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EXISTENCE OF NORMAL MEROMORPHIC FUNCTIONS WITH A PERFECT SET AS THE SET OF ESSENTIAL SINGULARITIES

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§1. Introduction

1. We are interested in whether there is a Cantor set E admitting no exceptionally ramified or normal meromorphic functions with E as the set of essential singularities. As for an exceptionally ramified meromorphic function, we [2] have recently given the following result.

THEOREM A. *Let E be a Cantor set with successive ratios ξ_n satisfying the condition*

$$\xi_{n+1} = o(\xi_n^5),$$

then the domain complementary to E admits no exceptionally ramified meromorphic functions with E as the set of essential singularities.

However, for a normal meromorphic function, S. Toppila [4] proved that if the set F is an infinite closed set, there exists a normal meromorphic function in the domain F^c complementary to F with at least one essential singularity in F. In [4], he gave a normal meromorphic function in F^c with one essential singularity in F.

In this paper, using the analogous method in S. Toppila [4], we shall give a normal meromorphic function with a Cantor set as the set of essential singularities.

Our result is stated as follows:

THEOREM. *Let E be a Cantor set with successive ratios ξ_n such that*

$$(1) \qquad \lim_{n\to\infty} \xi_n = 0$$

and

$$(2) \qquad \xi_{n+1} = O(\xi_n).$$

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Then there exists a normal meromorphic function in the domain complementary to E with E as the set of essential singularities.

Thus it follows from Theorem that the conclusion of Theorem A is false if we assume that a function is normal, instead of exceptionally ramified.

§ 2. Proof of Theorem

2. We form a Cantor set with successive ratios ξ_n, $0 < \xi_n < 2/3$, in the usual manner. We remove first an open interval of length $(1 - \xi_1)$ from the interval $I_{0,1}$: $[-1/2, 1/2]$, so that on both sides there remains a closed interval of length $\xi_1/2 \equiv \ell_1$. The remained intervals are denoted by $I_{1,1}$ and $I_{1,2}$. Inductively we remove an open interval of length $(1 - 2\eta_n) \prod_{p=1}^{n-1} \eta_p$, with $\xi_p/2 \equiv \eta_p$, $p = 1, 2, 3, \cdots$, from each $I_{n-1,k}$, $k = 1, 2, \cdots, 2^{n-1}$, so that on both sides there remains a closed interval of length $\prod_{p=1}^{n} \eta_p \equiv \ell_n$. The remaining intervals are denoted by $I_{n,2k-1}$ and $I_{n,2k}$. By repeating this procedure endlessly, we obtain an infinite sequence of closed intervals $\{I_{n,k}\}_{n=0,1,2,\cdots,\ k=1,2,\cdots,2^n}$. The set given by

$$E = \bigcap_{n=0}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}$$

is said to be the Cantor set in the interval $I_{0,1}$ with successive ratios ξ_n.

Denoting by $z_{n,k}$ the midpoint of $I_{n,k}$ and setting $\alpha_{n,k} = z_{n,k} + i\ell_n/2$, we shall give an infinite product

$$f(z) = \prod_{n=1}^{\infty} \prod_{k=1}^{2^n} \frac{z - \alpha_{n,k}}{z - \overline{\alpha}_{n,k}}.$$

Obviously this function f has the set E as the set of essential singularities. In order to prove Theorem it is enough to show that f is normal in the domain Ω complementary to E.

The proof of this is based on the following result due to O. Lehto and K. I. Virtanen [3].

THEOREM B. *A function f meromorphic in a domain G of hyperbolic type, is normal in G if and only if there exists a finite constant C so that for every $z \in G$*

$$\frac{|f'(z)|}{1 + |f(z)|^2} |dz| \leqq C \, d\sigma_G(z),$$

where $d\sigma_G(z)$ denotes the hyperbolic element of length of G.

In order to estimate $|dz|/d\sigma_\Omega(z)$, we need the following

LEMMA. *Let D be the domain complementary to the set $\{0, 1, \infty\}$. Then*

$$\lim_{w\to 0}\left(|w|\log\frac{1}{|w|}\right)\frac{d\sigma_D(w)}{|dw|} = \frac{1}{2}$$

(*see C. Constantinescu* [1]).

3. We first discuss $|dz|/d\sigma_\Omega(z)$. By Lemma, there exists a positive number δ_0, $1/8 > \delta_0 > 0$, such that

$$(3)\qquad \frac{|dw|}{d\sigma_D(w)} < 4|w|\log\frac{1}{|w|} \qquad \text{in } w\in\{w|\ 0<|w|<4\delta_0\}\equiv R_0 .$$

Applying the linear transformations $w = 1-\zeta$ and $w = 1/\zeta$ to (3), we have

$$(4)\qquad \frac{|dw|}{d\sigma_D(w)} < 4|w-1|\log\frac{1}{|w-1|}$$
$$\text{in } w\in\{w|\ 0<|w-1|<4\delta_0\}\equiv R_1$$

and

$$(5)\qquad \frac{|dw|}{d\sigma_D(w)} < 4|w|\log|w| \qquad \text{in } w\in\{w|\ 1/4\delta_0<|w|<\infty\}\equiv R_\infty ,$$

respectively. Since the set $R\equiv\{w|\ |w|\geqq\delta_0/4,\ |w-1|\geqq\delta_0/4,\ |w|\leqq 4/\delta_0\}$ is compact, there exists a positive number C_1 such that

$$(6)\qquad \frac{|dw|}{d\sigma_D(w)} < C_1 \qquad \text{in } w\in R .$$

We now set

$$\hat{\gamma}_{n,k} = \{z|\ |z-z_{n,k}| = \delta_0\ell_{n-1}\},$$
$$\check{\gamma}_{n,k} = \{z|\ |z-z_{n,k}| = \ell_n/\delta_0\},$$
$$\Gamma_{n,k} = \{z|\ |z-z_{n,k}| = \sqrt{\ell_n\ell_{n-1}}\}$$

and

$$(\Gamma_{n,k}) = \{z|\ |z-z_{n,k}| < \sqrt{\ell_n\ell_{n-1}}\},$$

for $n = 1, 2, 3, \cdots$, $k = 1, 2, \cdots, 2^n$. We denote by $S_{n,k}$ (resp. $T_{n,k}$) the closed ring domain bounded by $\hat{\gamma}_{n,k}$ (resp. $\check{\gamma}_{n,k}$) and $\Gamma_{n,k}$. The triply connected closed domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ (resp. $\check{\gamma}_{n,k}$, $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$) is denoted by $\Delta_{n,k}$ (resp. $\Delta'_{n,k}$), where $\Delta_{0,1}$ denotes the closed ring domain bounded by $\Gamma_{1,1}$ and $\Gamma_{1,2}$ in the extended complex plane $\hat{C}$. Immediately we have

$$\Omega = \bigcup_{\substack{k=1,2,\cdots,2^n \\ n=0,1,2,\cdots}} \Delta_{n,k} .$$

Denoting by $a_{n,k}$ (resp. $b_{n,k}$) the left (resp. right) endpoint of $I_{n,k}$, we write

$$D_{n,k} = \begin{cases} \text{the domain complementary to the set } \{a_{n,k},\ b_{n,k},\ 1\}, \\ \qquad\qquad \text{if } k = 1, 2, \cdots, 2^{n-1}, \\ \text{the domain complementary to the set } \{0,\ a_{n,k},\ b_{n,k}\}, \\ \qquad\qquad \text{if } k = 2^{n-1}+1,\ 2^{n-1}+2, \cdots, 2^n . \end{cases}$$

We take the conformal mapping $w = \phi_{n,k}(z)$ from $D_{n,k}$ onto D such that

$$\phi_{n,k}(a_{n,k}) = 0 , \qquad \phi_{n,k}(b_{n,k}) = 1$$

and

$$\begin{cases} \phi_{n,k}(1) = \infty , & \text{if } k = 1, 2, \cdots, 2^{n-1}, \\ \phi_{n,k}(0) = \infty , & \text{if } k = 2^{n-1}+1,\ 2^{n-1}+2, \cdots, 2^n . \end{cases}$$

From (1), there is a positive integer N, $N \geqq 2$,

$$(7) \qquad \xi_n < \delta_0^2/2 , \qquad \text{for } n \geqq N .$$

We denote by Ω_0 the closed domain bounded by the circles $\{\Gamma_{N,k}\}_{k=1,2,\cdots,2^N}$ in $\hat{C}$. For every $z \in \Omega - \Omega_0$, we choose the integers n and k such that $z \in \Delta_{n,k}$. Since $d\sigma_\Omega(z) \geqq d\sigma_{D_{n,k}}(z)$ and since the hyperbolic element of length is conformally invariant, we have for $z \in \Delta_{n,k}$

$$(8) \qquad \frac{|dz|}{d\sigma_\Omega(z)} \leqq \frac{|dz|}{d\sigma_{D_{n,k}}(z)} = \frac{|dz|}{|dw|} \cdot \frac{|dw|}{d\sigma_D(w)} < 9\,\ell_n \frac{|dw|}{d\sigma_D(w)} ,$$

where $w = \phi_{n,k}(z)$.

By elementary computations, we have

$$\phi_{n,k}(\hat{\gamma}_{n+1,2k-1}) \subset \{w \mid \delta_0/4 < |w| < 4\delta_0\} ,$$
$$\phi_{n,k}(\hat{\gamma}_{n+1,2k}) \subset \{w \mid \delta_0/4 < |w-1| < 4\delta_0\}$$

and

$$\phi_{n,k}(\check{\gamma}_{n,k}) \subset \{w \mid 1/4\delta_0 < |w| < 4/\delta_0\} ,$$

in view of (7). Thus

$$\phi_{n,k}(S_{n+1,2k-1}) \subset R_0\,,$$
$$\phi_{n,k}(S_{n+1,2k}) \subset R_1\,,$$
$$\phi_{n,k}(T_{n,k}) \subset R_\infty$$

and

$$\phi_{n,k}(\Delta'_{n,k}) \subset R\,.$$

Hence applying (3), (4), (5) and (6) to the image of $\Delta_{n,k}$ under $w = \phi_{n,k}(z)$, we deduce from (8) that

$$(9)\quad \begin{cases} \dfrac{|dz|}{d\sigma_\Omega(z)} < C_2|z-a_{n,k}|\log\dfrac{3\ell_n}{|z-a_{n,k}|}\,, & \text{for } z \in S_{n+1,2k-1}\,,\\ \dfrac{|dz|}{d\sigma_\Omega(z)} < C_3|z-b_{n,k}|\log\dfrac{3\ell_n}{|z-b_{n,k}|}\,, & \text{for } z \in S_{n+1,2k}\,,\\ \dfrac{|dz|}{d\sigma_\Omega(z)} < C_4|z-a_{n,k}|\log\dfrac{2|z-a_{n,k}|}{\ell_n}\,, & \text{for } z \in T_{n,k}\,,\\ \dfrac{|dz|}{d\sigma_\Omega(z)} < C_5\ell_n\,, & \text{for } z \in \Delta'_{n,k}\,, \end{cases}$$

where C_j are constant.

4. We next discuss the spherical derivative $\rho(f(z)) \equiv |f'(z)|/(1+|f(z)|^2)$ of f. We have for $z \in \Delta_{n,k}$, $n \geqq N$,

$$\begin{aligned} \rho(f(z)) &\leqq \frac{|f(z)|}{1+|f(z)|^2}\sum_{\substack{h=1,2,\cdots,2^m\\ m=1,2,3,\cdots}}\frac{\ell_m}{|z-\alpha_{m,h}||z-\bar\alpha_{m,h}|}\\ &\leqq \frac{1}{2}\sum_{\substack{h=1,2,\cdots,2^m\\ m=1,2,\cdots,n\\ (m,h)\neq(n,k)}}\frac{\ell_m}{|z-\alpha_{m,h}||z-\bar\alpha_{m,h}|}\\ &\quad+\frac{|f(z)|\ell_n}{(1+|f(z)|^2)|z-\alpha_{n,k}||z-\bar\alpha_{n,k}|}\\ &\quad+\frac{1}{2}\sum_{\substack{h=1,2,\cdots,2^m\\ m=n+1,n+2,\cdots}}\frac{\ell_m}{|z-\alpha_{m,h}||z-\bar\alpha_{m,h}|}\\ &\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}. \end{aligned}$$

The second term II is simply estimated as follows:

We have

$$\mathrm{II} < \frac{\ell_n}{|z-\bar\alpha_{n,k}|^2}\prod_{(m,h)\neq(n,k)}\left|\frac{z-\alpha_{m,h}}{z-\bar\alpha_{m,h}}\right| < \frac{\ell_n}{|z-\bar\alpha_{n,k}|^2} < \frac{36}{\ell_n}\,,$$
$$\text{for } z \in U_{n,k} \equiv \{z|\ |z-\alpha_{n,k}| \leqq \ell_n/6\}\,,$$

$$\mathrm{II} < \ell_n \Big/ \left\{ |z - \alpha_{n,k}|^2 \prod_{(m,h)\neq(n,k)} \left| \frac{z - \alpha_{m,h}}{z - \bar{\alpha}_{m,h}} \right| \right\} < \frac{\ell_n}{|z - \alpha_{n,k}|^2} < \frac{36}{\ell_n}, \quad \text{for } z \in U'_{n,k} \equiv \{z \mid |z - \bar{\alpha}_{n,k}| \leqq \ell_n/6\}$$

and

$$\mathrm{II} < \frac{\ell_n}{2|z - \alpha_{n,k}||z - \bar{\alpha}_{n,k}|} < 18/\ell_n, \quad \text{for } z \in \Delta'_{n,k} - (U_{n,k} \cup U'_{n,k}),$$

so that

$$\mathrm{II} < 36/\ell_n, \qquad \text{for } z \in \Delta'_{n,k}. \tag{10}$$

For $z \in S_{n+1,2k-1} \cup S_{n+1,2k} \cup T_{n,k}$ we have immediately

$$\mathrm{II} < \frac{\ell_n}{2|z - \alpha_{n,k}||z - \bar{\alpha}_{n,k}|}. \tag{11}$$

In order to estimate I and III, we take roughly a lower bound of $|z - \alpha_{m,h}|$ or $|z - \bar{\alpha}_{m,h}|$, $(m, h) \neq (n, k)$. We may without loss of generality suppose that $k = 1$, i.e. $z \in \Delta_{n,1}$.

(i) If $\alpha_{m,h} \in (\Gamma_{p,2})$, $p = 1, 2, 3, \cdots, n$, we have

$$|z - \alpha_{m,h}| \geqq d(\Gamma_{n,1}, \Gamma_{p,2}) \geqq d(\Gamma_{p,1}, \Gamma_{p,2}) \geqq \ell_{p-1}/3, \tag{12}$$

where $d(\Gamma_{p,q}, \Gamma_{r,s})$ denotes the distance between $\Gamma_{p,q}$ and $\Gamma_{r,s}$.

(ii) If $\alpha_{m,h} \in (\Gamma_{n+1,j})$, $j = 1, 2$, we have

$$\begin{cases} |z - \alpha_{m,h}| \geqq (\ell_n/\delta_0) - (\ell_n/2) > \ell_n, & \text{for } z \in T_{n,1}, \\ |z - \alpha_{m,h}| \geqq d(\Gamma_{n+1,j}, \alpha_{m,h}) \geqq \sqrt{\ell_n \ell_{n+1}}/3, & \text{for } z \in \Delta_{n,1} - T_{n,1}. \end{cases} \tag{13}$$

(iii) For the others, i.e. $\alpha_{1,1}, \alpha_{2,1}, \cdots, \alpha_{n-1,1}$, we have

$$|z - \alpha_{m,1}| \geqq d(\Gamma_{n,1}, \alpha_{m,1}) \geqq d(\Gamma_{m+1,1}, \alpha_{m,1}) \geqq \ell_m/3, \tag{14}$$

for $m = 1, 2, \cdots, n-1$. Here we may substitute $\bar{\alpha}_{m,h}$ for $\alpha_{m,h}$ in (12), (13) and (14). We need also

$$\ell_p/\ell_q = \eta_p \eta_{p-1} \cdots \eta_{q+1} < (1/3)^{p-q} \qquad (p > q). \tag{15}$$

Using (12) and (14) we deduce

$$\mathrm{I} = \sum_{\substack{m=1 \\ (m,h)\neq(n,1)}}^{n} \frac{\ell_m}{2} \left(\sum_{h=1,2,\cdots,2^m} \frac{1}{|z - \alpha_{m,h}||z - \bar{\alpha}_{m,h}|} \right)$$

$$= \sum_{m=1}^{n-1} \frac{\ell_m}{2} \left\{ \sum_{p=1}^{m} \left(\sum_{\alpha_{m,h} \in (\Gamma_{p,2})} \frac{1}{|z - \alpha_{m,h}||z - \bar{\alpha}_{m,h}|} \right) + \frac{1}{|z - \alpha_{m,1}||z - \bar{\alpha}_{m,1}|} \right\}$$

$$+ \frac{\ell_n}{2} \sum_{p=1}^{n} \sum_{\alpha_{n,h} \in (\Gamma_{p,2})} \frac{1}{|z - \alpha_{n,h}||z - \bar{\alpha}_{n,h}|}$$

$$\leqq \sum_{m=1}^{n-1} \frac{\ell_m}{2} \left\{ \left(\frac{3}{\ell_m}\right)^2 + \left(\frac{3}{\ell_{m-1}}\right)^2 + 2\left(\frac{3}{\ell_{m-2}}\right)^2 + 2^2\left(\frac{3}{\ell_{m-3}}\right)^2 + \cdots + 2^{m-1}\left(\frac{3}{\ell_0}\right)^2 \right\}$$

$$+ \frac{\ell_n}{2} \left\{ \left(\frac{3}{\ell_{n-1}}\right)^2 + 2\left(\frac{3}{\ell_{n-2}}\right)^2 + \cdots + 2^{n-1}\left(\frac{3}{\ell_0}\right)^2 \right\}.$$

Also in view of (15) we have

$$\text{(16)} \qquad \text{I} < C_6/\ell_{n-1} = C_6\eta_n/\ell_n .$$

Similarly we deduce from (12) and (13)

$$\text{III} < \sum_{m=n+1}^{\infty} 9 \cdot 2^{m-n-1} \frac{\ell_m}{\ell_n^2} \times \left\{ \frac{1}{9} + \left(\frac{\ell_n}{\ell_{n-1}}\right)^2 + 2\left(\frac{\ell_n}{\ell_{n-2}}\right)^2 + \cdots + 2^{n-1}\left(\frac{\ell_n}{\ell_0}\right)^2 \right\}, \quad \text{for } z \in T_{n,1},$$

$$\text{III} < \sum_{m=n+1}^{\infty} 9 \cdot 2^{m-n-1} \frac{\ell_m}{\ell_n \ell_{n+1}} \times \left\{ 1 + \frac{\ell_n \ell_{n+1}}{\ell_{n-1}^2} + 2\frac{\ell_n \ell_{n+1}}{\ell_{n-2}^2} + \cdots + 2^{n-1} \frac{\ell_n \ell_{n+1}}{\ell_0^2} \right\}, \quad \text{for } z \in \Delta_{n,1} - T_{n,1},$$

and so we have

$$\text{(17)} \qquad \begin{cases} \text{III} < C_7\eta_n/\ell_n , & \text{for } z \in T_{n,1}, \\ \text{III} < C_8/\ell_n , & \text{for } z \in \Delta_{n,1} - T_{n,1}, \end{cases}$$

in view of (2) and (15).

Thus summing (10), (11), (16) and (17), we have

$$\text{(18)} \qquad \begin{cases} \rho(f(z)) < \dfrac{C_9}{\ell_n} + \dfrac{\ell_n}{2|z - \alpha_{n,k}||z - \bar{\alpha}_{n,k}|}, & \text{for } z \in S_{n+1,2k-1} \cup S_{n+1,2k}, \\ \rho(f(z)) < \dfrac{C_{10}}{\ell_n}\eta_n + \dfrac{\ell_n}{2|z - \alpha_{n,k}||z - \bar{\alpha}_{n,k}|}, & \text{for } z \in T_{n,k}, \\ \rho(f(z)) < C_{11}/\ell_n , & \text{for } z \in \Delta'_{n,k}. \end{cases}$$

Hence combining (9) and (18), we deduce that

$$\rho(f(z))\frac{|dz|}{d\sigma_\Omega(z)} < C_2\left(3C_9 + \frac{3\ell_n^2}{2|z-\alpha_{n,k}||z-\bar{\alpha}_{n,k}|}\right)$$
$$\times \frac{|z-a_{n,k}|}{3\ell_n}\log\frac{3\ell_n}{|z-a_{n,k}|}, \qquad \text{for } z \in S_{n+1,2k-1},$$

$$\rho(f(z))\frac{|dz|}{d\sigma_\Omega(z)} < C_3\left(3C_9 + \frac{3\ell_n^2}{2|z-\alpha_{n,k}||z-\bar{\alpha}_{n,k}|}\right)$$
$$\times \frac{|z-b_{n,k}|}{3\ell_n}\log\frac{3\ell_n}{|z-b_{n,k}|}, \qquad \text{for } z \in S_{n+1,2k},$$

$$\rho(f(z))\frac{|dz|}{d\sigma_\Omega(z)} < C_4\left(\frac{C_{10}|z-a_{n,k}|^2}{\ell_n^2}\eta_n + \frac{|z-a_{n,k}|^2}{2|z-\alpha_{n,k}||z-\bar{\alpha}_{n,k}|}\right)$$
$$\times \frac{\ell_n}{2|z-a_{n,k}|}\log\frac{2|z-a_{n,k}|}{\ell_n}, \qquad \text{for } z \in T_{n,k}$$

and

$$\rho(f(z))\frac{|dz|}{d\sigma_\Omega(z)} < C_5 C_{11}, \qquad \text{for } z \in \Delta'_{n,k}.$$

Using the simple inequalities:

$$0 < x\log\frac{1}{x} < 1/e, \qquad \text{for } 0 < x < 1,$$
$$|z-a_{n,k}|/\ell_n < 1, \qquad \text{for } z \in S_{n+1,2k-1},$$
$$|z-b_{n,k}|/\ell_n < 1, \qquad \text{for } z \in S_{n+1,2k},$$
$$|z-\alpha_{n,k}| > \ell_n/3, \qquad \text{for } z \in S_{n+1,2k-1} \cup S_{n+1,2k},$$
$$|z-\bar{\alpha}_{n,k}| > \ell_n/3, \qquad \text{for } z \in S_{n+1,2k-1} \cup S_{n+1,2k},$$
$$\frac{1}{4} < \left|\frac{z-a_{n,k}}{z-\alpha_{n,k}}\right| < 4, \qquad \text{for } z \in T_{n,k},$$
$$\frac{1}{4} < \left|\frac{z-a_{n,k}}{z-\bar{\alpha}_{n,k}}\right| < 4, \qquad \text{for } z \in T_{n,k}$$

and

$$\frac{2}{3}\sqrt{\eta_n} < \frac{\ell_n}{|z-a_{n,k}|} < \frac{1}{4}, \qquad \text{for } z \in T_{n,k},$$

we are able to prove that $\rho(f(z))(|dz|/d\sigma_\Omega(z))$ is bounded in $\Omega - \Omega_0$. Further, since $\rho(f(z))(|dz|/d\sigma_\Omega(z))$ is also bounded in a compact set Ω_0, it is bounded in Ω. Thus by Theorem B, we deduce that f is normal in Ω. This completes the proof of Theorem.

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