V^+(t)\phi(X) - \phi(X)) \leq \inf_{v\in\Gamma_2}\sup_{u\in\Gamma_1}(A(u, v)\phi(X) + f(X, u, v)).$$

Since Γ_i is convex and compact, the right hand sides of (3.24) and (3.25) coincide. Hence we get

(3.26) $$\varliminf_{t\to 0} \frac{1}{t}(V(t)\phi(X) - \phi(X)) \geq \varlimsup_{t\to 0} \frac{1}{t}(V^+(t)\phi(X) - \phi(X)).$$

(3.26) turns out, by "$V(t)\phi(X) \leq V^+(t)\phi(X)$",

(3.27) $$\lim_{t\to 0} \frac{1}{t}(V(t)\phi(X) - \phi(X)) = \lim_{t\to 0} \frac{1}{t}(V^+(t)\phi(X) - \phi(X)).$$

Thus $\phi \in \mathscr{D}(G^-) \cap \mathscr{D}(G^+)$ and (1.8) holds. This completes the proof of Theorem 2.

Proof of Theorem 3. (1.11) is clear. For $\Delta = 2^{-N}$, we see

$$W(\Delta)\phi \geq \sup I(\Delta, u)\phi = V_N(\Delta)\phi$$

Hence

$$W(2\Delta)\phi = W(\Delta)W(\Delta)\phi \geq V_N(\Delta)W(\Delta)\phi \geq V_N(\Delta)V_N(\Delta)\phi = V_N(2\Delta)\phi.$$

Repeating this calculation, we get

$$W(k\Delta)\phi \geq V_N(k\Delta)\phi\,.$$

This derives

$$W(t)\phi \geq V(t)\phi\,, \qquad \text{for binary } t\,.$$

Since both sides are continuous in t, (1.12) holds. Now we can complete the proof of Theorem 3.

§ 4. Proof of Theorems 4 and 5

First we recall the definition of viscosity solution of equation (1.15), according to [11]. Let $W \in C([0, T] \times R^d)$ satisfy $W(0) = \phi$. W is called a viscosity solution, if the following holds, for any $\psi \in C_b^2((0, T) \times R^d)$,

$$\partial_t\psi(t_0, x_0) + F(\partial^2\psi(t_0, x_0)\,,\ \partial\psi(t_0, x_0)\,,\ W(t_0, x_0),\ x_0) \geq 0\,, \tag{4.1}$$

at any local minimum point $(t_0, x_0) \in (0, T) \times R^d$ of $W - \psi$, and

$$\partial_t\psi(t_0, x_0) + F(\partial^2\psi(t_0, x_0),\ \partial\psi(t_0, x_0),\ W(t_0, x_0), x_0) \leq 0\,, \tag{4.2}$$

at any local maximum point $(t_0, x_0) \in (0, T) \times R^d$ of $W - \psi$.

Remark. Equivalent definition is obtained by replacing the above statement "local" by "global". If W satisfies (4.1) (or (4.2) respectively), then W is called a subsolution (or supersolution respectively).

We will apply the similar method as [13]. Put $V(t, x) = V^-(t)\phi(x)$. Let $(t_0, x_0) \in (0, T) \times R^d$ be a global maximum point of $V - \psi$. For the proof we may assume

$$V(t_0, x_0) = \psi(t_0, x_0)\,. \tag{4.3}$$

Hence

$$V \leq \psi\,. \tag{4.4}$$

Now the monotone property of V^- implies

$$\psi(t_0, x_0) = V(t_0, x_0) = V^-(h)V(t_0 - h,\ \cdot)(x_0) \leq V^-(h)\psi(t_0 - h,\ \cdot)(x_0)\,, \quad \text{for } h < t_0\,. \tag{4.5}$$

On the other hand, there is a positive function δ on $(0, t_0)$, such that

$$\psi(t_0 - h, x) < \psi(t_0, x) - h\partial_t\psi(t_0, x) + h\delta(h) \tag{4.6}$$

and $\delta(h)$ is decreasing to 0, as $h \to 0$. So, for $\varepsilon_0 > 0$, there exists h_0 such that

$$\delta(h) < \varepsilon_0 , \qquad \text{for } h < h_0 . \tag{4.7}$$

Combining (4.6) and (4.7) with (4.5), we have

$$\psi(t_0, \chi_0) \leq V^-(h)(\psi(t_0, \cdot) + h\Phi)(\chi_0) , \qquad \text{for } h < t_0 , \tag{4.8}$$

where

$$\Phi(\chi) = - \partial_t \psi(t_0, \chi) + \varepsilon_0 . \in C_b^2(R^d) \tag{4.9}$$

Hereafter stressing the dependence on f, we denote V_N or $V^-(t)$ by $V_N(\cdot\, ; f)$ or $V^-(t; f)$ respectively.

Lemma. *Putting* $\lambda = \sup_{\chi u v}|A(u, v)\Phi(\chi)|$, *we have*

$$\| V^-(h; f)(\psi(t_0, \cdot) + h\Phi) - V^-(h; f + \Phi)\psi(t_0, \cdot)\| \leq \lambda h^2 , \quad \textit{for } h < h_0 . \tag{4.10}$$

Proof. Using Ito's formula we see

$$\begin{aligned} (4.11) \qquad & E\Big[s\Phi(X(s)) \exp\Big(-\int_0^s c(X, U_1, U_2)d\theta\Big) \\ & \qquad -\int_0^s \Phi(X(t)) \exp\Big(-\int_0^t c(X, U_1, U_2)d\theta\Big)dt\Big] \\ & = E\int_0^s \Big[\Phi(X(s)) \exp\Big(-\int_0^s c(X, U_1, U_2)d\theta\Big) - \Phi(X(t)) \exp\Big(-\int_0^t c(X, U_1, U_2)d\theta\Big)\Big]dt \\ & = E\int_0^s \Big[\int_t^s A(U_1(z), U_2(z)\Phi(X(z)) \exp\Big(-\int_0^2 c(X, U_1, U_2)d\theta\Big)dz\Big]dt \\ & \leq \lambda s^2 \end{aligned}$$

Hence, for $s < h_0$,

$$\begin{aligned} (4.12) \qquad & |J(s, \chi, U_1, U_2, \psi(t_0, \cdot) + h\Phi; f) - J(s, \chi, U_1, U_2, \psi(t_0, \cdot); f + \Phi)| \\ & \leq |J(s, \chi, U_1, U_2, \psi(t_0, \cdot) + h\Phi; f) - J(s, \chi, U_1, U_2, \psi(t_0, \cdot) + s\Phi; f)| \\ & + |J(s, \chi, U_1, U_2, \psi(t_0, \cdot) + s\Phi; f) - J(s, \chi, U_1, U_2, \psi(t_0, \cdot); f + \Phi)| \\ & \leq (h - s)\|\Phi\| + \lambda s^2 . \end{aligned}$$

By the definition of V^-, this yields

$$| V^-(s, \chi, \psi(t_0, \cdot) + h\Phi; f) V^-(s, \chi, \psi(t_0, \cdot); f + \Phi)| \leq (h - s)\|\Phi\| + \lambda s^2 . \tag{4.13}$$

Now, setting $s = h$, we complete the proof of Lemma.

Since $\psi(t_0, \cdot) \in C_b^2$, we see, as $h \to 0$,

$$\frac{1}{h}(V^-(h; f+\Phi)\psi(t_0, \cdot)(\chi) - \psi(t_0, \chi)) \tag{4.14}$$
$$\longrightarrow \sup_{u \in \Gamma_1} \inf_{v \in \Gamma_2} A(u, v)\psi(t_0, \cdot)(\chi) + f(\chi, u, v) + \Phi(\chi)\,.$$

Combining (4.14) with (4.8) and (4.10), we obtain as $h \to 0$,

$$0 \leq \frac{1}{h}(V^-(h; f)(\psi(t_0, \cdot) + h\Phi)(\chi_0) - \psi(t_0, \chi_0))$$
$$\longrightarrow \sup_{u \in \Gamma_1} \inf_{v \in \Gamma_2} A(u, v)\psi(t_0, \cdot)(\chi_0) + f(\chi_0, u, v) + \Phi(\chi_0)\,.$$

Hence, from (4.9).

$$0 \leq -F(\partial^2\psi(t_0, \chi_0), \partial\psi(t_0, \chi_0), \psi(t_0, \chi_0), \chi_0) - \partial_t\psi(t_0, \chi_0) + \varepsilon_0\,. \tag{4.15}$$

Since ε_0 is arbitrary, (4.3) and (4.15) conclude that V is a subsolution.

In the same way we can prove that V is a supersolution. Hence V is a viscosity solution. Applying the same argument to V^+, we complete the proof of Theorem 4.

Proof of Theorem 5. For $\varepsilon > 0$, there is a large $l = l(\varepsilon)$ such that

$$P(\sup_{t \leq T} |X(t, \chi, U_1, U_2) - \chi| > l) < \varepsilon$$

for any χ, U_1 and U_2. Hence (A4) implies

$$\sup_{tU_1U_2} |J(t, \chi, U_1, U_2, \phi)| \longrightarrow 0\,, \qquad \text{as } |\chi| \to \infty\,. \tag{4.16}$$

This derives (1.17).

Let $W \in C_0$ be a supersolution of (1.15). For any fixed $u \in \Gamma_1$, we put $q(t, \chi) = q(t, \chi; \phi) = I(t, u)\phi(\chi)$.

LEMMA. $W(t, \chi) \geq q(t, \chi)$

Proof. By (A4), there exists an approximate smooth function ϕ_n, with compact support, such that

$$\|\phi - \phi_n\| < 2^{-n}\,. \tag{4.17}$$

Now we choose a small positive number $\varepsilon(n)$, such that

$$(1 + \|\phi_n\|_{C^2})\varepsilon(n) < 2^{-n}\,. \tag{4.18}$$

We put $\alpha_n = \alpha + \varepsilon(n)I$. Replacing α by α_n, we define Φ_n, J_n, A_n and $I_n(t, u)$ in the same way as Φ, J, A and $I(t, u)$ respectively. Setting

$$q_n(t, \chi; \phi) = I_n(t, u)\phi(\chi) ,$$

we can easily see, from Proposition 3,

$$\|q_n(\cdot\,; \phi) - q(\cdot\,; \phi)\| \longrightarrow 0, \qquad \text{as } n \to \infty . \tag{4.19}$$

Moreover $q_n(\cdot) = q_n(\cdot\,; \phi_n)$ turns out a classical solution of Bellman equation and $q_n \in C_b^{2+\delta}((0, T) \times R^d)$ with some $\delta > 0$, according to [9].

Suppose that $W - q$ has a negative value at $(t', \chi') \in (0, T) \times R^d$, say

$$W(t', \chi') - q(t', \chi') = -2h < 0 . \tag{4.20}$$

$$\begin{aligned} \|q - q_n\| &\leq \|q(\cdot\,; \phi) - q_n(\cdot\,; \phi)\| + \|q_n(\cdot\,; \phi) - q_n(\cdot\,; \phi_n)\| \\ &\leq \|q(\cdot\,; \phi) - q_n(\cdot\,; \phi)\| + \|\phi - \phi_n\| . \end{aligned} \tag{4.21}$$

Since the right hand side of (4.21) tends to 0, as $n \to \infty$, we can choose a large N such that "$2^{-N} < 2h$" and

$$W(t', \chi') - q_n(t', \chi') \leq -h , \qquad \text{for } n \geq N. \tag{4.22}$$

Since W and q_n vanish at $\chi = \infty$, there is a compact set, $[\delta, T] \times \Lambda \subset [0, T] \times R^d$, such that, by virtue of (4.21)

$$W - q_n > -\frac{h}{2} \text{ outside } [\delta, T] \times \Lambda, \qquad \text{for } n \geq N. \tag{4.23}$$

A global minimum point $(t_n, \chi_n)\,(\in [\delta, T] \times \Lambda)$ of $W - q_n$ exists and

$$W(t_n, \chi_n) - q_n(t_n, \chi_n) \leq -h , \qquad \text{for } n \geq N. \tag{4.24}$$

Moreover we have

$$\partial_t q_n(t_n, \chi_n) + F(\partial^2 q_n(t_n, \chi_n), \partial q_n(t_n, \chi_n), W(t_n, \chi_n), \chi_n) \geq 0 \tag{4.25}$$

because W is a supersolution and q_n is smooth. Thus

$$\begin{aligned} 0 \leq\; & \partial_t q_n(t_n, \chi_n) \\ & -\inf_{v \in \Gamma_2} [A(u, v)q_n(t_n, \chi_n) + c(\chi_n, u, v)(q_n(t_n, \chi_n) - W(t_n, \chi_n)) + f(\chi_n, u, v)] . \end{aligned} \tag{4.26}$$

On the other hand q_n satisfies Bellman equation,

$$0 = \partial_t q_n - \inf_{v \in \Gamma_2} (A_n(u, v)q_n + f(\chi, u, v)) . \tag{4.27}$$

Substracting (4.27) from (4.26), we obtain

$$(4.28)\quad 0 \leq \inf_{v \in \Gamma_2} (A_n(u, v)q_n(t_n, \chi_n) + f(\chi_n, u, v))$$
$$- \inf_{v \in \Gamma_2} (A(u, v)q_n(t_n, \chi_n) + c(\chi_n, u, v)(q_n(t_n, \chi_n) - W(t_n, \chi_n)) + f(\chi_n, u, v))$$
$$\leq \sup_{v \in \Gamma_2} [A_n(u, v)q_n(t_n, \chi_n) - A(u, v)q_n(t_n, \chi_n) - c(\chi_n, u, v)(q_n(t_n, \chi_n) - W(t_n, \chi_n))]$$
$$\leq \sup_{v \in \Gamma_2} (A_n(u, v)q_n(t_n, \chi_n) - A(u, v)q_n(t_n, \chi_n)) - \bar{c}h$$

by (4.24). On the other hand

$$(4.29)\quad q_n(t, \chi + y) - 2q_n(t, \chi) + q_n(t, \chi - y)$$
$$\leq \inf_{U \in A_2} (J_n(t, \chi + y, u, U, \phi) + J_n(t, \chi - y, u, U, \phi)) - 2 \inf_{U \in A_2} J_n(t, \chi, u, U, \phi)$$
$$\leq \sup_{U \in A_2} (J_n(t, \chi + y, u, U, \phi) + J_n(t, \chi - y, u, U, \phi) - 2J_n(t, \chi, u, U, \phi))$$
$$\leq \sup_{U \in A_2} \|\partial^2 J_n(t, \cdot u, U, \phi)\| |y|^2 \leq \lambda_1(1 + \|\phi_n\|_{C^2})|y|^2 ,$$

where a constant λ_1 is independent of t, u and n, by (2.5). Appealing to "$\phi \in C_b^2$", (4.29) yields

$$(4.30)\quad \partial_i \partial_j q_n(t, \chi) \leq \lambda_1(1 + \|\phi_n\|_{C^2}) .$$

Thus we get, setting $\Delta =$ Laplacian

$$(4.31)\quad A_n(u, v)q_n(t, \chi) - A(u, v)q_n(t, \chi)$$
$$= 2\varepsilon(n) \sum \alpha_{ij}(\chi, u, x)\partial_i \partial_j q_n(t, \chi) + \varepsilon(n)^2 \Delta q_n(t, \chi)$$
$$\leq 2\varepsilon(n)(d^2\|\alpha\| + d\varepsilon(n))\lambda_1(1 + \|\phi_n\|_{C^2}) .$$

Combining (4.31) with (4.28), we get, with $\lambda_2 = 2d(d\|\alpha\| + 1)\lambda_1$

$$(4.32)\quad 0 < \lambda_2 \varepsilon(n)(1 + \|\phi_n\|_{C^2}) - \bar{c}h .$$

Recalling (4.18), (4.32) yields contradiction, as $n \to \infty$.

Now we will prove Theorem 5. Setting $\overline{W}(t, \chi) = W(t + s, \chi)$ for $t \leq T - s$, $\overline{W}$ turns out a supersolution of (1.15) with initial value $W(s)$. Hence Lemma derives

$$\overline{W}(t, \chi) \geq I(t, u)W(s)(\chi) , \qquad \text{for any } u \in \Gamma_1 .$$

Thus we get

$$W(t + s, \chi) \geq \sup_{u \in \Gamma_1} I(t, u)W(s)(\chi) .$$

So we have

$$W(2^{-N}, \chi) \geq V_N(2^{-N})\phi(\chi)$$

and

$$W(2^{-N+1}, \chi) \geq V_N(2^{-N})W(2^{-N})(\chi) \geq V_N(2^{-N})V_N(2^{-N})\phi(\chi) = V_N(2^{-N+1})\phi(\chi) .$$

Repeating this calculation, we have

$$W(t, \chi) \geq V_N(t)\phi(\chi) , \qquad \text{for } t = 2^{-N}k$$

Tending N to ∞, we see, for binary t

$$W(t, \chi) \geq V(t)\phi(\chi) . \tag{4.33}$$

Since both sides of (4.33) are continuous in t, (4.33) holds for any t. This means that V^- is a minimum supersolution.

For V^+ we can apply the same argument, using the inequality

$$\text{"}\sup \chi_\alpha - \sup y_\alpha + \sup z_\alpha \geq \inf (\chi_\alpha - y_\alpha + z_\alpha)\text{"}$$

instead of (4.29). Now we complete the proof of Theorem 5.

§ 5. Verification Theorem

In this section we prove the following Verification Theorem.

THEOREM 6. *Besides* (A1) *and* (A2), *we assume non-degeneracy.*

(A6) *there is* $\mu > 0$, *such that* $\alpha(x, u, v) \geq \mu I$, *for any* x, u, v. *Suppose that* $W \in W^{1,2}_\infty (= W^{1,2}_\infty((0, T) \times R^d))$ *is a solution of Cauchy problem of Isaacs equation* (1.15), *with* $W(0) = \phi$. *Then*

$$\begin{aligned} W(t, \chi) &= V^-(t, \chi, \phi) = V^+(t, \chi, \phi) \\ &= \sup_{U_1 \in A_1} \inf_{U_2 \in A_2} J(t, \chi, U_1, U_2, \phi) \\ &= \inf_{U_2 \in A_2} \sup_{U_1 \in A_1} J(t, \chi, U_1, U_2, \phi) . \end{aligned} \tag{5.1}$$

Proof. We fix bounded Borel measurable versions of $\partial_i W$ and $\partial_i \partial_j W$ arbitrarily and put

$$G(t, \chi, u, v) = A(u, v)W(t, \chi) + f(\chi, u, v) \tag{5.2}$$

and

$$\begin{aligned} M(t, \chi) = \{(\bar{u}, \bar{v}) \in \Gamma_1 \times \Gamma_2;\ &\text{for any } (u, v) \in \Gamma_1 \times \Gamma_2 , \\ &G(t, \chi, \bar{u}, v) \geq G(t, \chi, \bar{u}, \bar{v}) \geq G(t, \chi, u, \bar{v})\} . \end{aligned} \tag{5.3}$$

Since $G(t, \chi, \cdot)$ has a saddle point, $M(t, \chi)$ is a non-empty compact subset of $\Gamma_1 \times \Gamma_2$. Moreover its graph $= \{(t, \chi, \bar{u}, \bar{v});\ \inf_{v \in \Gamma_2} G(t, \chi, \bar{u}, v) = G(t, \chi, \bar{u}, \bar{v}) = \sup_{u \in \Gamma_1} G(t, \chi, u, \bar{v})\}$ is a Borel set. Therefore a Lebesgue measurable selector $(\bar{u}, \bar{v})$ of $M(t, \chi)$ exists. Thus we can choose a Borel function (u^*, v^*), such that

$$u^*(t, \chi) = \bar{u}(t, \chi) \text{ and } v^*(t, \chi) = \bar{v}(t, \chi) \text{ a.e.} \tag{5.4}$$

According to Proposition 5, following two CSDE have weak solutions;

$$\begin{cases} d\xi(t) = \alpha(\xi(t), u^*(t, \xi(t)), v^*(t, \xi(t)))dB(t) \\ \qquad\qquad + \Upsilon(\xi(t), u^*(t, \xi(t)), v^*(t, \xi(t))dt \\ \xi(0) = \chi \end{cases} \tag{5.5}$$

and for $U \in A_2$

$$\begin{cases} dX(t) = \alpha(X(t), u^*(t, X(t)), U(t))dB(t) + \Upsilon(X(t), u^*(t, X(t)), U(t))dt \\ X(0) = \chi . \end{cases} \tag{5.6}$$

Since we can apply Ito's formula to W, by (A6), we get

$$\begin{aligned} W(t, \chi) = E_x\int_0^t & f(\xi(s), u^*(s, \xi(s)), v^*(s, \xi(s))) \exp\left(-\int_0^s c(\xi, u^*, v^*)d\theta\right)ds \\ & + \phi(\xi(t)) \exp\left(-\int_0^t c(\xi(\theta), u^*(\theta, \xi(\theta)), v^*(\theta, \xi(\theta))d\theta\right) . \end{aligned} \tag{5.7}$$

Put $J(t, \chi, u^*(\cdot), v^*(\cdot), \phi) =$ the right hand side of (5.7). By (5.3) and (5.6),

$$W(t, \chi) \leq J(t, \chi, u^*(\cdot). U, \phi) , \qquad \text{for } U \in A_2 . \tag{5.8}$$

Hence we get

$$W(t, \chi) \leq \inf_{U \in A_2} J(t, \chi, u^*(\cdot), U, \phi) . \tag{5.9}$$

Let u_k be an approximate smooth function of u^*, such that $u_k(t, \chi) \in \Gamma_1$ and

$$\| u^* - u_k \|_{L^{d+1}_{((0,T)\times S_k)}} \leq 2^{-k} , \tag{5.10}$$

where $S_k =$ sphere with center 0 and radius 2^k. Again using Krylov's inequality, we have, as $k \to \infty$

$$\sup_{u \in A_2} | J(t, \chi, u^*(\cdot), U, \phi) - J(t, \chi, u_k(\cdot), U, \phi)| \longrightarrow 0 . \tag{5.11}$$

Replacing u^* by u_k, CSDE (5.6) has a unique strong solution X_k, which is B-adapted. So $u_k(t, X_k(t)) \in A_1$. This derives, by (5.9) and (5.11)

$$\begin{aligned} W(t, \chi) & \leq \lim_{k \to \infty} \inf_{U \in A_2} J(t, \chi, u_k(\cdot), U, \phi) \\ & \leq \sup_{U_1 \in A_1} \inf_{U_2 \in A_2} J(t, \chi, U_1, U_2, \phi) . \end{aligned} \tag{5.12}$$

Replaing u^* by v^*, we obtain

$$(5.13) \qquad W(t, \chi) \geq \inf_{U_2 \in A_2} \sup_{U_1 \in A_1} J(t, \chi, U_1, U_2, \phi).$$

By virtue of "$\sup_{U_1 \in A_1} \inf_{U_2 \in A_2} J(t, \chi, U_1, U_2, \phi) \leq \inf_{U_2 \in A_1} \sup_{U_1 \in A_1} J(t, \chi, U_1, U_2, \phi)$" (5.12) and (5.13) imply

$$(5.14) \qquad \begin{aligned} W(t, \chi) &= \inf_{U_2 \in A_2} \sup_{U_1 \in A_1} J(t, \chi, U_1, U_2, \phi) \\ &= \sup_{U_1 \in A_1} \inf_{U_2 \in A_2} J(t, \chi, U_1, U_2, \phi) . \end{aligned}$$

On the other hand Proposition 2 guarantees

$$(5.15) \qquad W(t, \chi) = \inf_{U_2 \in A_2} \sup_{U_1 \in A_1} J(t, \chi, U_1, U_2, \phi) \geq V^+(t, \chi, \phi)$$

where $\boldsymbol{B}_i = \cup_N \boldsymbol{B}_i(N)$, $\boldsymbol{B}_i(N) = \{U \in A_i,\ U(t) = U(2^{-N}[2^N t])\}$,

$$(5.16) \qquad W(t, \chi) = \sup_{U_1 \in \boldsymbol{B}_1} \inf_{U_2 \in A_2} J(t, \chi, U_1, U_2, \phi) \leq V^-(t, \chi, \phi) .$$

Since $V^+(t, \chi, \phi) \geq V^-(t, \chi, \phi)$ holds, we complete the proof.

Remark. By (5.7), a Borel modification (u^*, v^*) of selector of $M(t, \chi)$ provides a min-max policy i.e. for any $U_1 \in A_1$ and $U_2 \in A_2$

$$J(t, \chi, u^*(\cdot), U_2, \phi) \geq J(t, \chi, u^*(\cdot), v^*(\cdot), \phi) \geq J(t, \chi, U_1, v^*(\cdot), \phi) .$$

By the monotone property of V^-, we have

COROLLARY 1. *Let $W(\cdot\,; \phi) \in W_\infty^{1,2}$ be a solution of* (1.15) *with initial function ϕ. Then*

$$W(t, \chi; \phi) \leq W(t, \chi; \psi), \text{ whenever } \phi \leq \psi.$$

By the contraction of V^-, we have

COROLLARY 2. *If Isaacs equation has a solution in $W_\infty^{1,2}$, then it is unique and depends continuously on initial function.*

EXAMPLE (Bang-Bang control). Suppose that (A5) and (A6) hold and α is independent of u and v. Then (1.15) turns out a quasi-linear parabolic equation,

$$(5.17) \qquad \begin{cases} \partial_t W = \sum a_{ij}(\chi)\partial_i\partial_j W \\ \qquad\qquad + \sup_{u \in \Gamma_1} \inf_{v \in \Gamma_2} (\sum \gamma_i(\chi, u, v)\partial_i W - c(\chi, u, v)W + f(\chi, u, v)) \\ W(0) = \phi \ (\in C_b^2) \end{cases}$$

(5.17) has a unique solution in $W^{1,2}_{\infty}$. Furthermore we assume

(i) $\Upsilon(\chi, u, v) = \Upsilon_1(\chi)u + \Upsilon_2(\chi)v$

(ii) $c(\chi, u, v)$ is independent of u and v.

(iii) $f(\chi, u, v)$ is convex in u and concave in v.

Then $g(t, \chi, u, v) = \sum \Upsilon_i(\chi, u, v)\partial_i W(t, \chi) + f(\chi, u, v)$ is convex in u and concave in v, and continuous in (t, χ, u, v). Hence

$$K(t, \chi) = \{(\bar{u}, \bar{v})\} \in dby\Gamma_1 \times dby\Gamma_2;\ \text{for any } (u, v) \in \Gamma_1 \times \Gamma_2,$$
$$g(t, \chi, \bar{u}, v) \geq g(t, \chi, \bar{u}, \bar{v}) \geq g(t, \chi, u, \bar{v})\}$$

is a non-empty compact subset. Moreover there is a Borel selector (u^*, v^*) of $K(t, \chi)$, which is a min-max policy by Remark. Since $u^*(t, \chi) \in bdy\ \Gamma_1$ and $v^*(t, \chi) \in bdy\Gamma_2$, this is called a bang-bang policy [6].

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Department of Mathematics
Kobe University
Rokko, Kobe 657
Japan