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Nagoya Math. J.
Vol. 109 (1988), 151-157

ON THE JACOBIAN EQUATION $J(f,g)=0$ FOR POLYNOMIALS IN $k[x,y]$

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Let $k[x, y]$ be the ring of polynomials in two variables over a field k of characteristic zero.

If $f, g \in k[x, y]$ then we write $f \sim g$ in the case where $f = ag$, for some $a \in k^* = k \setminus \{0\}$, and we denote by $[f, g]$ the jacobian of (f, g), that is, $[f, g] = f_x g_y - f_y g_x$.

By a *direction* we mean a pair (p, q) of integers such that $\gcd(p, q) = 1$ and $p > 0$ or $q > 0$. If (p, q) is a direction then we say that a non-zero polynomial $f \in k[x, y]$ is a *(p, q)-form of degree n* if f is of the form

$$f = \sum_{pi+qj=n} a_{ij} x^i y^j ,$$

where $a_{ij} \in k$.

The following two facts are well known

THEOREM 0.1 ([1], [3], [2]). *Let (p, q) be a direction and let f and g be (p, q)-forms of positive degrees. If $[f, g] = 0$ then there exists a (p, q)-form h such that $f \sim h^m$ and $g \sim h^n$, for some natural m, n.*

THEOREM 0.2 ([2], [7]). *Let f and g be polynomials in $k[x, y]$ and assume that $[f, g]$ is a non-zero constant. Put $\deg(f) = dm > 1$, $\deg(g) = dn > 1$, where $\gcd(m, n) = 1$. Let W_f and W_g be the Newton's polygons of f and g, respectively. Then the polygons W_f and W_g are similar. More precisely, there exists a convex polygon W with vertices in $Z \times Z$ such that $W_f = mW$ and $W_g = nW$.*

Theorem 0.1 plays an essential role in considerations about the Jacobian Conjecture (see for example [1], [3], [2], [5]). Theorem 0.2 is also a consequence of Theorem 0.1.

In this note we show that Theorem 0.1 is a special case of a more general fact. We prove (see Section 1) that if f and g are non-constant

Received October 22, 1986.

polynomials in $k[x, y]$ such that $[f, g] = 0$, then there exist a polynomial $h \in k[x, y]$ and polynomials $u(t)$, $v(t) \in k[t]$ such that $f = u(h)$ and $g = v(h)$. Section 3 shows that the assertion of Theorem 0.2 is also true in the case where $[f, g] = 0$. Moreover, in Section 2, we examine closed polynomials in $k[x, y]$, that is, such polynomials $f \in k[x, y]$ for which the set $\{g \in k[x, y];\ [f, g] = 0\}$ is equal to $k[f]$.

§ 1. Ring $C_k(f)$

If $f \in k[x, y]$ then we denote by d_f the k-derivation of $k[x, y]$ defined by $d_f(g) = [f, g]$, for $g \in k[x, y]$. Denote also by $C_k(f)$ the ring of constants for d_f, that is,

$$C_k(f) = \{g \in k[x, y];\ [f, g] = 0\}.$$

Note the following obvious proposition

PROPOSITION 1.1. *Let $f \in k[x, y]$. Then*
(1) *$C_k(f)$ is a subring of $k[x, y]$ containing $k[f]$,*
(2) *$C_k(f) = k[x, y]$ if and only if $f \in k$.*

We see, by the above proposition, that the case "$f \in k$" is not interesting. In this case the derivation d_f is equal to zero. Now we shall consider only polynomials from $k[x, y] \setminus k$.

PROPOSITION 1.2. *Let $f, g \in k[x, y] \setminus k$. If $g \in C_k(f)$ then $C_k(f) = C_k(g)$.*

Proof. Assume that $g \in C_k(f)$. Then $[f, g] = 0$ and hence $g_x d_f = f_x d_g$ and $g_y d_f = f_y d_g$.

Since f and g do not belong to k, $f_x \neq 0$ or $f_y \neq 0$, and also $g_x \neq 0$ or $g_y \neq 0$. Assume that $f_x \neq 0$ and $g_y \neq 0$ (in the next cases we do the same procedure). Let $h \in C_k(f)$. Then $f_x d_g(h) = g_x d_f(h) = g_x 0 = 0$ and so, $h \in C_k(g)$. If $h \in C_k(g)$ then $q_y d_f(h) = f_y d_g(h) = 0$, that is, $h \in C_k(f)$.

Note also the following proposition which is a simple corollary to [6] Theorem 2.8.

PROPOSITION 1.3. *If $f \in k[x, y] \setminus k$ then there exists a polynomial $h \in k[x, y]$ such that $C_k(f) = k[h]$.*

As an immediate consequence of Propositions 1.2 and 1.3 we obtain

THEOREM 1.4. *Let $f, g \in k[x, y] \setminus k$. If $[f, g] = 0$ then there exist a polynomial $h \in k[x, y]$ and polynomials $u(t)$, $v(t) \in k[t]$ such that $f = u(h)$*

and $g = v(h)$.

§ 2. Closed polynomials in $k[x, y]$

We see, by Proposition 1.1, that if $f \in k[x, y]$ then $k[f] \subseteq C_k(f) \subseteq k[x, y]$. The case $C_k(f) = k[x, y]$ is trivial. Now we shall give a description of the case: $C_k(f) = k[f]$.

We shall say that a polynomial $f \in k[x, y] \setminus k$ is *closed* if the ring $k[f]$ is integrally closed in $k[x, y]$. Denote by $\mathscr{M}$ the family of subrings in $k[x, y]$ defined by

$$\mathscr{M} = \{k[f];\ f \in k[x, y] \setminus k\}.$$

If $k[f] \subsetneq k[g]$, for some $f, g \in k[x, y] \setminus k$, then $\deg(f) > \deg(g)$ and hence in the family $\mathscr{M}$ there exist maximal elements.

THEOREM 2.1. *Let* $f \in k[x, y] \setminus k$. *The following conditions are equivalent.*

(1) $C_k(f) = k[f]$,

(2) *f is closed,*

(3) *The ring $k[f]$ is a maximal element in $\mathscr{M}$.*

Proof. A proof of the equivalence (2) $\Leftrightarrow$ (3) is in [6] (Lemma 3.1). The implication (1) $\Rightarrow$ (2) is a consequence of [6] Proposition 2.2. Assume now that $k[f]$ is maximal in $\mathscr{M}$ and let h be such polynomial in $k[x, y]$ that $C_k(f) = k[h]$ (see Proposition 1.3). Then $k[f] \subseteq k[h]$ and, by the maximality of $k[f]$, we have $k[f] = k[h] = C_k(f)$.

Certain examples of closed polynomials may be obtained by the following two propositions.

PROPOSITION 2.2. *Let* $f, g \in k[x, y]$. *If* $[f, g] \in k^*$ *then f and g are closed.*

Proof. Without loss of any generality we may assume that f and g have no constant terms and that $[f, g] = 1$.

Consider the k-endomorphism F of the ring $k[\![x, u]\!]$ (the power series ring over k) defined by $F(x) = F(y) = g$. We know, by [4], that F is a k-automorphism of $k[\![x, y]\!]$.

Let d be the k-derivation of $k[\![x, y]\!]$ such that $d(x) = -f_y$ and $d(y) = f_x$, and let C be the ring of constants for d.

Observe that

$$k[\![x, y]\!] = F(k[\![x, y]\!]) = k[\![f, g]\!] = (k[\![f]\!])[\![g]\!] ,$$

and hence, it is easy to show that $C = k[\![f]\!]$. Now we have

$$C_k(f) = C \cap k[x, y] = k[\![f]\!] \cap k[x, y] = k[f] ,$$

and so, by Theorem 2.1, f is closed and, by symmetry, g is closed too.

Let (p, q) be a direction and let $f \in k[x, y] \setminus k$ be a (p, q)-form. We shall say that f is *primitive* if there is no (p, q)-form h such that $f \sim h^n$, with $n \geq 2$. For example, the (1.1)-forms x, y, xy, $x^2 + y^2$, $x^3 + xy^2 + 2y^3$ are primitive.

PROPOSITION 2.3. *Let (p, q) be a direction such that $p > 0$ and $q > 0$, and let f be a primitive (p, q)-form. Then f is a closed polynomial.*

Proof. Let d be the degree of f. We shall show that $C_k(f) = k[f]$. Assume that $g \in C_k(f)$ and let $g = g_0 + g_1 + \cdots + g_n$ be the (p, q)-decomposition of g, that is, each g_i, for $i = 1, \cdots, n$, is a (p, q)-form of degree i or is equal to zero, and g_0 is a constant. Then $[f, g_i]$, for $i = 1, \cdots, n$, is a (p, q)-form of degree $d + i - p - q$ (or is equal to zero), and hence the equality $0 = [f, g] = \sum [f, g_i]$ is the (p, q)-decomposition of zero. Hence $[f, g_1] = \cdots = [f, g_n] = 0$ and so, by Theorem 0.1, $g_1, \cdots, g_n \in k[f]$ and we see that $g \in k[f]$. Therefore $k[f] = C_k(f)$ and hence, by Theorem 2.1, f is closed.

§ 3. Newton's polygons

If f is a polynomial in $k[x, y]$ then S_f denotes the *support* of f, that is, S_f is the set of integer points (i, j) such that the monomial $x^i y^j$ appears in f with a non-zero coefficient. We denote by W_f the convex hull (in the real space $\boldsymbol{R}^2$) of $S_f \cup \{(0, 0)\}$. The set W_f is called (see [1]) the *Newton's polygon* of f.

Denote also by $k[x, y]^\circ$ the set $k[x, y] \setminus \bigcup_{a,b \geq 0} k[x^a, y^b]$. The set W_f is always a polygon or a line segment or a point, but it is easy to prove that W_f is a polygon if and only if $f \in k[x, y]^\circ$.

Note the following

LEMMA 3.1. *Let $f, g \in k[x, y] \setminus k$ and let $[f, g] = 0$. Then $f \in k[x, y]^\circ$ if and only if $g \in k[x, y]^\circ$*

Proof. Assume that $f \in k[x, y]^\circ$ and suppose that $g \notin k[x, y]^\circ$. Then $g \in k[x^b, y^b]$, for some non-negative integer a, b such that $a + b > 0$. If

$d = \gcd(a, b)$, $a = a'd$, $b = b'd$, then $g \in k[x^{a'}, y^{b'}]$ and hence, we may assume that $h = x^a y^b$ is a primitive (1, 1)-form (see Section 2) in $k[x, y]$. Now, by Proposition 2.3, $C_k(h) = k[h]$ and we see, by Proposition 1.2, that

$$f \in C_k(f) = C_k(g) = C_k(h) = k[x^a y^b]\,,$$

but it is a contradiction with our assumptions that $f \in k[x, y]^\circ$.

This lemma implies

COROLLARY 3.2. *If f and g are polynomials in $k[x, y] \setminus k$ such that $[f, g] = 0$ then W_f is a polygon if and only if W_g is a polygon.*

Let (p, q) be a direction. If h is a (p, q)-form then we denote by $d_{pq}(h)$ the degree of h. Every polynomial $f \in k[x, y]$ has a (p, q)-decomposition $f = \sum_n f_n$ into (p, q)-components f_n of degree n. We denote by f^*_{pq} the (p, q)-components of f of the highest degree. By *(p, q)-degree* $d_{pq}(f)$ of a polynomial f we mean the number $d_{pq}(f) = d_{pq}(f^*_{pq})$. In particular we have $d_{11}(f) = \deg(f)$. Note now some properties of (p, q)-forms.

LEMMA 3.3. *Let $f, g \in k[x, y] \setminus \{0\}$ and let (p, q) be a direction. Then*

(1) $(fg)^*_{pq} = f^*_{pq} g^*_{pq}$,

(2) $d_{pq}(fg) = d_{pq}(f) + d_{pq}(g)$,

(3) *If $d_{pq}(f) < d_{pq}(g)$ then $(f + g)^*_{pq} = g^*_{pq}$.*

LEMMA 3.4. *Let $f \in k[x, y]^\circ$ and let (a, b) be a non-zero integral point. The following properties are equivalent.*

(1) *The point (a, b) is a non-zero vertex of W_f,*

(2) *There exists a direction (p, q) such that $f^*_{pq} \sim x^a y^b$ and $ap + bq > 0$.*

The proofs of the above lemmas are straightforward.

Now we shall prove the following

LEMMA 3.5. *Let $h \in k[x, y] \setminus k$ and let $f = a_0 + a_1 h + \cdots + a_n h^n$, where $a_0, \cdots, a_n \in k$, $n \geq 1$ and $a_n \neq 0$. If (p, q) is a direction such that $d_{pq}(h) > 0$, then $f^*_{pq} \sim (h^*_{pq})^n$.*

Proof. Write $f = b_1 h^{i_1} + \cdots + b_t h^{i_t}$, where $b_1, \cdots, b_t$ are non-zero constants, $i_1 < \cdots < i_t$, $b_t = a_n$ and $i_t = n$. Then, for $j = 1, \cdots, t-1$,

$$d_{pq}(b_j h^{i_j}) = d_{pq}(h) i_j < d_{pq}(h) i_{j+1} = d_{pq}(b_{j+1} h^{i_{j+1}})$$

and hence, by Lemma 3.3,

$$f_{pq}^{*} \sim (h^{i_t})_{pq}^{*} = (h^n)_{pq}^{*} = (h_{pq}^{*})^n .$$

LEMMA 3.6. *Let* $h \in k[x, y]^{\circ} \setminus k$ *and let* $f = a_0 + a_1 h + \cdots + a_n h^n$, *where* $a_0, \cdots, a_n \in k$, $a_n \neq 0$, $n > 0$.

(1) *Let* A *be a non-zero vertex of* W_h. *Then there exists a unique non-zero vertex* B *of* W_f *such that the points* A, B *and* $(0, 0)$ *are collinear. Moreover* $|0B| = n|0A|$, *where* $0 = (0, 0)$ *and* $|0A|$, $|0B|$ *are the lengths of segments* $0A$ *and* $0B$, *respectively.*

(2) *For every non-zero vertex* D *of* W_f *there exists a unique non-zero vertex* C *of* W_h *such that the points* C, D *and* $(0, 0)$ *are collinear.*

Proof. We know, by Corollary 3.2, that W_h and W_f are polygons.

(1) Let $A = (a, b)$ be a non-zero vertex in W_h. Then, by Lemma 3.4, there exists a direction (p, q) such that $h_{pq}^{*} \sim x^a y^b$ and $d_{pq}(h) = pa + qb > 0$. Hence, by Lemma 3.5,

$$f_{pq}^{*} \sim (h_{pq}^{*})^n \sim x^{na} y^{nb}$$

and $(na)p + (nb)q = n(ap + bq) > 0$; so again by Lemma 3.4, $B = (na, nb)$ is a non-zero vertex of W_f. The points A, B, 0 lie on the line $bx - ay = 0$, $|0B| = n|0A|$, and it is clear that B is unique.

(2) Let $D = (u, v)$ be a non-zero vertex of W_f. Then (Lemma 3.4) $f_{pq}^{*} \sim x^u y^v$ and $pu + qv > 0$, for some direction (p, q). Consider the (p, q)-form h_{pq}^{*}. If $d_{pq}(h) \leq 0$ then $d_{pq}(a_i h^i) \leq 0$, for all $i = 0, 1, \cdots, n$ and we have a contradiction:

$$0 \geq d_{pq}(f) = d_{pq}(f_{pq}^{*}) = pu + qv > 0 .$$

Therefore, $d_{pq}(h) > 0$ and hence, by Lemma 3.5,

$$x^u y^v \sim f_{pq}^{*} \sim (h_{pq}^{*})^n \text{ and so},$$

h_{pq}^{*} is a monomial. Put $h_{pq}^{*} \sim x^s y^t$. Then $0 < d_{pq}(h) = ps + pt$ and hence, by Lemma 3.4, $C = (s, t)$ is a non-zero vertex of W_h. Moreover, the relation $x^u y^v \sim x^{ns} y^{nt}$ implies that $u = ns$ and $v = nt$. This means that the points $0, C, D$ lie on the line $tx - sy = 0$. It is clear that C is unique.

As an immediate consequence of Lemma 3.6 we obtain

COROLLARY 3.7. *Let* $h \in k[x, y]^{\circ}$ *and let* $f = a_0 + a_1 h + \cdots + a_n h^n$, *where* $a_0, \cdots, a_n \in k$, $a_n \neq 0$ *and* $n \geq 1$. *Then the polygons* W_h *and* W_f *are similar and the ratio of similarity is equal to* $1/n$.

From Corollaries 3.7, 3.2 and Theorem 1.4 we have

THEOREM 3.8. *Let $f, g \in k[x, y] \setminus k$ be such polynomials that $[f, g] = 0$.*

(1) *If W_f is a line segment then W_f too.*

(2) *Let W_f be a polygon. Then W_g is also a polygon, the polygons W_f and W_g are similar and the ratio of similarity is equal to* $\deg(f)/\deg(g)$.

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