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Nagoya Math. J.
Vol. 105 (1987), 147-151

FOR WHICH FINITE GROUPS G IS THE LATTICE $\mathscr{L}(G)$ OF SUBGROUPS GORENSTEIN?

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Introduction

Let G be a finite group and $\mathscr{L}(G)$ the lattice consisting of all subgroups of G. It is well known that $\mathscr{L}(G)$ is distributive if and only if G is cyclic (cf. [2, p. 173]). Moreover, the classical result of Iwasawa [8] says that $\mathscr{L}(G)$ is pure if and only if G is supersolvable. Here, a finite lattice is called pure if all of maximal chains in it have same length and a finite group G is called supersolvable if $\mathscr{L}(G)$ has a maximal chain which consists of normal subgroups of G.

On the other hand, some remarkable connections between commutative algebra and combinatorics have been discovered in recent years. One of the main topics in this area is the concept of Cohen-Macaulay and Gorenstein posets. See, for examples, Hochster [7] and Stanley [11].

Now, with the help of Stanley [10] and Iwasawa [8], Björner [3] proved that $\mathscr{L}(G)$ is Cohen-Macaulay if and only if G is supersolvable. So, it is natural to ask for which finite groups G the lattice $\mathscr{L}(G)$ is Gorenstein.

The purpose of this paper is to prove the following

THEOREM. *Let G be a finite group and $\mathscr{L}(G)$ its lattice of subgroups. Then $\mathscr{L}(G)$ is Gorenstein if and only if G is a cyclic group whose order is either square-free or a prime power.*

§ 1. Preliminaries from group theory, commutative algebra and combinatorics

We here summarize basic definitions and results on group theory, commutative algebra and combinatorics.

(1.1) Let G be a finite group whose order $\sharp(G)$ is $p_1 p_2 \cdots p_m$, where

Received October 7, 1985.

each p_i is a prime and $p_1 \leq p_2 \leq \cdots \leq p_m$. Then G is supersolvable if and only if G has a normal series

$$G = N_0 \supset N_1 \supset \cdots \supset N_m = 1$$

whose factor groups satisfy $\sharp(N_{i-1}/N_i) = p_i$ for all i. This is [5, Corollary 10.5.3].

(1.2) In [6] P. Hall has generalized Sylow theorems as follows; If G is a finite solvable group of order mn with $(m, n) = 1$, then

(i) G contains at least one subgroup of order m,

(ii) any two subgroups of order m are conjugate and

(iii) any subgroup whose order m' divides m is contained in a subgroup of order m.

(1.3) All posets (partially ordered sets) to be considered are finite. The length of a chain (totally ordered set) X is $\sharp(X) - 1$. The rank of a poset H, denoted by rank(H), is the supremum of lengths of chains contained in H. The height of an element α of a poset is the supremum of lengths of chains descending from α, and written by $\mathrm{ht}(\alpha)$.

(1.4) Let k be a field and H a poset. The Stanley-Reisner ring $k[H]$ is defined by

$$k[H] = k[X_\alpha | \alpha \in H]/(X_\alpha X_\beta | \alpha \not\sim \beta)$$

where $k[X_\alpha | \alpha \in H]$ is a polynomial ring over k and $\alpha \not\sim \beta$ means that α and β are incomparable in the partial order of H.

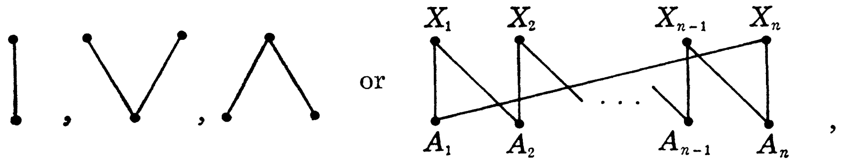
(1.5) A poset H is called Cohen-Macaulay (resp. Gorenstein) over a field k if $k[H]$ is a Cohen-Macaulay (resp. Gorenstein) ring. Some standard references of this topic are [1], [4], [7], [9] and [11].

(1.6) In the following we fix the base field k and omit the word "over k". If H is a Cohen-Macaulay (resp. Gorenstein) poset, then every open interval $I_{x,y} = \{\alpha \in H; x < \alpha < y\}$ is also Cohen-Macaulay (resp. Gorenstein). This result follows immediately from [7, (5.6)].

In particular (cf. [7, (5.5)]), if H is Gorenstein, then every open interval $I_{x,y}$ with $\mathrm{ht}(y) - \mathrm{ht}(x) = 2$ is of the form

(*) • or • • ,

and every open interval $I_{x,y}$ with $\mathrm{ht}(y) - \mathrm{ht}(x) = 3$ is of the form

(**) 

where n is a positive integer.

(1.7) Let G be a finite group. Thanks to (1.6), if $\mathscr{L}(G)$ is Cohen-Macaulay (resp. Gorenstein) then, for every subgroup K of G and every quotient group G/N, $\mathscr{L}(K)$ and $\mathscr{L}(G/N)$ are also Cohen-Macaulay (resp. Gorenstein).

(1.8) Finally, we remark that every Boolean lattice is Gorenstein (cf. [4, p. 615]).

§ 2. Proof of the theorem

(2.1) The "if" part is almost obvious. In fact, if G is a cyclic group whose order is square-free, say $\sharp(G) = p_1 p_2 \cdots p_d$, where $p_1 < p_2 < \cdots < p_d$ are primes, then the lattice $\mathscr{L}(G)$ is isomorphic to the Boolean lattice which consists of all subsets of a set of d elements, hence $\mathscr{L}(G)$ is Gorenstein by (1.8). On the other hand, if G is a cyclic group whose order is a prime power, then $\mathscr{L}(G)$ is a chain, hence Gorenstein.

(2.2) Now, let G be a finite group of order

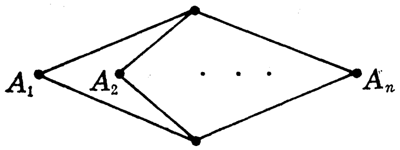
$$\sharp(G) = p_1^{r_1} p_2^{r_2} \cdots p_d^{r_d},$$

where $p_1 < p_2 < \cdots < p_d$ are primes and $r_i \geq 1$. We shall prove the "only if" part by induction on d.

(2.3) First, we consider the case $d = 1$ and put $\sharp(G) = p^r$. We shall show that G is a cyclic group by induction on r. The case $r = 1$ is trivial since p is a prime.

Assume $r > 1$. Since G is a p-group, G has a non-trivial center Z. Choose an element $x \in Z$ whose order is p. Let $\langle x \rangle$ be a cyclic group generated by x. The lattice $\mathscr{L}(G/\langle x \rangle)$ is Gorenstein by (1.7), hence $G/\langle x \rangle$ is cyclic by assumption of induction.

So, G is abelian. Hence, by the basis theorem for finite abelian groups, G must be the direct product $G_1 \times G_2 \times \cdots \times G_t$ of cyclic groups of order $p^{r_1}, p^{r_2}, \cdots, p^{r_t}$ ($r_i \geq 1$ and $r_1 + r_2 + \cdots + r_t = r$). We must prove $t = 1$. Suppose, on the contrary, $t \geq 2$ and $x \in G_1$, $y \in G_2$ are elements of order p. Then, the lattice $\mathscr{L}(\langle x \rangle \times \langle y \rangle)$ is Gorenstein by (1.7), but this lattice is of the form

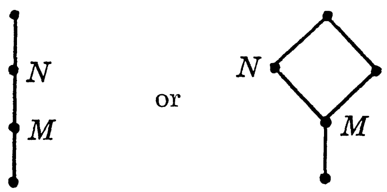


$(n \geq 3)$,

which contradicts (*) in (1.6).

(2.4) Secondly, we treat the case $d = 2$ and put $\sharp(G) = p^r q^s$ $(p < q)$. To begin with, we shall prove $r = s = 1$.

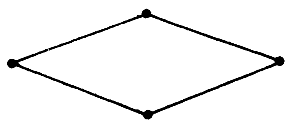
Case (i) Assume $s \geq 2$. Since G is supersolvable, G has a normal subgroup N of order q^2 by (1.1). In the quotient group G/N, consider a subgroup of order p, and we can obtain a subgroup K $(\supset N)$ of G of order $\sharp(K) = pq^2$. By (2.3), N is cyclic, hence there exists only one subgroup M of K which is properly contained in N. Thus $\mathscr{L}(K)$ must be



by (**) in (1.6), but it is impossible since K has a subgroup of order p.

Case (ii) Assume $r \geq 2$. In this case, G must contain a normal subgroup N of order q and a subgroup K $(\supset N)$ of order $p^2 q$. Note that N is a unique subgroup of K of order q and that, since $L(K/N)$ is Gorenstein, K/N is a cyclic group of order p^2. Hence K has a unique subgroup M of order pq, and M is the only proper subgroup of K which contains N. So, $\mathscr{L}(K)$ cannot be Gorenstein by the same argument as in case (i).

Now, $\sharp(G) = pq$ and $L(G)$ is Gorenstein, hence $L(G)$ must be



by (*) in (1.6). This implies G is cyclic.

(2.5) Now, suppose that $d \geq 3$. Since G is supersolvable, G contains a subgroup K_i of order $p_1^{r_1} \cdots p_{i-1}^{r_{i-1}} p_{i+1}^{r_{i+1}} \cdots p_d^{r_d}$ by (i) in (1.2). Since $\mathscr{L}(K_i)$ is Gorenstein, by assumption of induction we have $r_j = 1$ for all j $(\neq i)$. Hence $r_i = 1$ for all i, and $\sharp(G) = p_1 p_2 \cdots p_d$ $(p_1 < p_2 < \cdots < p_d)$.

By (1.1), G contains a normal subgroup N_d of order p_d, and N_d is the unique subgroup of G of order p_d. By assumption of induction G/N_d is a cyclic group of order $p_1p_2 \cdots p_{d-1}$. Hence there exists only one subgroup M_i of G of order $\sharp(M_i) = p_ip_d$ $(i \neq d)$. By virtue of (iii) in (1.2), every subgroup of G of order p_i must be contained in M_i, hence G has a unique subgroup N_i of order p_i by (*) in (1.6), and N_i must be a normal subgroup of G.

Consequently, $G = N_1 \times N_2 \times \cdots \times N_d$ and G is a cyclic group of order $p_1p_2 \cdots p_d$ as desired.

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