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Nagoya Math. J.
Vol. 106 (1987), 143-162

CERTAIN UNITARY REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP, II

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Introduction

The *infinite symmetric group* $\mathfrak{S}_\infty$ is the discrete group of all finite permutations of the set X of all natural numbers. Among discrete groups, it has distinctive features from the viewpoint of representation theory and harmonic analysis. First, it is one of the most typical ICC-groups as well as free groups and known to be a group of non-type I. Secondly, it is a locally finite group, namely, the inductive limit of usual symmetric groups $\mathfrak{S}_n$. Furthermore it is contained in infinite dimensional classical groups $GL(\infty)$, $O(\infty)$ and $U(\infty)$ and their representation theories are related each other.

Our present interest lies in irreducible unitary representations of $\mathfrak{S}_\infty$. Its factor representations of type II have been studied considerably in [6]. While, its irreducible representations have been investigated only in a few particular cases, see [1] and [4]. So it is important to have a large stock of irreducible representations. The present paper is a continuation of the author's previous one [3], where we have discussed irreducible representations of $\mathfrak{S}_\infty$ parametrized by certain automorphisms of X.

Let $\mathrm{Aut}(X)$ be the group of all automorphisms of X. For each $\theta \in \mathrm{Aut}(X)$ we denote by $H(\theta)$ the subgroup of all finite permutations $g \in \mathfrak{S}_\infty$ which commute with θ. We define unitary representations $U^{\theta,\chi}$ as the induced representations $\mathrm{Ind}_{H(\theta)}^{\mathfrak{S}_\infty}\chi$, where χ is a unitary character of $H(\theta)$. The results in [3] are restricted to particular automorphisms θ to discuss their irreducibility and equivalence. In the present paper, we first determine the class of automorphisms $\theta \in \mathrm{Aut}(X)$ for which the unitary representation $U^{\theta,\chi}$ is irreducible. Next we give a complete classification of the irreducible representations $U^{\theta,\chi}$.

Received March 5, 1986.

We shall now give a brief sketch of the contents. In Section 1, we recall the structure of the subgroups $H(\theta)$.

In Section 2, we find the class of automorphisms θ satisfying the following property:

(A) $|H(\theta): H(g\theta g^{-1}) \cap H(\theta)| = \infty$ for all $g \in \mathfrak{S}_\infty - H(\theta)$.

Let P be a subset of $\{2, 3, \cdots\}$, possibly $P = \phi$ (the empty set). We denote by $\mathrm{Aut}_P^0(X)$ the set of all automorphisms $\theta \in \mathrm{Aut}(X)$ written in cycle-notation as follows:

$$\theta = \prod_{p \in P} \prod_{n=1}^{\infty} (i_{n0}^p i_{n1}^p \cdots i_{np-1}^p),$$

where the cycles $(i_{n0}^p i_{n1}^p \cdots i_{np-1}^p)$ are pairwise disjoint. If P does not contain 2, we denote by $\mathrm{Aut}_P^1(X)$ the set of all automorphisms $\theta \in \mathrm{Aut}(X)$ of the form:

$$\theta = (j_0 j_1) \prod_{p \in P} \prod_{n=1}^{\infty} (i_{n0}^p i_{n1}^p \cdots i_{np-1}^p),$$

where the cycles $(j_0 j_1)$ and $(i_{n0}^p i_{n1}^p \cdots i_{np-1}^p)$, $p \in P$, $n \geq 1$, are pairwise disjoint and $|X - \mathrm{supp}\,\theta| = 0$ or ∞. If P contains 2, we tacitly understand $\mathrm{Aut}_P^1(X)$ to be empty. Put $\mathrm{Aut}_P(X) = \mathrm{Aut}_P^0(X) \cup \mathrm{Aut}_P^1(X)$. Then it is proved that $\theta \in \mathrm{Aut}(X)$ has the property (A) if and only if it belongs to $\mathrm{Aut}_P(X)$ for some P.

In Section 3, irreducibility of the unitary representations $U^{\theta,\chi}$ will be discussed. The property (A) is relevant to the following assertion, (see Theorem 3.2).

THEOREM (Irreducibility). *Let P be a subset of $\{2, 3, \cdots\}$ and let θ be a member of $\mathrm{Aut}_P(X)$. Then the unitary representation $U^{\theta,\chi}$ is irreducible for any unitary character χ of $H(\theta)$.*

The next step is to discuss unitary equivalence between two irreducible representations $U^{\theta,\chi}$ and $U^{\theta',\chi'}$. Section 4 contains a proof of the following result, (see Theorem 4.12).

THEOREM (Equivalence). *Let θ and θ' be members of $\mathrm{Aut}_P(X)$ and $\mathrm{Aut}_{P'}(X)$, respectively. And let χ and χ' be unitary characters of $H(\theta)$ and $H(\theta')$, respectively. Then two unitary representations $U^{\theta,\chi}$ and $U^{\theta',\chi'}$ are equivalent if and only if the following three conditions are satisfied*: (i) $P = P'$; (ii) $H(\theta') = \gamma H(\theta)\gamma^{-1}$ *for some* $\gamma \in \mathfrak{S}_\infty$; (iii) $\chi'(\gamma h \gamma^{-1}) = \chi(h)$ *for all* $h \in H(\theta)$.

Finally, in Section 5, we shall discuss the relationship among three classes of irreducible representations Π^+, Π^- and $\boldsymbol{U}$. Here we denote by Π^+ and Π^- the set of all irreducible representations $\rho * 1 = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho \times 1$ and $\rho * \mathrm{sgn} = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho \times \mathrm{sgn}$, respectively, where ρ runs over all equivalence classes of irreducible representations of $\mathfrak{S}_n$, $0 \leq n < \infty$. If θ belongs to $\mathrm{Aut}_\phi(X)$, the irreducible representations $U^{\theta,\chi}$ are contained in either Π^+ or Π^-. We denote by $\boldsymbol{U}$ the set of all irreducible representations $U^{\theta,\chi}$, where $\theta \in \mathrm{Aut}_P(X)$ with *non-empty* P. Then we have the following result, (see Theorem 5.3).

THEOREM. *Two irreducible representations are not equivalent if they belong to distinct classes* Π^+, Π^- *or* $\boldsymbol{U}$.

In Appendix A we give an explicit expression of endomorphisms of $\mathfrak{S}_\infty$. In particular, the result proves that $\mathrm{Aut}(X)$ is isomorphic to the automorphism group $\mathrm{Aut}(\mathfrak{S}_\infty)$.

Appendix B contains two remarks on representations of $\Pi^\pm$. We shall give irreducible decompositions of certain induced representations and tensor products.

§1. The structure of the subgroups $H(\theta)$

Let X be the set of all natural numbers and let $\mathfrak{S}_\infty$ be the group of all finite permutations of X. The group $\mathfrak{S}_\infty$, equipped with the discrete topology, is called the *infinite symmetric group*. If Y is a subset of X, we denote by $\mathfrak{S}(Y)$ the group of all finite permutations of X which act identically outside Y. For simplicity, we write $\mathfrak{S}_n$ and $\mathfrak{S}_{\infty-n}$ for $\mathfrak{S}(\{1, 2, \cdots, n\})$ and $\mathfrak{S}(\{n+1, n+2, \cdots\})$, respectively.

Let $\mathrm{Aut}(X)$ be the group of all automorphisms of X. Each $\theta \in \mathrm{Aut}(X)$ can be written in *cycle-notation*, i.e. as a product of pairwise disjoint cycles. For each $p = \infty, 2, 3, \cdots$, we denote by θ_p the product of all cycles of length p appearing in the cycle-notation of θ, and by $N(p, \theta)$ the number of such cycles. Thus, each $\theta \in \mathrm{Aut}(X)$ admits the expression: $\theta = \theta_\infty \theta_2 \theta_3 \cdots$, where θ_p is a product of $N(p, \theta)$ cycles of length p. We call it the *canonical expression* of θ.

For each $\theta \in \mathrm{Aut}(X)$ we denote by $H(\theta)$ the subgroup of all finite permutations of X which commute with θ:

$$H(\theta) = \{g \in \mathfrak{S}_\infty ; g\theta = \theta g\}.$$

In order to describe the structure of $H(\theta)$, we introduce several subgroups. We set

$$H'(\theta) = \{g \in \mathfrak{S}_\infty;\, g\theta = \theta g \quad \text{and} \quad \operatorname{supp} g \subset \operatorname{supp} \theta\},$$

where $\operatorname{supp} \theta = \{i \in X;\, \theta(i) \neq i\}$. Obviously, $H(\theta) = \mathfrak{S}(X - \operatorname{supp} \theta) \times H'(\theta)$ and $H'(\theta_\infty) = \{e\}$. Let $\theta = \theta_\infty \theta_2 \theta_3 \cdots$ be the canonical expression of $\theta \in \operatorname{Aut}(X)$ and let $\theta_p = \prod_{n=1}^{N(p,\theta)} (i_{n0} i_{n1} \cdots i_{np-1})$ be a cycle-notation of θ_p, $2 \leq p < \infty$. We define $A(\theta_p)$ to be the subgroup generated by all the cyclic permutations $(i_{n0} i_{n1} \cdots i_{np-1})$, $n = 1, 2, \cdots, N(p, \theta)$. We denote by $S(\theta_p)$ the subgroup of all permutations $g \in \mathfrak{S}_\infty$ having the following properties: (i) $\operatorname{supp} g \subset \operatorname{supp} \theta_p$; (ii) there exists some $\sigma \in \mathfrak{S}_{N(p,\theta)}$ such that $g(i_{nk}) = i_{\sigma(n)k}$ for all n and k. Obviously, $A(\theta_p)$ is isomorphic to the restricted direct product $Z_p \times Z_p \times \cdots$ ($N(p, \theta)$-times) and $S(\theta_p)$ is isomorphic to $\mathfrak{S}_{N(p,\theta)}$. The following result was proved in [3].

Proposition 1.1. *Let $\theta = \theta_\infty \theta_2 \theta_3 \cdots$ be the canonical expression of $\theta \in$* Aut (X). *Then*

$$H(\theta) = \mathfrak{S}(X - \operatorname{supp} \theta) \times H'(\theta_2) \times H'(\theta_3) \times \cdots$$

in the sense of restricted direct product. Furthermore, for each p, $2 \leq p < \infty$, we have

$$H'(\theta_p) = S(\theta_p) \ltimes A(\theta_p) \qquad (\textit{semidirect product}).$$

Remark. Since $\mathfrak{S}_\infty$ is a normal subgroup of Aut (X), for any $\theta \in$ Aut (X) the map $g \mapsto \theta g \theta^{-1}$, $g \in \mathfrak{S}_\infty$, induces an automorphism $\hat{\theta}$ of $\mathfrak{S}_\infty$. As is seen in Appendix A, every automorphism of $\mathfrak{S}_\infty$ is obtained in this manner. Therefore $H(\theta)$ coincides with the subgroup of all permutations of $\mathfrak{S}_\infty$ fixed under the automorphism $\hat{\theta}$.

§ 2. Characterization of certain automorphisms

This section will be devoted to the study of automorphisms $\theta \in$ Aut (X) satisfying the conditions (A) and (B) below:

(A) $|H(\theta) : H(g\theta g^{-1}) \cap H(\theta)| = \infty$ for all $g \in \mathfrak{S}_\infty - H(\theta)$;

(B) the normalizer of $H(\theta)$ in $\mathfrak{S}_\infty$ coincides with $H(\theta)$ itself, i.e. $gH(\theta)g^{-1} = H(\theta)$ implies $g \in H(\theta)$.

Obviously, the condition (A) implies (B). The main results are the following

Theorem 2.1. *The condition* (B) *is satisfied for $\theta \in$* Aut (X) *if and only if the following three conditions are fulfilled*:

(i) $N(\infty, \theta) = 0$;

(ii) *$N(p, \theta) = 0$ or ∞ for any $p \geq 3$;*
(iii) $|X - \operatorname{supp}\theta| \neq 2$ *or* $N(2, \theta) \neq 1$.

THEOREM 2.2. *The condition* (A) *is satisfied for* $\theta \in \operatorname{Aut}(X)$ *if and only if the following three conditions are fulfilled*:
(i) $N(\infty, \theta) = 0$;
(ii) *$N(p, \theta) = 0$ or ∞ for any $p \geq 3$;*
(iii) *one of the next three conditions*:
(a) $N(2, \theta) = 0$;
(b) $N(2, \theta) = \infty$;
(c) *$N(2, \theta) = 1$ and $|X - \operatorname{supp}\theta| = 0$ or ∞.*

We begin with the following

PROPOSITION 2.3. *The condition* (B) *is not satisfied for any $\theta \in \operatorname{Aut}(X)$ with $N(\infty, \theta) \geq 1$.*

Proof. Take distinct i and $j \in \operatorname{supp}\theta_\infty$ and consider the transposition $g = (ij)$. Then $H(\theta) = gH(\theta)g^{-1}$ though g does not belong to $H(\theta)$. Q.E.D.

LEMMA 2.4. *Let $\theta = \prod_{n=1}^{N}(i_{n0}i_{n1}\cdots i_{np-1})$, $1 \leq N \leq \infty$, and let $\gamma = (j_0 j_1 \cdots j_{p-1})$ be a cycle of length p. Then γ commutes with θ if and only if* (i) $\operatorname{supp}\gamma \subset X - \operatorname{supp}\theta$; *or* (ii) *$\gamma = (i_{n0}i_{n1}\cdots i_{np-1})^q$ for some $n \geq 1$ and q, $1 \leq q < p$. In case of* (ii), *n and q are uniquely determined and $(p, q) = 1$.*

LEMMA 2.5. *Let $\theta = \prod_{m=1}^{\infty}(i_{m0}i_{m1}\cdots i_{mp-1})$ and $\theta' = \prod_{n=1}^{\infty}(j_{n0}j_{n1}\cdots j_{np-1})$. Assume that $\operatorname{supp}\theta = \operatorname{supp}\theta' = X$. Then $H(\theta) = H(\theta')$ if and only if $\theta' = \theta^q$ for some $q \geq 1$ with $(p, q) = 1$.*

LEMMA 2.6. *Let $\theta = \prod_{n=1}^{N}(i_{n0}i_{n1}\cdots i_{np-1})$, $1 \leq N \leq \infty$. Then the normalizer of $H'(\theta)$ in $\mathfrak{S}(\operatorname{supp}\theta)$ is equal to $H'(\theta)$ itself if and only if $N = \infty$ or $p = 2$.*

The proofs of the above three lemmas are easy and omitted. The following result is an immediate consequence of Lemma 2.6.

PROPOSITION 2.7. *Let $\theta \in \operatorname{Aut}(X)$ with $N(\infty, \theta) = 0$. If $1 \leq N(p, \theta) < \infty$ for some $p \geq 3$, the condition* (B) *is not satisfied.*

LEMMA 2.8. *Let $\theta \in \operatorname{Aut}(X)$ with $N(\infty, \theta) = 0$. Then the next three conditions are eqivalent*:
(i) *the normalizer of $H(\theta)$ in $\mathfrak{S}_\infty$ is a subgroup of $\mathfrak{S}(X - \operatorname{supp}\theta) \times \mathfrak{S}(\operatorname{supp}\theta)$,*

(ii) $gH(\theta)g^{-1} = H(\theta)$ *implies* $g(\operatorname{supp}\theta) \subset \operatorname{supp}\theta$,

(iii) $|X - \operatorname{supp}\theta| \neq 2$ or $N(2, \theta) \neq 1$.

Proof. Evidently, (i) and (ii) are equivalent. According to (iii), consider the following three possibilities:

(I) $|X - \operatorname{supp}\theta| \neq 2$;

(II) $|X - \operatorname{supp}\theta| = 2$ and $N(2, \theta) \neq 1$;

(III) $|X - \operatorname{supp}\theta| = 2$ and $N(2, \theta) = 1$.

In order to prove the assertion, we have only to show that (I) implies (i) (or (ii)), that (II) implies (i) (or (ii)) and that (ii) fails under (III).

Case (I). Suppose that $gH(\theta)g^{-1} = H(\theta)$, $g \in \mathfrak{S}_\infty$. We first note that

$$\mathfrak{S}(X - \operatorname{supp}\theta) \times H'(\theta) = \mathfrak{S}(X - g(\operatorname{supp}\theta)) \times H'(g\theta g^{-1}).$$

If $|X - \operatorname{supp}\theta| = 0$, obviously (ii) is satisfied. We suppose that $|X - \operatorname{supp}\theta| = 1$, say, $X - \operatorname{supp}\theta = \{i\}$. It is sufficient to show that $g(i) = i$. Suppose otherwise, then $(g^{-1}(i)j_1j_2\cdots j_k) \in H(\theta)$ for some $j_1, j_2, \cdots, j_k \in \operatorname{supp}\theta$. By assumption, $(ig(j_1)g(j_2)\cdots g(j_k))$ also belongs to $H(\theta)$. Since i is fixed under θ, so are $g(j_1), g(j_2), \cdots, g(j_k)$. This contradiction implies that $g(i) = i$.

Finally we assume that $|X - \operatorname{supp}\theta| > 2$. Consider arbitrary distinct three elements $i_1, i_2, i_3 \in X - \operatorname{supp}\theta$. Obviously, (i_1i_2) and (i_1i_3) belong to $H(\theta)$. By assumption, $(g(i_1)g(i_2))$ and $(g(i_1)g(i_3))$ also belong to $H(\theta)$. This implies that $g(i_1)$ and $g(i_2)$ are fixed under θ or that θ contains the cycle $(g(i_1)g(i_2))$. The latter is impossible because $(g(i_1)g(i_3)) \in H(\theta)$. Thus, $g(i_1)$, $g(i_2)$ and $g(i_3)$ are fixed under θ. Hence $X - \operatorname{supp}\theta \subset X - g(\operatorname{supp}\theta)$ as desired.

Case (II). Suppose that $gH(\theta)g^{-1} = H(\theta)$, $g \in \mathfrak{S}_\infty$. We put $X - \operatorname{supp}\theta = \{i, j\}$. Viewing that $(ij) \in H(\theta) = gH(\theta)g^{-1}$, we see that both $g^{-1}(i)$ and $g^{-1}(j)$ are fixed under θ or that θ contains the cycle $(g^{-1}(i)g^{-1}(j))$. It is sufficient to prove that the latter does not occur.

If $N(2, \theta) = 0$, obviously θ contains no cycle of length 2.

We assume that $N(2, \theta) \geq 2$ and that θ contains the cycle $(g^{-1}(i)g^{-1}(j))$. Take another cycle (k_1k_2) which is contained in θ and put $\gamma = (g^{-1}(i)k_1)(g^{-1}(j)k_2) \in H(\theta)$. Then, also $g\gamma g^{-1} \in H(\theta)$. This implies that $(i\,\theta g(k_1))(j\,\theta g(k_2)) = (ig(k_1))(jg(k_2))$, therefore, $g(k_1)$ and $g(k_2)$ are fixed under θ. This contradicts the choice of k_1 and k_2.

Case (III). We put $X - \operatorname{supp}\theta = \{i_1, i_2\}, \theta_2 = (j_1j_2)$ and $g = (i_1j_1)(i_2j_2)$.

Then, $g(X-\operatorname{supp}\theta)=\{j_1, j_2\}\neq X-\operatorname{supp}\theta$, namely, $g(\operatorname{supp}\theta)\neq\operatorname{supp}\theta$. While, we have $H(\theta)=gH(\theta)g^{-1}$ because $H(\theta)=\mathfrak{S}(\{i_1, i_2\})\times\mathfrak{S}(\{j_1, j_2\})\times H'(\theta_3)\times\cdots$. Q.E.D.

PROPOSITION 2.9. *Let* $\theta\in\operatorname{Aut}(X)$ *with* $N(\infty,\theta)=0$. *If* $|X-\operatorname{supp}\theta|=2$ *and* $N(2,\theta)=1$, *the condition* (B) *is not satisfied.*

This follows immediately from Lemma 2.8. With these preparations, we now give a proof of our first main assertion.

Proof of Theorem 2.1. In view of Propositions 2.3, 2.7 and 2.9, we have only to show that the conditions (i), (ii) and (iii) implies (B). Suppose that $gH(\theta)g^{-1}=H(\theta), g\in\mathfrak{S}_\infty$. By assumptions (i) and (iii) we see that $g\in\mathfrak{S}(X-\operatorname{supp}\theta)\times\mathfrak{S}(\operatorname{supp}\theta)$ with the help of Lemma 2.8. If $\operatorname{supp}\theta=\phi$ (the empty set), i.e. $\theta=e$, obviously the condition (B) is satisfied. We assume that $\operatorname{supp}\theta\neq\phi$. Let p be the smallest number with $\theta_p\neq e$ and put $\theta_p=\prod_{n=1}^{N}(i_{n0}i_{n1}\cdots i_{np-1})$, where $N=\infty$ if $p\geq 3$ and $1\leq N\leq\infty$ if $p=2$. The cycle $(i_{n0}i_{n1}\cdots i_{np-1})$ belongs to $H(\theta)=gH(\theta)g^{-1}$ for every n, so the cycle $(g^{-1}(i_{n0})g^{-1}(i_{n1})\cdots g^{-1}(i_{np-1}))$ belongs to $H(\theta)$. In particular, $g^{-1}(i_{n0}), g^{-1}(i_{n1}), \cdots, g^{-1}(i_{np-1})$ belong to $\operatorname{supp}\theta_p$ because of the choice of p. In other words, we have $\operatorname{supp}\theta_p=g(\operatorname{supp}\theta_p)$. In a similar way we can show that $\operatorname{supp}\theta_p$ is invariant under g for every p, namely, $g\in\mathfrak{S}(X-\operatorname{supp}\theta)\times\mathfrak{S}(\operatorname{supp}\theta_2)\times\mathfrak{S}(\operatorname{supp}\theta_3)\times\cdots$. It follows from Lemma 2.6 that $g\in\mathfrak{S}(X-\operatorname{supp}\theta)\times H'(\theta_2)\times H'(\theta_3)\times\cdots=H(\theta)$. Q.E.D.

Now we come to a proof of Theorem 2.2. Since the condition (A) implies (B), we may assume the conditions (i), (ii) and (iii) in Theorem 2.1. According to the condition (iii), we shall divide Theorem 2.2 into two propositions below.

PROPOSITION 2.10. *Let* $\theta\in\operatorname{Aut}(X)$ *have the following properties*: (i) $N(\infty,\theta)=0$; (ii) $N(p,\theta)=0$ *or* ∞ *for any* $p\geq 3$; (iii) $N(2,\theta)\neq 1$. *Then the condition* (A) *is satisfied if and only if* $N(2,\theta)=0$ *or* ∞.

Proof. First we assume that $N(2,\theta)=0$ or ∞. We put $P=\{p\geq 2; N(p,\infty)=\infty$. If $P=\phi$, i.e. $\theta=e$, the condition (A) is obviously satisfied. We now assume that $P\neq\phi$. Then θ can be written in the form:

$$\theta=\prod_{p\in P}\prod_{n=1}^{\infty}(i_{n0}^{p}i_{n1}^{p}\cdots i_{np-1}^{p}).$$

For a given $g\in\mathfrak{S}_\infty-H(\theta)$, we choose a sufficiently large N such that supp g does not intersect with $\bigcup_{p\in P}\bigcup_{n>N}\{i^p_{n0}, i^p_{n1}, \cdots, i^p_{np-1}\}$. For $p\in P$, $n>N$ and m, $1\le m\le N$, we put

$$\sigma^p_{mn}=\prod_{k=0}^{p-1}(i^p_{mk}i^p_{nk})\in H(\theta)\,.$$

We now pose the following hypothesis:

(H) for all $p\in P$ and for all m, $1\le m\le N$, there exist two distinct numbers n and $n'>N$ such that $(\sigma^p_{mn'})^{-1}\sigma^p_{mn}\in H(g\theta g^{-1})$.

If (H) is false, the set $\{\sigma^p_{mn}(H(\theta)\cap H(g\theta g^{-1}));\ n>N\}$ contains infinitely many cosets for some $p\in P$ and m, $1\le m\le N$ and this implies (A). Now suppose (H). Since $(\sigma^p_{mn'})^{-1}\sigma^p_{mn}$ commutes with $g\theta g^{-1}$,

$$\begin{aligned}
&\prod_{k=0}^{p-1}(i^p_{mk}i^p_{n'k})(i^p_{mk}i^p_{nk})\\
&\quad=\prod_{k=0}^{p-1}(g\theta g^{-1}(i^p_{mk})\,g\theta g^{-1}(i_{n'k}))(g\theta g^{-1}(i^p_{mk})\,g\theta g^{-1}(i^p_{nk}))\\
&\quad=\prod_{k=0}^{p-1}(g\theta g^{-1}(i^p_{mk})\,i^p_{n'k+1})(g\theta g^{-1}(i^p_{mk})\,i^p_{nk+1})\,.
\end{aligned}$$

Hence

$$\prod_{k=0}^{p-1}(i^p_{mk}i^p_{nk}i^p_{n'k})=\prod_{k=0}^{p-1}(g\theta g^{-1}(i^p_{mk-1})\,i^p_{nk}i^p_{n'k})\,.$$

This implies that $g\theta g^{-1}(i^p_{mk})=i^p_{mk+1}$, $0\le k\le p-1$, and that θ contains the cycle $(g^{-1}(i^p_{m0})g^{-1}(i^p_{m1})\cdots g^{-1}(i_{mp-1}))$. Then there exists some $m'=m'(m)$, $1\le m'\le N$, such that

$$(g^{-1}(i^p_{m0})g^{-1}(i^p_{m1})\cdots g^{-1}(i^p_{mp-1}))=(i^p_{m'0}i^p_{m'1}\cdots i^p_{m'p-1})\,,$$

or equivalently,

$$(i^p_{m0}i^p_{m1}\cdots i^p_{mp-1})=(g(i^p_{m'0})g(i^p_{m'1})\cdots g(i^p_{m'p-1}))\,.$$

Since the correspondence $m\leftrightarrow m'$ is one-to-one, we conclude that

$$\begin{aligned}
\theta&=\prod_{p\in P}\prod_{m=1}^{N}(i^p_{m0}i^p_{m1}\cdots i^p_{mp-1})\prod_{m>N}(i^p_{m0}i^p_{m1}\cdots i^p_{mp-1})\\
&=\prod_{p\in P}\prod_{m'=1}^{N}(g(i^p_{m'0})\cdots g(i^p_{m'p-1}))\prod_{m>N}(i^p_{m0}i^p_{m1}\cdots i^p_{mp-1})\\
&=g\theta g^{-1}\,.
\end{aligned}$$

This contradicts the assumption $g\in\mathfrak{S}_\infty-H(\theta)$, namely, (H) is false.

Next we assume that $2\le N(2,\theta)<\infty$. We take two cycles of length 2 contained in θ, say (i_0i_1) and (j_0j_1). Consider the cycle $g=(i_0j_0)\in\mathfrak{S}_\infty-$

$H(\theta)$. Obviously we have

$$H(\theta) \cap H(g\theta g^{-1}) \supset \mathfrak{S}(X - \operatorname{supp}\theta) \times H'(\theta_3) \times H'(\theta_4) \times \cdots,$$

which implies that $|H(\theta): H(\theta) \cap H(g\theta g^{-1})| \leq |H'(\theta_2)| < \infty$. Consequently, the condition (A) is not satisfied. Q.E.D.

PROPOSITION 2.11. *Let $\theta \in \operatorname{Aut}(X)$ have the following three properties*: (i) $N(\infty, \theta) = 0$; (ii) $N(p, \theta) = 0$ *or* ∞ *for all* $p \geq 3$; (iii) $|X - \operatorname{supp}\theta| \neq 2$ *and* $N(2, \theta) = 1$. *Then the condition* (A) *is satisfied if and only if* $|X - \operatorname{supp}\theta| = 0$ *or* ∞.

The proof is modeled after the previous one. Theorem 2.2 is a direct consequence of Propositions 2.10 and 2.11.

§ 3. Irreducible representations

In this section we shall discuss irreducibility of the induced representation $U^{\theta,\chi} = \operatorname{Ind}_{H(\theta)}^{\mathfrak{S}_\infty}\chi$, where χ is a unitary character of $H(\theta)$, $\theta \in \operatorname{Aut}(X)$. With the help of Proposition 1.1, we can describe a complete stock of unitary characters of $H(\theta)$.

By virtue of Theorem 2.2, any automorphism $\theta \in \operatorname{Aut}(X)$ satisfying the condition (A) can be written in the following forms:

(i) $\theta = \prod_{p \in P} \prod_{n=1}^{\infty} (i_{n0}^p i_{n1}^p \cdots i_{np-1}^p)$, where $P \subset \{2, 3, \cdots\}$;

(ii) $\theta = (j_0 j_1) \prod_{p \in P} \prod_{n=1}^{\infty} (i_{n0}^p i_{n1}^p \cdots i_{np-1}^p)$, where $P \subset \{3, 4, \cdots\}$ and $|X - \operatorname{supp}\theta| = 0$ or ∞.

If $P = \phi$ (the empty set), θ is e (the identity) or a transposition according as (i) or (ii). We denote by $\operatorname{Aut}_P^0(X)$ and $\operatorname{Aut}_P^1(X)$ the sets of all automorphisms $\theta \in \operatorname{Aut}(X)$ of the form (i) and (ii), respectively. For any subset $P \subset \{2, 3, \cdots\}$, we set $\operatorname{Aut}_P(X) = \operatorname{Aut}_P^0(X) \cup \operatorname{Aut}_P^1(X)$ with the convention that $\operatorname{Aut}_P^1(X)$ is empty if $2 \in P$.

The following general result is easy to see, (e.g. [2] or [3]).

LEMMA 3.1. *Let G be a discrete group and H a subgroup such that every H-orbit in the quotient space G/H is an infinite set except $\{H\}$. Then the induced representation $\operatorname{Ind}_H^G \chi$ is irreducible for any unitary character χ of H. Moreover, two representations $\operatorname{Ind}_H^G \chi$ and $\operatorname{Ind}_H^G \chi'$ are equivalent if and only if $\chi = \chi'$.*

From Theorem 2.2 and the definition of $\operatorname{Aut}_P(X)$ we see that $\bigcup \operatorname{Aut}_P(X)$, where P runs over all subsets of $\{2, 3, \cdots\}$, coincides with the set of all automorphisms $\theta \in \mathrm{Aut}(X)$ satisfying

(A′) every $H(\theta)$-orbit in $\mathfrak{S}_\infty/H(\theta)$ is an infinite set except $\{H(\theta)\}$.

The following result is then immediate from Lemma 3.1.

THEOREM 3.2. *If $\theta \in \mathrm{Aut}_P(X)$, $P \subset \{2, 3, \cdots\}$, the unitary representation $U^{\theta,\chi}$ is irreducible for any unitary character χ of $H(\theta)$. Moreover, two unitary representations $U^{\theta,\chi}$ and $U^{\theta,\chi'}$ are equivalent if and only if $\chi = \chi'$.*

In the rest of this section, we shall give several remarks on representations $U^{\theta,\chi}$, where $\theta \in \mathrm{Aut}(X)$ does not enjoy the property (A), i.e. $\theta \notin \mathrm{Aut}_P(X)$ for any subset P of $\{2, 3, \cdots\}$.

Some notation is needed. Let Y be an infinite subset of X. If $\tilde{\theta}$ is an automorphism of Y, we set $\tilde{H}(\tilde{\theta}) = \{g \in \mathfrak{S}(Y); g\tilde{\theta} = \tilde{\theta}g\}$. Identifying $\mathfrak{S}(Y)$ with $\mathfrak{S}_\infty$, we agree to put $U^{\tilde{\theta},\tilde{\chi}} = \mathrm{Ind}_{\tilde{H}(\tilde{\theta})}^{\mathfrak{S}(Y)}\tilde{\chi}$, where $\tilde{\chi}$ is a unitary character of $\tilde{H}(\tilde{\theta})$. If Y is an arbitrary countable set, we denote by $\mathfrak{S}(Y)$ the group of all finite permutations of Y. After usual terminology of finite symmetric groups (e.g. [5]), we give the following

DEFINITION. Let Y and Z be disjoint countable sets. Let U and V be unitary representations of $\mathfrak{S}(Y)$ and $\mathfrak{S}(Z)$, respectively. The *outer product* $U*V$ is the induced representation $\mathrm{Ind}_{\mathfrak{S}(Y)\times\mathfrak{S}(Z)}^{\mathfrak{S}(Y\cup Z)} U \times V$.

Suppose that $\theta \in \mathrm{Aut}(X)$ does not satisfy the condition (A). To begin with, we shall give a result for the case of $N(\infty, \theta) \geq 1$.

PROPOSITION 3.3. *Let $\theta \in \mathrm{Aut}(X)$ be such that $N(\infty, \theta) \geq 1$. Then $\tilde{\theta} = \theta\theta_\infty^{-1}$ is an automorphism of $X - \mathrm{supp}\,\theta_\infty$ and any unitary character χ of $H(\theta)$ is of the form $\chi = 1 \times \tilde{\chi}$ according to the decomposition $H(\theta) = \{e\} \times \tilde{H}(\tilde{\theta})$. Moreover.*

$$U^{\theta,\chi} \simeq R * U^{\tilde{\theta},\tilde{\chi}},$$

where R denotes the regular representation of $\mathfrak{S}(\mathrm{supp}\,\theta_\infty) \simeq \mathfrak{S}_\infty$. In particular, $U^{\theta,\chi}$ is not irreducible.

Assume that $\theta \in \mathrm{Aut}(X)$ does not satisfy the condition (A) and that $N(\infty, \theta) = 0$. Viewing Theorem 2.2, we shall consider the following three cases:

(I) $1 \leq N(p, \theta) < \infty$ for some $p \geq 3$;

(II) $2 \leq N(2, \theta) < \infty$;

(III) $N(2, \theta) = 1$ and $1 \leq |X - \mathrm{supp}\,\theta| < \infty$.

The following result is corresponding to (III).

PROPOSITION 3.4. *Let* $\theta \in \mathrm{Aut}(X)$ *satisfy* $X - \mathrm{supp}\,\theta = \{1, 2, \cdots, N\}$ *and* $\theta_2 = (N+1\ N+2)$ *with* $1 \leq N < \infty$. *Then any unitary character* χ *of* $H(\theta)$ *is of the form* $\chi_0 \times \chi_2 \times \tilde{\chi}$, *according to* $H(\theta) = \mathfrak{S}_N \times \mathfrak{S}(\{N+1, N+2\}) \times H'(\tilde{\theta})$, *where* $\tilde{\theta} = \theta\theta_2^{-1}$. *Furthermore, we have*

$$U^{\theta,\chi} \simeq \chi_0 * \chi_2 * U^{\tilde{\theta},\tilde{\chi}} \simeq \sum_{\rho \in \mathfrak{S}_{N+1}^{\wedge}} [\chi_0 * \chi_2 \colon \rho] \rho * U^{\tilde{\theta},\tilde{\chi}},$$

where $\mathfrak{S}_{N+2}^{\wedge}$ *denotes the set of all equivalence classes of irreducible representations of* $\mathfrak{S}_{N+2}$. *In particular,* $U^{\theta,\chi}$ *is not irreducible.*

Proof. The decomposition is obtained from transitivity of induced representations. With the help of the branching rule ([4], Chapter III), we can see the representation $\chi_0 * \chi_2$ of $\mathfrak{S}_{N+2}$ is never irreducible. Therefore $U^{\theta,\chi}$ is not irreducible. Q.E.D.

In case of (I) and (II), it can be verified that $U^{\theta,\chi}$ is a sum of representations of the form $\rho * U^{\tilde{\theta},\tilde{\chi}}$, where ρ is an irreducible representation of finite symmetric groups $\mathfrak{S}_n$. However, we can not conclude that $U^{\theta,\chi}$ is not irreducible with the help of indices only, as we did in Proposition 3.4. We conjecture that $U^{\theta,\chi}$ is not irreducible if pN is large.

Remark. The question of irreducibility of $\rho * U^{\tilde{\theta},\tilde{\chi}}$ is left to be solved. For particular cases, see Corollary 3.5 below and Appendix B.

The following two results are immediate from Theorem 3.2.

COROLLARY 3.5. *Let* $\theta \in \mathrm{Aut}_P^0(X)$ *and* Y *a countable set. Let* χ *and* ε *be unitary characters of* $H(\theta)$ *and* $\mathfrak{S}(Y)$, *respectively. Then* $\varepsilon * U^{\theta,\chi}$ *is irreducible whenever* $\mathrm{supp}\,\theta = X$.

COROLLARY 3.6. *Let* $\theta \in \mathrm{Aut}_P^0(X)$ *and* Y *a countably infinite set. Let* ε_1 *and* ε_2 *be unitary characters of* $\mathfrak{S}_2$ *and* $\mathfrak{S}_{\infty-2}$, *respectively. Identifying* $\mathfrak{S}(Y)$ *with* $\mathfrak{S}_\infty$, *we regard* $\varepsilon_1 * \varepsilon_2$ *as a representation of* $\mathfrak{S}(Y)$. *Then, for any unitary character* χ *of* $H(\theta)$, $\varepsilon_1 * \varepsilon_2 * U^{\theta,\chi}$ *is irreducible whenever* $\mathrm{supp}\,\theta = X$ *and* $2 \notin P$.

§ 4. Equivalence

Having in Section 3 proved irreducibility of the unitary representations $U^{\theta,\chi}$, $\theta \in \mathrm{Aut}_P(X)$, $P \subset \{2, 3, \cdots\}$, we shall now discuss equivalence among them. The following general result plays an essential role. The proof is modeled after that of Lemma 3.1.

LEMMA 4.1. *Let K and H be subgroups of a discrete group G and let χ and χ' be unitary characters of K and H, respectively. If* $|H: H \cap gKg^{-1}| = \infty$ *for all* $g \in G$, *or if* $|K: K \cap gHg^{-1}| = \infty$ *for all* $g \in G$, *then two representations* $\mathrm{Ind}_K^G \chi$ *and* $\mathrm{Ind}_H^G \chi'$ *are disjoint.*

Accordingly, for two automorphisms θ and $\theta' \in \mathrm{Aut}(X)$ we consider the following condition:

(C) $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$
or $|H(\theta'): H(\theta') \cap H(g\theta g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$.

Our first aim is to show the following

PROPOSITION 4.2. *Let* $\theta \in \mathrm{Aut}_P(X)$ *and* $\theta' \in \mathrm{Aut}_{P'}(X)$. *And let* χ *and* χ' *be unitary characters of* $H(\theta)$ *and* $H(\theta')$, *respectively. Then two unitary representations* $U^{\theta,\chi}$ *and* $U^{\theta',\chi'}$ *are not equivalent whenever* $P \neq P'$.

Obviously, this is a direct consequence of Lemma 4.1 and the following

LEMMA 4.3. *Let* $\theta \in \mathrm{Aut}_P(X)$ and $\theta' \in \mathrm{Aut}_{P'}(X)$. *Then the condition* (C) *is satisfied whenever* $P \neq P'$.

Proof. We may assume that one of the following two possibilities occurs:

(I) there exists some $p \geq 3$ such that $p \in P - P'$;

(II) $2 \in P$ and $P - \{2\} = P'$.

In order to avoid repeating almost the same proof twice, we only show that (I) implies (C).

Let $\theta_p = \prod_{n=1}^{\infty} a_n$ be the cycle-notation of θ_p, where each a_n is a cycle of length p. Then we have (I-a) $a_n \notin H(\theta')$ for infinitely many n; or (I-b) $a_n \in H(\theta')$ except finitely many n.

Case (I-a). Given $g \in \mathfrak{S}_\infty$, we put $J = \{n \geq 1;\, a_n \notin H(\theta')$ and $\mathrm{supp}\, a_n \cap \mathrm{supp}\, g = \phi\}$, which is an infinite set by assumption. Suppose that $a_{n'}^{-1} a_n \in H(g\theta' g^{-1})$ for distinct n and $n' \in J$. Then, as is easily seen, we have $\theta' a_n \theta'^{-1} = a_{n'}^{-1}$. In particular, n' is uniquely determined by n if it exists. This shows that the set $\{a_n(H(\theta) \cap H(g\theta' g^{-1}));\, n \in J\}$ contains infinitely many cosets. Hence the condition (C) is satisfied.

Case (I-b). Since a_n is a cycle of length $p \geq 3$, the condition $a_n \in H(\theta')$ implies that $a_n \in \mathfrak{S}(X - \mathrm{supp}\, \theta')$ or $a_n \in H'(\theta'_q)$ for some $q \in P'$ which is necessarily a divisor of p. Thus, infinitely many a_n's belong to $\mathfrak{S}(X -$

$\operatorname{supp}\theta'$) or $H'(\theta'_q)$ for some q.

First we assume that infinitely many a_n's belong to $\mathfrak{S}(X-\operatorname{supp}\theta')$. Then, for a given $g\in\mathfrak{S}_\infty$ the set $J=\{n\geq 1;\, a_n\in\mathfrak{S}(X-\operatorname{supp}\theta')$ and $\operatorname{supp}a_n\cap\operatorname{supp}g=\phi\}$ is infinite. We put $a_n=(i_{n0}i_{n1}\cdots i_{np-1})$, $n\geq 1$, and fix an arbitrary $N\in J$. Now we consider the cycles $b_n=(i_{N0}i_{n0})$, $n\in J-\{N\}$, which belong to $H(\theta')$. Since $b_{n'}^{-1}b_n=(i_{N0}i_{n0}i_{n'0})$ for distinct n and $n'\in J-\{N\}$, $b_{n'}^{-1}b_n$ does not belong to $H(g\theta g^{-1})$. This implies $|H(\theta')\colon H(\theta')\cap H(g\theta g^{-1})|=\infty$.

Next we assume that infinitely many a_n's belong to $H'(\theta'_q)$, where q is a divisor of p and belongs to P'. For a given $g\in\mathfrak{S}_\infty$ we put $J=\{n\geq 1;\ a_n\in H'(\theta'_q)$ and $\operatorname{supp}a_n\cap\operatorname{supp}g=\phi\}$, which is an infinite set by assumption. Let $\theta'_q=\prod_{m=1}^{\infty}b_m$ be the cycle-notation of θ'_q. For each $n\in J$ there exists some $m=m(n)$ such that $\operatorname{supp}b_m\subset\operatorname{supp}a_n$. Obviously, $m(n)\neq m(n')$ if $n\neq n'$. We see that $b_{m(n')}^{-1}b_{m(n)}$ does not belong to $H(g\theta g^{-1})$ whenever $n\neq n'$. In fact, $b_{m(n')}^{-1}b_{m(n)}\in H(g\theta g^{-1})$ implies $\theta b_{m(n)}\theta^{-1}=b_{m(n)}$ or $=b_{m(n')}^{-1}$, but both are impossible because $\operatorname{supp}b_{m(n)}\subset\operatorname{supp}a_n$. Thus we conclude that $|H(\theta')\colon H(\theta')\cap H(g\theta g^{-1})|=\infty$. Q.E.D.

We are now in a position to discuss the case when both θ and θ' belong to $\operatorname{Aut}_P(X)$, $P\subset\{2,3,\cdots\}$. Let $\theta=\varepsilon\prod_{p\in P}\theta_p$ be the canonical expression of θ, where ε denotes the identity or a transposition according as $\theta\in\operatorname{Aut}_P^0(X)$ or $\theta\in\operatorname{Aut}_P^1(X)$. Similarly we put $\theta'=\varepsilon'\prod_{p\in P}\theta'_p$. Let $\theta_p=\prod_{n=1}^{\infty}a_{pn}$ and $\theta'_p=\prod_{m=1}^{\infty}b_{pm}$ be the cycle-notations of θ_p and θ'_p, respectively. Put

$$J_p(\theta,\theta')=\{n\geq 1;\ a_{pn}=(b_{pm})^q \text{ for some } m \text{ and } q\},$$
$$J_p(\theta',\theta)=\{m\geq 1;\ b_{pm}=(a_{pn})^q \text{ for some } n \text{ and } q\}.$$

Then we have a natural bijective correspondence between $J_p(\theta,\theta')$ and $J_p(\theta',\theta)$, namely, for each $m\in J_p(\theta',\theta)$ there exist a unique $n=n(m)\in J_p(\theta,\theta')$ and a unique $q=q(m)$, $1\leq q<p$, such that $b_{pm}=(a_{pn})^q$. In this case, necessarily q is relatively prime to p. The complement of $J_p(\theta,\theta')$ will be denoted by $K_p(\theta,\theta')$.

LEMMA 4.4. *Let θ and $\theta'\in\operatorname{Aut}_P(X)$. If $\sum_{p\in P}|K_p(\theta,\theta')|=\infty$ or if $\sum_{p\in P}|K_p(\theta',\theta)|=\infty$, the condition* (C) *is satisfied.*

Proof. Without loss of generality we may assume that $\sum_{p\in P}|K_p(\theta,\theta')|=\infty$. Then there occurs (I) $|K_p(\theta,\theta')|=\infty$ for some $p\in P$; or (II) $|K_p(\theta,\theta')|$

$< \infty$ for all $p \in P$.

Case (I). If $a_{pn} \notin H(\theta')$ except finitely many $n \in K_p(\theta, \theta')$, we can show that $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$. Otherwise, one can verify that $|H(\theta'): H(\theta') \cap H(g\theta g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$.

Case (II). We put $P_0 = \{p \in P; K_p(\theta, \theta') \text{ is not empty}\}$, which is an infinite set by assumption. For each $p \in P_0$, choose and fix an $n \in K_p(\theta, \theta')$ and put $c_p = a_{pn} \in H'(\theta_p) \subset H(\theta)$. If $c_p \in \mathfrak{S}(X - \operatorname{supp} \theta')$ for infinitely many $p \in P_0$, or if there exists some $q \in P$ such that $c_p \in H'(\theta'_q)$ for infinitely many $p \in P_0$, we can see that $|H(\theta'): H(\theta') \cap H(g\theta g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$. Otherwise, we can show that $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$. Q.E.D.

Keeping the notations introduced before Lemma 4.4, we note that $b_{pm} = (a_{pn(m)})^{q(m)}$ for each $m \in J_p(\theta', \theta)$. Put

$$M(q, p) = \{m \in J_p(\theta', \theta); q(m) = q\}.$$

Then we can easily prove the following result.

LEMMA 4.5. *If there exists some $p \in P$ such that $|M(q; p)| = |M(q', p)| = \infty$ with distinct q and q', then the condition* (C) *is satisfied.*

Viewing Lemmas 4.4 and 4.5, we have only to consider two automorphisms $\theta = \varepsilon \prod_{p \in P} \theta_p$ and $\theta' = \varepsilon' \prod_{p \in P} \theta'_p \in \operatorname{Aut}_P(X)$, where θ_p and θ'_p are given by

$$\theta_p = \prod_{n=1}^{\infty} a_{pn} \prod_{n=1}^{A(p)} b_{pn} \prod_{m=1}^{C(p)} c_{pm},$$
$$\theta'_p = \prod_{n=1}^{\infty} a_{pn}^q \prod_{n=1}^{A(p)} b_{pn}^{q'(n)} \prod_{m=1}^{D(p)} d_{pm}, \qquad q'(n) \neq q,$$

where $0 \leq A(p), C(p), D(p) < \infty$ and $\sum_{p \in P} \{C(p) + D(p)\} < \infty$.

LEMMA 4.6. *If there exists some $p \in P$ such that $C(p) \neq D(p)$, the condition* (C) *is satisfied.*

Proof. Assume that $0 \leq D(p) < C(p) < \infty$. Then one can show that $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for any $g \in \mathfrak{S}_\infty$. Q.E.D.

LEMMA 4.7. *If $\sum_{p \in P} A(p) = \infty$, the condition* (C) *is satisfied.*

Proof. We can show that $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for all $g \in \mathfrak{S}_\infty$. Q.E.D.

Suppose now that the condition (C) is not satisfied for θ and $\theta' \in$

$\mathrm{Aut}_P(X)$. Then it follows from Lemmas 4.6 and 4.7 that θ_p and θ'_p are related as follows:

$$\theta'_p = \gamma_p \theta_p^{q(p)} \gamma_p^{-1}, \quad \gamma_p \in \mathfrak{S}_\infty, \quad p \in P,$$

where $\gamma_p = e$ except finitely many p. The following result is then immediate.

PROPOSITION 4.8. *Let $\theta = \varepsilon \prod_{p \in P} \theta_p$ and $\theta' = \varepsilon' \prod_{p \in P} \theta'_p$ be the canonical expressions of θ and $\theta' \in \mathrm{Aut}_P(X)$, respectively. Assume that both θ and θ' belong to $\mathrm{Aut}_P^0(X)$ (resp. $\mathrm{Aut}_P^1(X)$). Then the condition* (C) *is not satisfied if and only if there exist a sequence $(q(p))_{p \in P}$ and $\gamma \in \mathfrak{S}_\infty$ such that $\theta' = \gamma(\prod_{p \in P} \theta_p^{q(p)})\gamma^{-1}$ (resp. $\theta' = \gamma(\varepsilon \prod_{p \in P} \theta_p^{q(p)})\gamma^{-1}$).*

LEMMA 4.9. *Let $\theta \in \mathrm{Aut}_P^0(X)$ and $\theta' \in \mathrm{Aut}_P^1(X)$. Assume that θ_p and θ'_p are related as follows:*

$$\theta'_p = \gamma_p \theta_p^{q(p)} \gamma_p^{-1}, \quad \gamma_p \in \mathfrak{S}_\infty, \quad p \in P,$$

where $\gamma_p = e$ except finitely many p. Then the condition (C) *is satisfied if and only if $|X - \mathrm{supp}\,\theta'| = \infty$.*

Proof. Without loss of generality, we may put $\theta'_2 = \varepsilon' = (1\ 2)$. First we assume that $|X - \mathrm{supp}\,\theta'| = \infty$. In order to see that $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$ for any $g \in \mathfrak{S}_\infty$, it suffices to consider the following two cases: (I) $g(1) \in X - \mathrm{supp}\,\theta$; (II) $g(1) \in \mathrm{supp}\,\theta_p$ for some $p \in P$.

Case (I). We put $Y = \{i \in X - \mathrm{supp}\,\theta; g(i) = i,\ i \neq g(1) \text{ and } i \neq g(2)\}$, which is an infinite set by assumption. We put $\sigma_i = (g(1)\, i)$, $i \in Y$. Assume that $\sigma_j^{-1}\sigma_i \in H(g\theta' g^{-1})$ for distinct i and $j \in Y$. Then we have

$$(2\, \theta' g^{-1}(i)\, \theta' g^{-1}(j)) = (1\, g^{-1}(i)\, g^{-1}(j)).$$

This implies that $\theta' g^{-1}(i) = 1$ or $\theta' g^{-1}(j) = 1$. But this is impossible because $i \neq g(1)$ and $i \neq g(2)$. Thus we have $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$.

Case (II). Let $\theta_p = \prod_{n=1}^{\infty} (i_{n0} i_{n1} \cdots i_{np-1})$ be the cycle-notation of θ_p. We may assume $g(1) = i_{10}$. We fix a sufficiently large N such that $\mathrm{supp}\, g$ and $\mathrm{supp}\,\gamma_p$ do not intersect with $\{i_{n0}, i_{n1}, \cdots, i_{np-1}\}$ for any $n > N$. Now we put $\sigma_n = \prod_{k=0}^{p-1} (i_{1k} i_{nk})$, $n > N$. Suppose that $\sigma_{n'}^{-1}\sigma_n \in H(g\theta' g^{-1})$ for distinct n and n'. Then by a standard argument, we get contradiction. Hence $|H(\theta): H(\theta) \cap H(g\theta' g^{-1})| = \infty$.

Next we assume that $|X - \mathrm{supp}\,\theta'| = 0$. Then, obviously $|X - \mathrm{supp}\,\theta| = 2$. Therefore we conclude that $H(\theta') = \gamma H(\theta)\gamma^{-1}$ with $\gamma = \prod_{p \in P} \gamma_p \in \mathfrak{S}_\infty$.

Hence (C) is not satisfied. Q.E.D.

PROPOSITION 4.10. *Let* $\theta = \prod_{p\in P}\theta_p$ *and* $\theta' = \varepsilon' \prod_{p\in P}\theta'_p$ *be the canonical expressions of* $\theta \in \mathrm{Aut}^0_P(X)$ *and* $\theta' \in \mathrm{Aut}^1_P(X)$, *respectively. Then the condition* (C) *is not satisfied if and only if* $|X - \mathrm{supp}\,\theta'| = 0$ *and there exist a sequence* $(q(p))_{p\in P}$ *and* $\gamma \in \mathfrak{S}_\infty$ *such that* $\theta' = \varepsilon'\gamma(\prod_{p\in P}\theta_p^{q(p)})\gamma^{-1}$.

This follows directly from Lemma 4.9. Combining the results of Propositions 4.8 and 4.10, we have

COROLLARY 4.11. *Let* θ *and* $\theta' \in \mathrm{Aut}_P(X)$. *Then the condition* (C) *is satisfied if and only if* $H(\theta)$ *and* $H(\theta')$ *are not conjugate in* $\mathfrak{S}_\infty$.

With these preparations, we can now discuss equivalence relation between two unitary representations $U^{\theta,\chi}$ and $U^{\theta',\chi'}$, where θ and θ' belong to $\mathrm{Aut}_P(X)$. It follows from Lemma 4.1 and Corollary 4.11 that they can be equivalent only when $H(\theta)$ and $H(\theta')$ are conjugate in $\mathfrak{S}_\infty$. Consequently, by Lemma 3.1 and Proposition 4.2 we have the following final result.

THEOREM 4.12. *Let* $\theta \in \mathrm{Aut}_P(X)$ *and* $\theta' \in \mathrm{Aut}_{P'}(X)$. *Let* χ *and* χ' *be unitary characters of* $H(\theta)$ *and* $H(\theta')$, *respectively. Then two unitary representations* $U^{\theta,\chi}$ *and* $U^{\theta',\chi'}$ *are equivalent if and only if* (i) $P = P'$; (ii) *there exists* $\gamma \in \mathfrak{S}_\infty$ *such that* $H(\theta') = \gamma H(\theta)\gamma^{-1}$; (iii) $\chi'(\gamma h \gamma^{-1}) = \chi(h)$ *for all* $h \in H(\theta)$.

§ 5. Relationship among Π^+, Π^- and U

For an irreducible representation ρ of $\mathfrak{S}_n$, $0 \le n < \infty$, we form the outer products $\rho * \mathbf{1}$ and $\rho * \mathrm{sgn}$, where $\mathbf{1}$ is the trivial representation and sgn the alternating representation of $\mathfrak{S}_{\infty-n}$. In the author's previous paper [3], $\rho*\mathbf{1}$ and $\rho*\mathrm{sgn}$ were denoted by π^ρ and $\bar{\pi}^\rho$, respectively. We denote by Π^+ and Π^- the collections of $\rho*\mathbf{1}$ and $\rho*\mathrm{sgn}$, respectively. As was shown in [1] and [4], the representations in Π^+ are irreducible and mutually inequivalent. The same assertion is true for Π^- and it can be shown that any two representations of Π^+ and Π^- are mutually inequivalent. Here we recall the following fact proved in [1] and [4].

PROPOSITION 5.1. *Introduce the weakest topology in* $\mathfrak{S}_\infty$ *in such a way that* $\mathfrak{S}_n \times \mathfrak{S}_{\infty-n}$ *is an open set for all* $n = 0, 1, 2, \cdots$. *Then* Π^+ *coincides with the set of all equivalence classes of continuous irreducible unitary rep-*

resentations.

The following result is easily verified.

PROPOSITION 5.2. (1) *If $\theta \in \mathrm{Aut}_\phi^0(X)$, i.e. $\theta = e$, unitary characters of $H(\theta) = \mathfrak{S}_\infty$ are* $\mathbf{1}$ *and* sgn. *Furthermore,*

$$U^{e,1} \simeq \mathbf{1} \text{ (the trivial representation)},$$
$$U^{e,\mathrm{sgn}} \simeq \mathrm{sgn} \text{ (the alternating representation)}.$$

(2) *If $\mathrm{Aut}_\phi^1(X)$, i.e. $\theta =$ transposition, $H(\theta)$ is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_{\infty-2}$ by an inner automorphism of $\mathfrak{S}_\infty$ and its unitary characters are $\varepsilon \times \mathbf{1}$ and $\varepsilon \times$* sgn, *where ε is a unitary character of $\mathfrak{S}_2$. Moreover,*

$$U^{\theta,\varepsilon\times\mathbf{1}} \simeq \varepsilon * \mathbf{1}, \qquad U^{\theta,\varepsilon\times\mathrm{sgn}} \simeq \varepsilon * \mathrm{sgn}\ .$$

We denote by $\boldsymbol{U}$ the collection of all unitary representations $U^{\theta,\chi}$, where $\theta \in \mathrm{Aut}_P(X)$ with *non-empty* subset P and χ is a unitary character of $H(\theta)$.

THEOREM 5.3. *Two irreducible representations are not equivalnt if they belong to distinct classes Π^+, Π^- or $\boldsymbol{U}$.*

Proof. It is sufficient to prove that $U^{\theta,\chi} \in \boldsymbol{U}$ does not belong to Π^+. We realize the representation $U^{\theta,\chi}$ on the Hilbert space of all complex-valued square summable functions on $\mathfrak{S}_\infty/H(\theta)$, (see [3]). We denote by δ the delta-function concentrated at the single point $H(\theta)$ of $\mathfrak{S}_\infty/H(\theta)$. Then,

$$(U^{\theta,\chi}(g)\delta, \delta) = \begin{cases} \chi(g) & \text{if } g \in H(\theta), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, as is easily shown, $H(\theta)$ is not open with respect to the topology introduced in Lemma 5.1. Therefore $U^{\theta,\chi}$ is not continuous with respect to this topology and does not belong to Π^+. Q.E.D.

Appendix A. The structure of endomorphisms of $\mathfrak{S}_\infty$

Here we shall give an explicit expression of endomorphisms of $\mathfrak{S}_\infty$. Let $\mathfrak{A}_\infty$ be the subgroup of all even permutations in $\mathfrak{S}_\infty$, namely, the kernel of the alternating representation: $g \mapsto \mathrm{sgn}\, g$. The group $\mathfrak{A}_\infty$ is called the *infinite alternating group*. We write $\mathfrak{A}_{\infty-n}$ for $\mathfrak{A}_\infty \cap \mathfrak{S}_{\infty-n}$.

LEMMA A.1. (1) *The only normal subgroups of $\mathfrak{S}_\infty$ are $\{e\}$, $\mathfrak{A}_\infty$ and $\mathfrak{S}_\infty$ itself.*

(2) *The only subgroups of* $\mathfrak{S}_\infty$ *of finite index are* $\mathfrak{A}_\infty$ *and* $\mathfrak{S}_\infty$ *itself.*

(3) *The only subgroups of* $\mathfrak{S}_\infty$ *containing* $\mathfrak{A}_{\infty-2}$ *are* $\mathfrak{H}_\infty$, $\mathfrak{H}_{\infty-1}$, $\mathfrak{H}_{\infty-2}$, $(12)\mathfrak{H}_{\infty-1}(12)$, $\mathfrak{S}_2\times\mathfrak{H}_{\infty-2}$ *and* $\mathfrak{A}_{\infty-2}\cup(12)(34)\mathfrak{A}_{\infty-2}$, *where* $\mathfrak{H}$ *denotes* $\mathfrak{S}$ *or* $\mathfrak{A}$.

This follows by an elementary observation of generators. With the help of Lemma A.1 we get the following

LEMMA A.2. *Let* K *be a subgroup of* $\mathfrak{S}_\infty$ *and* Ω *the quotient space* $\mathfrak{S}_\infty/K$. *Assume that* $\mathrm{s}(g)\equiv\{\omega\in\Omega;\, g\omega\neq\omega\}$ *is finite for any* $g\in\mathfrak{S}_\infty$. *Then the subgroup* K *is necessarily equal to* $\mathfrak{S}_\infty$, $\mathfrak{A}_\infty$ *or conjugate subgroups of* $\mathfrak{S}_{\infty-1}$.

Let $\mathrm{End}(\mathfrak{S}_\infty)$ denote the set of all endomorphisms of $\mathfrak{S}_\infty$. Given $f\in\mathrm{End}(\mathfrak{S}_\infty)$, an action of $\mathfrak{S}_\infty$ on X is defined by means of the maps $i\mapsto f(g)i$, $i\in X$. Under this action X is decomposed into a disjoint union of $\mathfrak{S}_\infty$-orbits $\mathcal{O}$. It follows from Lemma A.2 that $|\mathcal{O}|=1$, 2 or ∞.

LEMMA A.3. *Let* $\mathcal{O}$ *be an orbit containing infinitely many points. Then the points of* $\mathcal{O}$ *are parametrized so that* $f(g)i_k=i_{g(k)}$ *for all* $k=1,2,\cdots$.

Proof. Since $\mathcal{O}$ is a transitive $\mathfrak{S}_\infty$-space, there exists a subgroup K such that $\mathcal{O}\simeq\mathfrak{S}_\infty/K$. Since $\mathrm{s}(f(g))$ in $\mathcal{O}$ is finite, we see by Lemma A.2 that K is a conjugate subgroup of $\mathfrak{S}_{\infty-1}$. The desired assertion is then immediate. Q.E.D.

Since $\mathrm{supp}\, f(g)$ is finite for all $g\in\mathfrak{S}_\infty$, the number of orbits containing two or infinitely many points is finite. We denote by $\mathcal{O}_{2,1},\cdots,\mathcal{O}_{2,s}$ the orbits containing two points and $\mathcal{O}_{\infty,1},\cdots,\mathcal{O}_{\infty,t}$ the orbits containing infinitely many points. We denote by $\mathcal{O}_{\mathrm{fix}}$ the set of all points fixed under the action $i\mapsto f(g)i$, $i\in X$. Then we have a partition of X:

$$X=\mathcal{O}_{\mathrm{fix}}\cup\bigcup_{m=1}^{s}\mathcal{O}_{2,m}\cup\bigcup_{n=1}^{t}\mathcal{O}_{\infty,n}\,.$$

Viewing Lemma A.3, we regard $\mathcal{O}_{\infty,n}$ as an infinite (ordered) sequence. With these observations, we have the following

THEOREM A.4. *For each* $f\in\mathrm{End}(\mathfrak{S}_\infty)$ *there exist unordered pairs of points* $\mathcal{O}_{2,m}=\{j_{m1},j_{m2}\}$, $m=1,2,\cdots,s$, *and ordered sequences* $\mathcal{O}_{\infty,n}=\{i_{n1},i_{n2},\cdots\}$, $n=1,2,\cdots,t$, *(possibly* $s=0$ *or* $t=0$*) having the following properties:*

(i) $\mathcal{O}_{2,1},\cdots,\mathcal{O}_{2,s},\mathcal{O}_{\infty,1},\cdots,\mathcal{O}_{\infty,t}$ *are mutually disjoint as subsets of* X;

(ii) *for any* $g\in\mathfrak{S}_\infty$,

$$f(g) = \left[\prod_{m=1}^{s} (j_{m1} j_{m2})\right]^{\varepsilon(g)} \prod_{n=1}^{t} \begin{pmatrix} i_{n1} & i_{n2} & \cdots \\ i_{ng(1)} & i_{ng(2)} & \cdots \end{pmatrix},$$

where $\varepsilon(g) = 0$ *if* $g \in \mathfrak{A}_\infty$ *and* $\varepsilon(g) = 1$ *otherwise.*

This theorem has many applications. For example, we can determine certain extensions of $\mathfrak{S}_\infty$-spaces. Here we only mention the following

THEOREM A.5. *Let* $f \in \mathrm{End}(\mathfrak{S}_\infty)$. *Then* f *is an automorphism of* $\mathfrak{S}_\infty$ *if and only if* $f(g) = \theta g \theta^{-1}$ *for some* $\theta \in \mathrm{Aut}(X)$. *In particular,* $\mathrm{Aut}(X)$ *is isomorphic to the automorphism group* $\mathrm{Aut}(\mathfrak{S}_\infty)$.

Appendix B. Some remarks on representations of Π^+

Let ρ be an irreducible representation of $\mathfrak{S}_n$, $0 \leq n < \infty$, and U an irreducible representation of $\mathfrak{S}_\infty$. As we remarked in Section 3, the question of irreducibility or irreducible decomposition of the outer product $\rho * U$ is open in a general situation. If $U \in \Pi^\pm$, however, we can give a decomposition formula for $\rho * U$ with the help of transitivity of induced representations.

THEOREM B.1. *Let* ρ *and* ρ' *be finite dimensional unitary representations of* $\mathfrak{S}_n$ *and* $\mathfrak{S}_m$, *respectively. Then we have*

$$\rho*(\rho'*\mathbf{1}) \simeq (\rho*\rho')*\mathbf{1} \simeq \sum_{\tau \in \mathfrak{S}_{n+m}^\wedge} [\rho*\rho' : \tau]\tau*\mathbf{1}.$$

Applying the above result to the regular representation, we have the following

COROLLARY B.2. *If* τ *is an irreducible representation of* $\mathfrak{S}_{n+m}$, *the following identity holds*:

$$\dim \tau = \sum_{\alpha \in \mathfrak{S}_n^\wedge} \sum_{\beta \in \mathfrak{S}_m^\wedge} (\dim \alpha)(\dim \beta)[\alpha*\beta : \tau].$$

Finally we shall give a decomposition formula for the tensor product of representations of Π^+. The proof is omitted because we need rather complicated and combinatorial arguments about orbits of certain $\mathfrak{S}_n$-spaces. An analogous assertion for Π^- can be obtained easily.

THEOREM B.3. *Let* ρ *and* ρ' *be finite dimensional unitary representations of* $\mathfrak{S}_n$ *and* $\mathfrak{S}_m$, *respectively. Then*

$$(\rho*\mathbf{1}) \otimes (\rho'*\mathbf{1}) \simeq \sum_{j=0}^{\min(n,m)} \tau(j)*\mathbf{1},$$

where $\tau(j)$ *is a unitary representation of* $\mathfrak{S}_{n+m-j}$ *given by*

$$\tau(j) = \mathrm{Ind}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j} \times \mathfrak{S}_{m-j}}^{\mathfrak{S}_{n+m-j}} \rho_j, \qquad \rho_j(\sigma_1, \sigma_2, \sigma_3) = \rho(\sigma_1\sigma_2) \otimes \rho'(\sigma_1\sigma_3).$$

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