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THE THETA FUNCTIONS OF SUBLATTICES OF THE LEECH LATTICE

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To the memory of late Takehiko Miyata

Introduction

Let Λ be the Leech lattice which is an even unimodular lattice with no vectors of squared length 2 in 24-dimensional Euclidean space $\boldsymbol{R}^{24}$. Then the Mathieu Group M_{24} is a subgroup of the automorphism group $\cdot 0$ of Λ and the action on Λ of M_{24} induces a natural permutation representation of M_{24} on an orthogonal basis $\{e_i \mid 1 \leqq i \leqq 24\}$ of $\boldsymbol{R}^{24}$. For $m \in M_{24}$, let Λ_m be the sublattice of vectors invariant under m:

$$\Lambda_m = \{x \in \Lambda \mid x^m = x\}$$

and $\Theta_m(z)$ be the theta function of Λ_m:

$$\Theta_m(z) = \sum_{x \in \Lambda_m} e^{\pi i z \ell(x)}$$

where $\ell(x) = \ell(x, x)$ and $\ell(x, y)$ $(x, y \in \boldsymbol{R}^{24})$ is the inner product of $\boldsymbol{R}^{24}$ with $\ell(e_i, e_j) = 2\delta_{ij}$.

One of the purposes of this note is to express $\Theta_m(z)$ explicitly by the classical Jacobi theta functions $\theta_i(z)$ $(i = 2, 3, 4)$ and the Dedekind eta-function. The results are given in Table 2 of Section 2. Furthermore, by using these expressions of $\Theta_m(z)$, we will prove the following theorem:

THEOREM 2.1. *Let $\Theta_m(z)$ $(m \in M_{24})$ be as above and let*

$$\eta_m(z) = \prod_t \eta(tz)^{r_t}$$

where $\eta(z)$ is the Dedekind eta-function

$$\eta(z) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) \qquad (q = e^{\pi i z})$$

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and m has a cycle decomosition $\prod t^{r_t} = 1^{r_1}2^{r_2}\cdots$ *Then the functions* $\Theta_m(z)/\eta_m(z)$ *are modular functions which appear in a moonshine of Fischer-Griess's Monster* [3].

For the statement of this theorem, we refer the readers to [3; p. 315] and Remarks 2.1–2.2 in Section 2 of this paper. In Section 1, we explain how to describe $\Theta_m(z)$ in terms of Jacobi theta functions, where a presentation (1.1) of the Leech lattice (cf. Tasaka [9]) and Table 1 which can be obtained from Todd [11] will be very important. In Section 2 we will prove the results in Table 2 and Theorem 2.1. We note that, in the proof of Theorem 2.1, Table 3 of [3] and a result of Koike [4] are useful. But the main works of Section 2 are the calculations of Jacobi theta functions in which several formulas between them are applied effectively. Some of these formulas can be found in [6] and [7], but we will also use those which may be new, for example

$$\theta_2(z)\theta_2(7z) + \theta_3(z)\theta_3(7z) + \theta_4(z)\theta_4(7z) = 2\{\theta_2(2z)\theta_2(14z) + \theta_3(2z)\theta_3(14z)\}\,,$$

$$4\eta(z)\eta(11z) = \theta_3(z)\theta_3(11z) - \theta_2(z)\theta_2(11z) - \theta_4(z)\theta_4(11z)$$

(cf. (T15) and (T24) of Appendix respectively).

Such formulas are collected and proved in Appendix. In the proofs, Lemma A.1–2 will be fundamental.

§1. Leech lattice and its sublattices

The Leech lattice Λ in the Euclidean space $\boldsymbol{R}^{24}$ can be described as disjoint sum in the following way;

$$\Lambda = \bigcup_{X\in\mathscr{G}} \{(\tfrac{1}{2}e_X + L_0) \cup (\tfrac{1}{4}e_\Omega + \tfrac{1}{2}e_X + L_1)\}\,. \tag{1.1}$$

Some explanations will be needed.

A) The set $\Omega = \{1, 2, \cdots, 24\}$ is a 24-point set and $\mathscr{G} \subset P(\Omega)$ is the (binary) Golay code on Ω. For codes and Golay code, see [2] or [6].

B) The system of vectors $\{e_i;\ i\in\Omega\}$ is the orthogonal 2-frame of $\boldsymbol{R}^{24}$, that is, denoting by $\ell(x)$ the squared length of a vector $x\in\boldsymbol{R}^{24}$, and by $\ell(x, y)$ the corresponding inner product of vectors x and y,

$$\ell(e_i, e_j) = 2\delta_{ij}\,. \tag{1.2}$$

C) We put $L = \sum_{i\in\Omega}\boldsymbol{Z}e_i$, and for $\delta = 0$ or 1, we define

(1.3) $$L_\delta = \{x = \sum x_i e_i \in L;\ \sum x_i \equiv \delta \pmod 2\}\ .$$

Note that, after scaling by $1/\sqrt{2}$, L_0 is isomorphic to the (even) lattice of type D_{24}.

D) For a subset X of Ω, we put

(1.4) $$e_X = \sum_{i \in X} e_i\ .$$

E) The characterization of Leech lattice (cf. [1]) shows that the lattice Λ defined by (1.1) is (isomorphic to) the Leech lattice. (See [9] p. 708). Also the formula

(1.5) $$\ell(\tfrac{1}{2}e_X + \sum x_i e_i) = 2 \sum_{j \notin X} x_j^2 + 2 \sum_{i \in X} x_i(x_i + 1) + \tfrac{1}{2}|X|$$

is useful, where $|X|$ denotes the cardinality of the set X.

The Mathieu group M_{24} is the subgroup of the symmetric group $S_{24} \cong S(\Omega)$ which leaves invariant the Golay code $\mathscr{G}$. The element m of M_{24} operates on the lattice Λ in natural way, that is, $(e_i)^m = e_{im}$ for $i \in \Omega$. Thus

(1.6) $$(\tfrac{1}{2}e_X + \sum x_i e_i)^m = \tfrac{1}{2}e_{Xm} + \sum x_i e_{im}\ ,$$

(1.7) $$(\tfrac{1}{4}e_\Omega + \tfrac{1}{2}e_X + \sum x_i e_i)^m = \tfrac{1}{4}e_\Omega + \tfrac{1}{2}e_{Xm} + \sum x_i e_{im}\ .$$

In this way, the group M_{24} is a subgroup of the group $\cdot 0$ of Conway which is the automorphism group of the Leech lattice Λ. In view of [10] and [3; p. 315], it is important to study the invariant sublattice Λ_m and its theta function $\Theta_m(z)$ for all element m of $\cdot 0$. Here we restrict ourselves to the element m of the Mathieu group M_{24}. That is, for twenty-one "rational" conjugate classes of M_{24}, the theta functions $\Theta_m(z)$ of invariant sublattices Λ_m will be expressed as homogeneous polynomials of Jacobi's theta functions.

For an element m of M_{24}, considered as an element of S_{24}, let

(1.8) $$m = (U_1)(U_2) \cdots (U_s)$$

be its cycle decomposition, where U_j are subsets of Ω, giving a disjoint sum decomposition of Ω, and (U_j) are certain cyclic permutations on U_j. That is, if we write $U_j = \{i_1, i_2, \cdots, i_t\}$ in appropriate order, then $(U_j) = (i_1 i_2 \cdots i_t)$. The class of m can be written as

$$m = |U_1||U_2| \cdots |U_s|\ ,$$

where $|U_j|$ means the cardinality of U_j. Thus $m = 1^8 2^8$ means that m is a product of eight mutually commutative transpositions, fixing the remaining eight points. Also $m = 3^8$ means that m is a product of eight mutually disjoint cycles of length three, fixing no points, and so on.

From (1.6) and (1.7), it follows that $x = \frac{1}{2}e_X + \sum x_i e_i$ (or $y = \frac{1}{4}e_\Omega + \frac{1}{2}e_X + \sum y_i e_i$) is invariant under m if and only if, first the code word X (the subset X contained in the Golay code $\mathscr{G}$) is invariant under m, secondly $x_i = x_j$ (or $y_i = y_j$) if $i, j \in U_k$, and finally $\sum x_i \equiv 0 \pmod 2$ (or $\sum y_i \equiv 1$ (mod 2)). In this case, we have

$$\sum_i x_i = \sum_k |U_k| x_{i(k)} ,$$

for example, where $i(k)$ is a representative in each U_k. On the other hand, it is clear that a code word X is invariant under m if and only if the disjoint sum decomposition $\Omega = \bigcup U_k$ is a refinement of the decomposition $\Omega = X \cup (\Omega - X)$. We devide the subsets U_k into four categories with respect to the code word X. That is, if $U_k \subset X$ and $|U_k|$ is even, then U_k is called first category (type I). If $U_k \subset X$ and $|U_k|$ is odd, then U_k is called second category (type II). If $U_k \subset (\Omega - X)$ and $|U_k|$ is even, then U_k is called third category (type III). Finally if $U_k \subset (\Omega - X)$ and $|U_k|$ is odd, then U_k is called fourth category (type IV).

Under these notations, the m-invariant vector x (or y) can be written as

$$x = \tfrac{1}{2}e_X + \sum x_k e_{U_k} \qquad (\text{or } y = \tfrac{1}{4}e_\Omega + \tfrac{1}{2}e_X + \sum y_k e_{U_k}) ,$$

where the condition $\sum x_i \equiv 0 \pmod 2$ (or $\sum y_i \equiv 1 \pmod 2$) is rewritten as

$$\textstyle\sum^{(\mathrm{II})} x_k + \sum^{(\mathrm{IV})} x_k \equiv 0 \pmod 2 \ (\text{or } \sum^{(\mathrm{II})} y_k + \sum^{(\mathrm{IV})} y_k \equiv 1 \pmod 2) .$$

Thus, denoting by $\mathscr{G}_m$ the m-invariant subgroup (subcode) of the Golay code $\mathscr{G}$, we have

$$\Lambda_m = \bigcup_{X \in \mathscr{G}_m} \{(\tfrac{1}{2}e_X + (L_0)_m) \cup (\tfrac{1}{4}e_\Omega + \tfrac{1}{2}e_X + (L_1)_m)\} , \tag{1.9}$$

(disjoint sum decomposition), where

$$(L_0)_m = \{\textstyle\sum x_k e_{U_k};\ \sum^{(\mathrm{II})} x_k + \sum^{(\mathrm{IV})} x_k \equiv 0 \pmod 2\} , \tag{1.10}$$

$$(L_1)_m = \{\textstyle\sum y_k e_{U_k};\ \sum^{(\mathrm{II})} y_k + \sum^{(\mathrm{IV})} y_k \equiv 1 \pmod 2\} . \tag{1.11}$$

Note that if the type II and the type IV are void then the set $(L_1)_m$ is an empty set. Thus if m does not contain cycles of odd length, then

(1.9)′ $$\Lambda_m = \bigcup_{X \in \mathscr{C}_m} (\tfrac{1}{2}e_X + (L_0)_m) ,$$

where, in this case

(1.10)′ $$(L_0)_m = \sum \boldsymbol{Z} e_{U_k} .$$

For a (discrete) point set X in Euclidean space $\boldsymbol{R}^N$, we define its theta function $\Theta(X, z) = \Theta_X(z)$ (with respect to the origin 0) as

$$\Theta(X, z) = \sum_{x \in X} e^{\pi i z \ell(x)} = \sum q^{\ell(x)} ,$$

where z is a complex number such that $\mathrm{Im}(z) > 0$ and $q = e^{\pi i z}$, so that $|q| < 1$. Note that we are interested in the cases where the right hand side is convergent. It is easy to see that

(1.12) $$\Theta(X \cup Y, z) = \Theta(X, z) + \Theta(Y, z) ,$$

for "disjoint sum" $X \cup Y$. And also

(1.13) $$\Theta(X \times Y, z) = \Theta(X, z)\Theta(Y, z) ,$$

if X and Y are contained in mutually orthogonal (linear) subspaces.

Jacobi's theta functions (theta zeros) are defined in the following way:

(1.14) $$\theta_3(z) = \sum_{n \in \boldsymbol{Z}} e^{\pi i z n^2} = \sum q^{n^2} ,$$

(1.15) $$\theta_4(z) = \sum_n (-1)^n q^{n^2} ,$$

(1.16) $$\theta_2(z) = \sum_n q^{(n+1/2)^2} ,$$

where $\mathrm{Im}(z) > 0$ and $q = e^{\pi i z}$. Here we define two more functions $\rho_0(z)$ and $\rho_1(z)$ as

(1.17) $$\rho_0(z) = \sum_n q^{(n+1/4)^2} ,$$

(1.18) $$\rho_1(z) = \sum_n (-1)^n q^{(n+1/4)^2} .$$

It is clear that $\theta_3(z)$, $\theta_2(z)$ and $\rho_0(z)$ are the theta functions of $\boldsymbol{Z}$, $\boldsymbol{Z} + \frac{1}{2} = \{n + \frac{1}{2};\, n \in \boldsymbol{Z}\}$, and $\boldsymbol{Z} + \frac{1}{4}$, respectively. It is easy to see that $\boldsymbol{Z} - \frac{1}{4}$ has $\rho_0(z)$ as its theta function, from its symmetry. Using these functions and $\theta_4(z)$ and $\rho_1(z)$, we can express the theta functions of point sets of various type.

Assume that m contains cycles of odd length. Then from (1.11), it follows that, for $Y = \frac{1}{4}e_\Omega + \frac{1}{2}e_X + (L_1)_m$,

$$Y = \sum^{(\mathrm{I})} (Z - \tfrac{1}{4})e_{U_k} + \sum^{(\mathrm{III})} (Z + \tfrac{1}{4})e_{U_k}$$
$$+ \{\sum^{(\mathrm{II})}(Z - \tfrac{1}{4})e_{U_k} + \sum^{(\mathrm{IV})} (Z + \tfrac{1}{4})e_{U_k}\}_1 .$$

Thus we have

$$(1.19) \qquad \Theta(Y, z) = \prod{}^{(\mathrm{I})\cup(\mathrm{III})}\rho_0(2|U_k|z)$$
$$\times \tfrac{1}{2} \times \{\prod{}^{(\mathrm{II})\cup(\mathrm{IV})}\rho_0(2|U_k|z) - \prod{}^{(\mathrm{II})\cup(\mathrm{IV})}\rho_1(2|U_k|z)\} ,$$

for $Y = \frac{1}{4}e_\Omega + \frac{1}{2}e_X + (L_1)_m$. See the remarks below for the details. The right hand side of this formula is independent of the code word X. So the contribution of these sets to the theta function of Λ_m is $|\mathscr{G}_m|$ times of (1.19).

For the set $X = \frac{1}{2}e_X + (L_0)_m$, its theta function $\Theta(X, z)$ can be described in the similar way. That is,

$$(1.20) \qquad \Theta(X, z) = \prod{}^{(\mathrm{I})} \theta_2(2|U_k|z) \times \prod{}^{(\mathrm{III})} \theta_3(2|U_k|z)$$
$$\times \tfrac{1}{2} \times \prod{}^{(\mathrm{II})} \theta_2(2|U_k|z) \times \prod{}^{(\mathrm{IV})} \theta_3(2|U_k|z) ,$$

if the type II is not void, and

$$(1.21) \qquad \Theta(X, z) = \prod{}^{(\mathrm{I})} \theta_2(2|U_k|z) \times \prod{}^{(\mathrm{III})}\theta_3(2|U_k|z)$$
$$\times \tfrac{1}{2} \times \{\prod{}^{(\mathrm{IV})} \theta_3(2|U_k|z) + \prod{}^{(\mathrm{IV})} \theta_4(2|U_k|z)\} ,$$

if the type II is void. Note that if the type II and IV are void (that is, m does not contain cycles of odd length), then

$$(1.22) \qquad \Theta(X, z) = \prod{}^{(\mathrm{I})} \theta_2(2|U_k|z) \times \prod{}^{(\mathrm{III})} \theta_3(2|U_k|z) .$$

If the type I or III 'is void, the corresponding terms are to be replaced by 1.

Summing up all these contributions, we get the theta function $\Theta(\Lambda_m, z) = \Theta_m(z)$. That is,

(Ξ) *The theta function* $\Theta_m(z)$ *is expressed as the sum of terms given by* (1.19) *and* (1.20) (*or* (1.21) *or* (1.22)) *for all code words* $X \in \mathscr{G}_m$.

Remark 1. The exact structure of invariant subcode $\mathscr{G}_m$ for each m is discussed in the subsequent paragraphs,

Remark 2. It is clear that $\theta_3(z)^n$ is the theta function of Z^n with respect to the standard metric. The function $\theta_4(z)^n$ is the "theta function" of Z^n with weight $(-1)^{\Sigma x_i}$ at each point $x = (x_1, x_2, \cdots, x_n) \in Z^n$. Thus $\frac{1}{2}(\theta_3(z)^n + \theta_4(z)^n)$ is the "normal" theta function of $(Z^n)_0$, and $\frac{1}{2}(\theta_3(z)^n - \theta_4(z)^n)$ is the one of $(Z^n)_1$, where $(Z^n)_\delta = \{x = (x_1, \cdots, x_n) \in Z^n; \sum x_i \equiv \delta$

(mod 2)}, for $\delta = 0$ or 1. Note that $(\boldsymbol{Z}^n)_0$ is the even lattice of type D_n.

Concerning to our 2-frame $\{e_i\}$, as $\ell(e_i) = 2$, the theta function of $L = \sum \boldsymbol{Z}e_i$ is $\theta_3(2z)^{24}$, for example.

Remark 3. The theta function of $(\sum(\boldsymbol{Z} + \frac{1}{2})e_{U_k})_0$ is derived in the similar way. But, in this case, as

$$\sum(-1)^n q^{(n+1/2)^2} = 0 ,$$

this theta function is equal to $\frac{1}{2}\prod\theta_2(2|U_k|z)$. The same reasoning is used for the formula (1.19).

Remark 4. Similarly, for a natural number p, we define

$$\Theta^{(p)}(z) = \theta_3(z)\theta_3(pz) + \theta_2(z)\theta_2(pz) . \tag{1.23}$$

This is the theta function of $(\boldsymbol{Z}e + \boldsymbol{Z}f) \cup \{\frac{1}{2}(e+f) + \boldsymbol{Z}e + \boldsymbol{Z}f\}$, where $\ell(e) = 1$, $\ell(f) = p$ and $\ell(e, f) = 0$. If p is a prime number such that $p \equiv 3 \pmod 4$, then this set is the integer ring of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-p})$, considered as a lattice in $\boldsymbol{C} \cong \boldsymbol{R} \times \boldsymbol{R}$ in natural way. The cases $p = 3, 7, 11$ and 23 will appear in the next section.

We call 8-point subset X of Ω an octad if X belongs to the Golay code $\mathscr{G}$. Also 12-point subset belonging to $\mathscr{G}$ is called a dodecad. Next 16-point subset belonging to $\mathscr{G}$ will be called co-octad. A co-octad is actually the complementary subset of an octad. The Golay code $\mathscr{G}$ consists of one $0 = \emptyset$ (the empty subset), 759 octads and co-octads, 2576 dodecads and one Ω (the full subset). This will be written as

$$\begin{aligned} \mathscr{G} &= 1(\emptyset) + 759(\text{octad}) + 2576(\text{dodecad}) + 579(\text{co-octad}) + 1(\Omega) \\ &= 1 + 759 + 2576 + 759 + 1. \end{aligned} \tag{1.24}$$

For each class m, the invariant subcode $\mathscr{G}_m$ is described in the similar way, specifying its code words (octads, dodecads or co-octads) by its cycle types. For example, if $m = 1^8 2^8$, then

$$\begin{aligned} \mathscr{G}_m = 1\{\emptyset\} &+ \{(1^8) + 14(2^4) + 56(1^4 2^2)\} + 112\{(1^4 2^4)\} \\ &+ \{(2^8) + 14(1^8 2^4) + 56(1^4 2^6)\} + 1\{\Omega\} . \end{aligned}$$

This means that the set of octads in $\mathscr{G}_m$ consists of one 1^8 (the fixed point set of m), fourteen 2^4 and fifty-six $1^4 2^2$, for example. Also if $m = 1^6 3^6$, then

$$\mathscr{G}_m = 1\{\emptyset\} + \{6(1^5 3) + 15(1^2 3^2)\} + 20\{(1^3 3^3)\} + \{6(1^1 3^5) + 15(1^4 3^4)\} + 1\{\Omega\} .$$

These can be obtained from the table of Todd's paper [11]. In the Table 1, the description of $\mathscr{G}_m$ for each class m is given in this fashion. It is notable that $|\mathscr{G}_m| = 2^{s/2}$, where s is the even integer determined in (1.8). Using this table and (Ξ), we can describe the theta function $\Theta_m(z)$ completely. (This will be done in the next section).

Table 1

1^{24}	$1 + 759 + 2576 + 759 + 1$
$1^8 2^8$	$1 + \{1^8 + 14(2^4) + 56(1^4 2^2)\} + 112\{1^4 2^4\} + \{2^8 + 14(1^8 2^4) + 56(1^4 2^6)\} + 1$
$1^6 3^6$	$1 + \{6(1^5 3) + 15(1^2 3^2)\} + 20\{1^3 3^3\} + \{6(1.3^5) + 15(1^4 3^4)\} + 1$
$1^4 2^2 4^4$	$1 + \{1^4 2^2 + 2(4^2) + 8(1^2 2\cdot 4)\} + \{4(1^4 4^2) + 4(2^2 4^2)\}$ $+ \{4^4 + 2(1^4 2^2 4^2) + 8(1^2 2\cdot 4^3)\} + 1$
$1^4 5^4$	$1 + 4(1^3 5) + 6(1^2 5^2) + 4(1.5^3) + 1$
$1^2 2^2 3^2 6^2$	$1 + \{1^2 3^2 + 2(1\cdot 2^2 3) + 2(2.6)\} + 4(1\cdot 2\cdot 3\cdot 6)$ $+ \{2^2 6^2 + 2(1\cdot 3\cdot 6^2) + 2(1^2 2\cdot 3^2 6)\} + 1$
$1^3 7^3$	$1 + 3(1.7) + 0 + 3(1^2 7^2) + 1$
$1^2 2\cdot 4\cdot 8^2$	$1 + (1^2 2\cdot 4) + \{2(4\cdot 8) + 2(1^2 2\cdot 8)\} + (8^2) + 1$
$1^2 11^2$	$1 + 0 + 2(1\cdot 11) + 0 + 1$
$1\cdot 2\cdot 7\cdot 14$	$1 + (1\cdot 7) + 0 + (2\cdot 14) + 1$
$1\cdot 3\cdot 5\cdot 15$	$1 + (3\cdot 5) + 0 + (1\cdot 15) + 1$
$1\cdot 23$	$1 + 0 + 0 + 0 + 1$
2^{12}	$1 + 15(2^4) + 32(2^6) + 15(2^8) + 1$
3^8	$1 + 0 + 14(3^4) + 0 + 1$
$2^4 4^4$	$1 + \{2^4 + 6(4^2)\} + 0 + \{4^4 + 6(2^4 4^2)\} + 1$
4^6	$1 + 3(4^2) + 0 + 3(4^4) + 1$
6^4	$1 + 0 + 2(6^2) + 0 + 1$
$2^2 10^2$	$1 + 0 + 2(2\cdot 10) + 0 + 1$
$2\cdot 4\cdot 6\cdot 12$	$1 + (2\cdot 6) + 0 + (4\cdot 12) + 1$
12^2	$1 + 0 + 0 + 0 + 1$
$3\cdot 21$	$1 + 0 + 0 + 0 + 1$

EXAMPLE 1.1. For $m = 2^{12}$, we use (1.22) and

$$\mathscr{G}_m = 1 + 15(2^4) + 32(2^6) + 15(2^8) + 1.$$

So we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(4z)^{12} + 15 \times \theta_3(4z)^8\theta_2(4z)^4 + 32 \times \theta_3(4z)^6\theta_2(4z)^6 \\ &\quad + 15 \times \theta_3(4z)^4\theta_2(4z)^8 + \theta_2(4z)^{12} \\ &= \tfrac{1}{2}\{(\theta_3(4z)^2 + \theta_2(4z)^2)^6 + (\theta_3(4z)^2 - \theta_2(4z)^2)^6\} + 32\theta_3(4z)^6\theta_2(4z)^6 .\end{aligned}$$

From (T4–6) of appendix, we have $\theta_3(4z)^2+\theta_2(4z)^2=\theta_3(2z)^2$ and $\theta_3(4z)^2-\theta_2(4z)^2=\theta_4(2z)^2$ and $2\theta_2(4z)\theta_3(4z)=\theta_2(2z)^2$. So we have

$$\Theta_m(z)=\tfrac{1}{2}\{\theta_3(2z)^{12}+\theta_2(2z)^{12}+\theta_4(2z)^{12}\}\ . \tag{1.25}$$

EXAMPLE 1.2. For the class $m=1^8 2^8$, the contributions of types $\frac{1}{4}e_\Omega+\frac{1}{2}e_X+(L_1)_m$ is $|\mathcal{G}_m|=2^8$ times of

$$P=\rho_0(4z)^8\times\tfrac{1}{2}(\rho_0(2z)^8-\rho_1(2z)^8)\ ,$$

by the formula (1.19). Using (T3) and (T8–9) and also (T11), we have

$$256P=128\times 2^{-12}\theta_2(z)^{12}(\theta_3(z)^4-\theta_4(z)^4)=2^{-5}\theta_2(z)^{16}\ , \tag{1.26}$$

For the calculus of remaining terms, we put

$$E_4(z)=\tfrac{1}{2}\{\theta_3(z)^8+\theta_2(z)^8+\theta_4(z)^8\}\ .$$

From code word $\{\emptyset\}+\{\Omega\}$ and $\{1^8\}+\{2^8\}$, we have

$$\begin{aligned}Q_1&=\theta_3(4z)^8\times\tfrac{1}{2}(\theta_3(2z)^8+\theta_4(2z)^8)+\theta_2(4z)^8\times\tfrac{1}{2}\theta_2(2z)^8\\&\quad+\tfrac{1}{2}\theta_3(4z)^8\theta_2(2z)^8+\tfrac{1}{2}\theta_2(4z)^8(\theta_3(2z)^8+\theta_4(2z)^8)\\&=E_4(2z)(\theta_3(4z)^8+\theta_2(4z)^8)\ .\end{aligned}$$

From $14\{2^4\}+14\{1^8 2^4\}$, we have

$$\begin{aligned}Q_2&=7\theta_2(4z)^4\theta_3(4z)^4(\theta_3(2z)^8+\theta_4(2z)^8)+7\theta_2(4z)^4\theta_3(4z)^4\theta_2(2z)^8\\&=14E_4(2z)\theta_2(4z)^4\theta_3(4z)^4\ .\end{aligned}$$

From $56\{1^4 2^2\}+56\{1^4 2^6\}$ and $112\{1^4 2^4\}$, we have

$$\begin{aligned}Q_3&=28\theta_2(2z)^4\theta_2(4z)^2\theta_3(2z)^4\theta_3(4z)^6+28\theta_2(2z)^4\theta_2(4z)^6\theta_3(2z)^4\theta_3(4z)^2\\&\quad+56\theta_2(2z)^4\theta_2(4z)^4\theta_3(2z)^4\theta_3(4z)^4\\&=28\theta_2(2z)^4\theta_2(4z)^2\theta_3(2z)^4\theta_3(4z)^2(\theta_2(4z)^2+\theta_3(4z)^2)^2\\&=7\theta_2(2z)^8\theta_3(2z)^8=\tfrac{7}{256}\theta_2(z)^{16}\ ,\end{aligned}$$

using (T4) and (T5). Summing up all terms, we have

$$\Theta_m(z)=E_4(2z)\{\theta_3(4z)^8+14\theta_2(4z)^4\theta_3(4z)^4+\theta_2(4z)^8\}+\tfrac{15}{256}\theta_2(z)^{16}\ .$$

As one can see easily from (T4–7) that

$$\theta_3(4z)^8+14\theta_3(4z)^4\theta_2(4z)^4+\theta_2(4z)^8=\tfrac{1}{2}\{\theta_3(2z)^8+\theta_2(2z)^8+\theta_4(2z)^8\}\ ,$$

so we have

$$\Theta_m(z)=E_4(2z)^2+\frac{15}{256}\theta_2(z)^{16}\ . \tag{1.27}$$

§ 2. Conway-Norton's conjecture

In this section, we will prove the following theorem:

THEOREM 2.1. *For* $m \in M_{24}$, *let* $\Theta_m(z)$ *be the theta function of the invariant sublattice* Λ_m *as in* Section 1 *and let*

$$\eta_m(z) = \prod_t \eta(tz)^{r_t}$$

where $\eta(z) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n})$ $(q = e^{\pi i z})$ *and* m *has a cycle decomposition* $\prod_t t^{r_t} = 1^{r_1} 2^{r_2} \cdots$. *Then the functions* $\Theta_m(z)/\eta_m(z)$ *are modular functions which appear in a moonshine of Fischer-Griess's Monster constructed in* [3].

Remark 2.1. In [3], the statement of this theorem was conjectured for any elements of $\cdot 0$ (= the automorphism group of Leech lattice) [3; p. 315]. But Koike has checked that, for some elements of $\cdot 0$, similar statements are not necessarily true.

Remark 2.2. In [4], Koike proved that, for all $m \in M_{24}$, there exist modular forms $\theta_m(z)$ such that $\theta_m(z)/\eta_m(z)$ are modular functions which appear in a moonshine of Fischer-Griess's Monster. These modular forms $\theta_m(z)$ exactly coincide with our theta-functions $\Theta_m(z)$ (cf. [4; Table I and Table II]).

The proof of this theorem will be done by showing that $\Theta_m(z)$ can be expressed as in the following Table 2 and then using Table 3 of [3] or a result of Koike [4] (see Theorem 2.2 below). But for an element m of M_{24} with a cycle decomposition $1^4 5^4$, this method does not work well and so we will check the case $m = 1^4 5^4$ by comparing the Fourier coefficients of our $\Theta_m(z)$ and Koike's $\theta_m(z)$ in [4].

Now we will give a table of expressions of $\Theta_m(z)$ by Jacobi theta functions. Also, in this table, discrete subgroups Γ_m for function fields $\boldsymbol{C}(\Theta_m(z)/\eta_m(z))$ and the corresponding conjugacy classes in Fischer-Griss's Monster are given by using the notations in [3]. Also we use the following notations:

(2.1) $$E_4(z) = \tfrac{1}{2}\{\theta_2(z)^8 + \theta_3(z)^8 + \theta_4(z)^8\}$$
= the theta function of the E_8-lattice (cf. [6; p. 134])

(2.2) $$\theta_1'(z) = \theta_2(z)\theta_3(z)\theta_4(z) = 2\eta(z)^3 \qquad \text{(cf. (A22))}$$

(2.3) $$\Theta^{(p)}(z) = \theta_2(z)\theta_2(pz) + \theta_3(z)\theta_3(pz)$$

Table 2

m	$\Theta_m(z)$	Γ_m
1^{24}	$E_4(z)^3 - \frac{45}{16}\theta_1'(z)^8$	1+ (1A)
$1^8 2^8$	$E_4(2z)^2 + \frac{15}{256}\theta_2(z)^{16}$	
	$= \{\frac{1}{2}(\theta_3(z)^4 + \theta_4(z)^4)\}^4 - \frac{3}{8}(\theta_2(z)\theta_4(2z))^8$	14+ (14A)
$1^6 3^6$	$\Theta^{(3)}(2z)^6 - \frac{9}{4}(\theta_1'(z)\theta_1'(3z))^2$	3+ (3A)
$1^4 2^2 4^4$	$\theta_3(2z)^{10} - \frac{5}{4}\theta_2(2z)^4\theta_4(2z)^2\theta_4(4z)^4$	4+ (4A)
$1^4 5^4$	$\frac{1}{2}(\varphi_2^4\hat{\varphi}_2^4 + \varphi_3^4\hat{\varphi}_3^4 + \varphi_4^4\hat{\varphi}_4^4) + 3\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3(2\varphi_2^2\hat{\varphi}_3^2 + \varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3$	
	$+ 2\varphi_3^2\hat{\varphi}_2^2) \quad \varphi_i = \theta_i(2z), \quad \hat{\varphi}_i = \theta_i(10z)$	5+ (5A)
$1^2 2^2 3^2 6^2$	$(\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2 - \frac{3}{4}(\theta_2(z)\theta_2(3z)\theta_4(2z)\theta_4(6z))^2$	6+ (6A)
$1^3 7^3$	$\Theta^{(7)}(2z)^3 - \frac{3}{2}\theta_1'(z)\theta_1'(7z)$	7+ (7A)
$1^2 2\cdot 4\cdot 8^2$	$\theta_3(2z)^3\theta_3(4z)^3 - \frac{3}{4}\theta_2(2z)^2\theta_2(4z)\theta_4(2z)\theta_4(4z)^2$	8+ (8A)
$1^2 11^2$	$\Theta^{(11)}(2z)^2 - \frac{1}{4}(\theta_2\hat{\theta}_2 - \theta_3\hat{\theta}_3 + \theta_4\hat{\theta}_4)^2 \quad \hat{\theta}_i = \theta_i(11z)$	11+ (11A)
$1\cdot 2\cdot 7\cdot 14$	$\Theta^{(7)}(2z)\Theta^{(7)}(4z) - \frac{1}{2}\theta_2(z)\theta_2(7z)\theta_4(2z)\theta_4(14z)$	14+ (14A)
$1\cdot 3\cdot 5\cdot 15$	$\Theta^{(3)}(2z)\Theta^{(3)}(10z) - \frac{3}{2}\psi(2z)\psi(6z)$	
	$\psi(z) = \theta_2(z)\theta_3(5z) - \theta_3(z)\theta_2(5z)$	15+ (15A)
$1\cdot 23$	$\Theta^{(23)}(2z) - 2\eta_m(z)$	23+ (23A)
2^{12}	$\frac{1}{2}(\theta_2(2z)^{12} + \theta_3(2z)^{12} + \theta_4(2z)^{12}) = \theta_3(2z)^{12} - \frac{3}{2}\theta_1'(2z)^4$	4+ (4A)
3^8	$E_4(3z)$	3/3 (3C)
$2^4 4^4$	$(\frac{1}{2}(\theta_3(2z)^4 + \theta_4(2z)^4))^2$	4/2 (4B)
4^6	$\theta_3(4z)^6$	8/2 (8B)
6^4	$\theta_3(6z)^4$	12/3+ (12D)
$2^2 10^2$	$(\frac{1}{2}(\theta_3(z)\theta_3(5z) + \theta_4(z)\theta_4(5z)))^2$	20+ (20A)
$2\cdot 4\cdot 6\cdot 12$	$\Theta^{(3)}(4z)\Theta^3(8z)$	12/2+ (12C)
12^2	$\theta_3(12z)^2$	24/6+ (24E)
$3\cdot 21$	$\Theta^{(7)}(6z)$	21/3+ (21C)

The following theorem is a consequence of Koike [4; Proposition 2.2] which is useful for our proof of Theorem 2.1.

THEOREM 2.2. *Let* m, $\theta_m(z)$ *and* Γ_m *be elements of* M_{24}, *functions and discrete subgroups of* $SL(2.\mathbf{R})$ *defined in the following table respectively. Then* $\theta_m(z)/\eta_m(z)$ *is a generator of a function field corresponding to* Γ_m *which is of genus* 0:

m	$\theta_m(z)$	Γ_m
$1^8 2^8$	$(\frac{1}{2}(\theta_3(z)^4 + \theta_4(z)^4))^4$	$2+$
$1^6 3^6$	$\Theta^{(3)}(2z)^6$	$3+$
$1^2 2^2 3^2 6^2$	$(\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2$	$6+$
$1^3 7^3$	$\Theta^{(7)}(2z)^3$	$7+$
$1^2 11^2$	$\Theta^{(11)}(2z)^2$	$11+$
$1\cdot 2\cdot 7\cdot 14$	$\Theta^{(7)}(2z)\Theta^{(7)}(4z)$	$14+$
$1\cdot 3\cdot 5\cdot 15$	$\Theta^{(3)}(2z)\Theta^{(3)}(10z)$	$15+$
$1\cdot 23$	$\Theta^{(23)}(2z)$	$23+$

Proof. We see from Table 3 of [3] that Γ_m is of genus 0. Let $\theta(z; A)$ be the theta function of an even integral, positive definite matrix A:

$$\theta(z; A) = \sum_{x \in \boldsymbol{Z}^n} e^{\pi i z({}^t x A x)} \qquad (n = \text{the degree of } A)$$

A result of Koike [4; Proposition 2.2] implies that a generator of a function field for Γ_m can be expressed in terms of $\theta(z; A)$, where

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 1 \\ 1 & (p+1)/2 \end{pmatrix} \quad (p = 3, 7, 11 \text{ or } 23)\,.$$

These are positive definite symmetric matrices associated with a lattice of type D_4 or lattices of the ring of integers of $\boldsymbol{Q}(\sqrt{-p})$ (cf. Remark 2 or 4 in § 1) and so we have

$$(*) \qquad \theta(z; A) = \tfrac{1}{2}(\theta_3(z)^4 + \theta_4(z)^4) \quad \text{or} \quad \Theta^{(p)}(2z)$$

respectively. Now Theorem 2.2 follows from Koike's result and $(*)$.

q.e.d.

Now we will begin the proof of Theorem 2.1.

(1) Let $m = 1^8 2^8$. Then by (1.27), we have

$$(\sharp) \qquad \Theta_m(z) = E_4(2z)^2 + \frac{15}{256}\theta_2(z)^{16}\,.$$

On the other hand, we have, putting $\theta_i = \theta_i(z)$ $(i = 2, 3, 4)$,

$$\begin{aligned} E_4(2z) &= \tfrac{1}{2}\sum_{i=2}^{4} \theta_i(2z)^8 \\ &= (\theta_3^8 + 14\theta_3^4\theta_4^4 + \theta_4^8)/16 \qquad \text{by (T5–6)} \end{aligned}$$

and

$$\eta_m(z) = \theta_2(z)^8\theta_4(2z)^8/256 \qquad \text{by (T20) \& (T22)}$$

Then, by using (T7) and (T11), we get

$$\Theta_m(z) + 96\eta_m(z) = \{\tfrac{1}{2}(\theta_3(z)^4 + \theta_4(z)^4)\}^4 .$$

Then from Theorem 2.2 we get Theorem 2.1 for $m = 1^8 2^8$.

Now we will give another proof of Theorem 2.1 for $m = 1^8 2^8$. We have

$$\begin{aligned} E_4(2z) &= (\theta_3^8 + 14\theta_3^4\theta_4^4 + \theta_4^8)/16 \\ &= \theta_4(2z)^8 + \{\tfrac{1}{4}(\theta_3^4 - \theta_4^4)\}^2 && \text{(T7)} \\ &= \theta_4(2z)^8 + \theta_2^8/16 && \text{(T11)} \\ &= \eta(z)^{16}/\eta(2z)^8 + 16\eta(2z)^{16}/\eta(z)^8. && \text{(T20) \& (T22)} \end{aligned}$$

Then from this and (♯), we get directly

$$\Theta_m(z)/\eta_m(z) = \eta(z)^{24}/\eta(2z)^{24} + 4096\eta(2z)^{24}/\eta(z)^{24} + 32$$

which is a generator for 2+ by Table 3 of [3].

(2) Let $m = 1^6 3^6$. Set

$$\varphi_i = \theta_i(2z) \quad \text{and} \quad \hat{\varphi}_i = \theta_i(6z) .$$

By the statement (Ξ) in Section 1 and Table 1, we have

$$(\sharp) \qquad \begin{aligned} \Theta_m(z) = \tfrac{1}{2}\{&(\varphi_3\hat{\varphi}_3)^6 + (\varphi_4\hat{\varphi}_4)^6\} \\ &+ 6 \times \tfrac{1}{2}\varphi_2^5\hat{\varphi}_2\varphi_3\hat{\varphi}_3^5 + 15 \times \tfrac{1}{2}(\varphi_2\hat{\varphi}_2)^2(\varphi_3\hat{\varphi}_3)^4 + 20 \times \tfrac{1}{2}(\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3)^3 \\ &+ 6 \times \tfrac{1}{2}\varphi_2\hat{\varphi}_2^5\varphi_3^5\hat{\varphi}_3 + 15 \times \tfrac{1}{2}(\varphi_2\hat{\varphi}_2)^4(\varphi_3\hat{\varphi}_3)^2 + \tfrac{1}{2}(\varphi_2\hat{\varphi}_2)^6 \\ &+ 64 \times \tfrac{1}{2}\{\rho_0(2z)^6\rho_0(6z)^6 - \rho_1(2z)^6\rho_1(6z)^6\} . \end{aligned}$$

Now we will show

$$(*) \qquad \Theta_m(z) = \Theta^{(3)}(2z)^6 - 36\eta_m(z) \qquad (m = 1^6 3^6)$$

which, by Theorem 2.2, yields Theorem 2.1 for $m = 1^6 3^6$. In the proof of this equation, the identity

$$\text{(T12)} \qquad \theta_2(z)\theta_2(3z) + \theta_4(z)\theta_4(3z) = \theta_3(z)\theta_3(3z)$$

will be useful. Now we will calculate parts of the right hand side of (♯) in the following (i), (ii) and (iii).

$$\text{(i)} \qquad \begin{aligned} &\tfrac{1}{2}\{(\rho_0(2z)\rho_0(6z))^6 - (\rho_1(2z)\rho_1(6z))^6\} \\ &= 2^{-7}(\theta_2\hat{\theta}_2)^3\{(\theta_3\hat{\theta}_3)^3 - (\theta_4\hat{\theta}_4)^3\} && \text{(T8–9)} \\ &= 2^{-7}(\theta_2\hat{\theta}_2)^3\{(\theta_3\hat{\theta}_3 - \theta_4\hat{\theta}_4)^3 + 3\theta_3\hat{\theta}_3\theta_4\hat{\theta}_4(\theta_3\hat{\theta}_3 - \theta_4\hat{\theta}_4)\} \end{aligned}$$

$$= 2^{-7}\{(\theta_2\hat{\theta}_2)^6 + 3(\theta_2\hat{\theta}_2)^4\theta_3\hat{\theta}_3\theta_4\hat{\theta}_4\} \qquad \text{(T12)}$$
$$= \tfrac{1}{2}(\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3)^3 + \tfrac{3}{8}(\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3)^2(\varphi_4\hat{\varphi}_4)^2 \qquad \text{(T4) \& (T7)}$$
$$= \tfrac{1}{8}\{3(\varphi_2\hat{\varphi}_2)^2(\varphi_3\hat{\varphi}_3)^4 - 2(\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3)^3 + 3(\varphi_2\hat{\varphi}_2)^2(\varphi_3\hat{\varphi}_3)^4\} \qquad \text{(T12)}\,.$$

(ii)
$$6 \times \tfrac{1}{2}(\varphi_2\hat{\varphi}_2^5\varphi_3^5\hat{\varphi}_3 + \varphi_2^5\hat{\varphi}_2\varphi_3\hat{\varphi}_3^5)$$
$$= 3\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3((\varphi_2\hat{\varphi}_2)^4 + (\varphi_3\hat{\varphi}_3)^4 - (\varphi_3^4 - \varphi_2^4)(\hat{\varphi}_3^4 - \hat{\varphi}_2^4)\}$$
$$= 3\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3\{(\varphi_2\hat{\varphi}_2)^4 + (\varphi_3\hat{\varphi}_3)^4 - (\varphi_3\hat{\varphi}_3 - \varphi_2\hat{\varphi}_2)^4\} \qquad \text{(T11) \& (T12)}$$
$$= 12(\varphi_2\hat{\varphi}_2)^2(\varphi_3\hat{\varphi}_3)^4 - 18(\varphi_2\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_3)^3 + 12(\varphi_2\hat{\varphi}_2)^4(\varphi_3\hat{\varphi}_3)^2$$

(iii)
$$\tfrac{1}{2}\{(\varphi_2\hat{\varphi}_2)^6 + (\varphi_3\hat{\varphi}_3)^6 + (\varphi_4\hat{\varphi}_4)^6\}$$
$$=\tfrac{1}{2}\{(\varphi_2\hat{\varphi}_2)^6 + (\varphi_3\hat{\varphi}_3)^6 - (\varphi_3\hat{\varphi}_3 - \varphi_2\hat{\varphi}_2)^6\} \qquad \text{(T12)}$$

By (i), (ii) and (iii), we get

$$\begin{aligned}\Theta_m(z) &= (\varphi_2\hat{\varphi}_2)^6 - 3(\varphi_2\hat{\varphi}_2)^5(\varphi_3\hat{\varphi}_3)\\ &\quad + 51(\varphi_2\hat{\varphi}_2)^4(\varphi_3\hat{\varphi}_3)^2 - 34(\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3)^3\\ &\quad + 51(\varphi_2\hat{\varphi}_2)^2(\varphi_3\hat{\varphi}_3)^4 - 3(\varphi_2\hat{\varphi}_2)(\varphi_3\hat{\varphi}_3)^5 + (\varphi_3\hat{\varphi}_3)^6\\ &= (\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^6 - 9\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3(\varphi_3\hat{\varphi}_3 - \varphi_2\hat{\varphi}_2)^4\\ &= (\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^6 - 9\varphi_2\hat{\varphi}_2\varphi_3\varphi_3(\varphi_4\hat{\varphi}_4)^4\\ &= \Theta^{(3)}(2z)^6 - 36\eta_m(z)\,.\end{aligned}$$

Then it follows from Theorem 2.2 that $\Theta_m(z)/\eta_m(z)$ ($m = 1^6 3^6$) is a generator for $3+$.

(3) Let $m = 1^4 2^2 4^4$. By (Ξ) in Section 1 and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}(\theta_3(2z)^4 + \theta_4(2z)^4)\theta_3(4z)^2\theta_3(8z)^4\\ &\quad + \tfrac{1}{2}\theta_2(2z)^4\theta_2(4z)^2\theta_3(8z)^4\\ &\quad + 2 \times \tfrac{1}{2}\theta_2(8z)^2(\theta_3(2z)^4 + \theta_4(2z)^4)\theta_3(4z)^2\theta_3(8z)^2\\ &\quad + 8 \times \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(8z)\theta_3(2z)^2\theta_3(4z)\theta_3(8z)^3\\ &\quad + 4 \times \tfrac{1}{2}\theta_2(4z)^2\theta_2(8z)^2(\theta_3(2z)^4 + \theta_4(2z)^4)\theta_3(8z)^2\\ &\quad + 4 \times \tfrac{1}{2}\theta_2(2z)^4\theta_2(8z)^2\theta_3(4z)^2\theta_3(8z)^2\\ &\quad + \tfrac{1}{2}\theta_2(8z)^4(\theta_3(2z)^4 + \theta_4(2z)^4)\theta_3(4z)^2\\ &\quad + 2 \times \tfrac{1}{2}\theta_2(2z)^4\theta_2(4z)^2\theta_2(8z)^2\theta_3(8z)^2\\ &\quad + 8 \times \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(8z)^3\theta_3(2z)^2\theta_3(4z)\theta_3(8z)\\ &\quad + \tfrac{1}{2}\theta_2(2z)^4\theta_2(4z)^2\theta_2(8z)^4\\ &\quad + 32 \times \tfrac{1}{2}\rho_0(8z)^4\rho_0(4z)^2(\rho_0(2z)^4 - \rho_1(2z)^4)\,.\end{aligned}$$

Let $\varphi_i = \theta_i(2z)$. By (T1–2) and (T5–6), we have

$$\theta_2(8z) = (\varphi_3 - \varphi_4)/2, \qquad \theta_3(8z) = (\varphi_3 + \varphi_4)/2$$
$$\theta_2(4z)^2 = (\varphi_3^2 - \varphi_4^2)/2, \qquad \theta_3(4z)^2 = (\varphi_3^2 + \varphi_4^2)/2$$

Then, expressing $\Theta_m(z)$ by φ_3 and φ_4, it is not difficult to see

$$\Theta_m(z) = \varphi_3^{10} + \tfrac{5}{4}\varphi_3^2\varphi_4^8 - \tfrac{5}{4}\varphi_3^6\varphi_4^4 = \varphi_3^{10} - \tfrac{5}{4}\varphi_2^4\varphi_3^2\varphi_4^4 = \varphi_3^{10} - 20\eta_m$$

since $\eta_m(z) = \varphi_2^4\varphi_3^2\varphi_4^4/16$ by (T20) & (T22). Then, by using

(T21) $$\varphi_3(z) = \eta(2z)^5/\eta(z)^2\eta(4z)^2$$

we get

$$\Theta_m(z)/\eta_m(z) = \eta(2z)^{48}/\eta(z)^{24}\eta(4z)^{24} - 20 ,$$

which is a generator for 4+ by Table 3 of [3].

(4) Let $m = 1^2 2^2 3^2 6^2$. By (Ξ) in Section 1 and Table 1, we have

$$\begin{aligned}
\Theta_m(z) = {} & \tfrac{1}{2}\{\theta_3(2z)^2\theta_3(6z)^2 + \theta_4(2z)^2\theta_4(6z)^2\}\theta_3(4z)^2\theta_3(12z)^2 \\
& + \tfrac{1}{2}\theta_2(2z)^2\theta_2(6z)^2\theta_3(4z)^2\theta_3(12z)^2 \\
& + 2 \times \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_3(6z)\theta_3(2z)\theta_3(6z)\theta_3(12z)^2 \\
& + 2 \times \tfrac{1}{2}\theta_2(4z)\theta_2(12z)\theta_3(4z)\theta_3(12z)\{\theta_3(2z)^2\theta_3(6z)^2 + \theta_4(2z)^2\theta_4(6z)^2\} \\
& + 4 \times \tfrac{1}{2}\theta_2(2z)\theta_2(4z)\theta_2(6z)\theta_2(12z)\theta_3(2z)\theta_3(4z)\theta_3(6z)\theta_3(12z) \\
& + \tfrac{1}{2}\theta_2(4z)^2\theta_2(12z)^2\{\theta_3(3z)^2\theta_3(6z)^2 + \theta_4(2z)^2\theta_4(6z)^2\} \\
& + 2 \times \tfrac{1}{2}\theta_2(2z)\theta_2(6z)\theta_2(12z)^2\theta_3(2z)\theta_3(4z)^2\theta_3(6z) \\
& + 2 \times \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(6z)^2\theta_2(12z)\theta_3(4z)\theta_3(12z) \\
& + \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)^2\theta_2(6z)^2\theta_2(12z)^2 \\
& + 16 \times \tfrac{1}{2}\rho_0(4z)^2\rho_0(12z)^2\{\rho_0(2z)^2\rho_0(6z)^2 - \rho_1(2z)^2\rho_1(6z)^2\} .
\end{aligned}$$

Then the calculations similar to the case $m = 1^6 3^6$ yield

$$\begin{aligned}
\Theta_m(z) &= (\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2 - \tfrac{3}{4}(\theta_2(z)\theta_2(3z)\theta_4(2z)\theta_4(6z))^2 \\
&= (\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2 - 12\eta_m(z)
\end{aligned}$$

as $\eta_m(z) = (\theta_2(z)\theta_2(3z)\theta_4(2z)\theta_4(6z))^2/16$ by (T20) & (T22).

The details are omitted, just noting that the formal (T12) should be used. Now Theorem 2.1 for $m = 1^2 2^2 3^2 6^2$ follows from Theorem 2.2.

(5) The cases $m = 1^3 7^3$, $1\cdot 2\cdot 7\cdot 14$ and $3\cdot 21$. Dealing with these cases, the formulas (T15–16) and (T19) will be particularly useful. Set

$$\varphi_i = \theta_i(2z) \quad \text{and} \quad \hat{\varphi}_i = \theta_i(14z) .$$

Then we have

$$\theta_2(z)\theta_2(7z) = \varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 - \varphi_4\hat{\varphi}_4$$

which can be derived from (T15–16).

(5–1) Let $m = 1^3 7^3$. By (Ξ) and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{(\varphi_3\hat{\varphi}_3)^3 + (\varphi_4\hat{\varphi}_4)^3\} + \tfrac{3}{2}\varphi_2\hat{\varphi}_2(\varphi_3\hat{\varphi}_3)^2 + \tfrac{3}{2}(\varphi_2\hat{\varphi}_2)^2\varphi_3\hat{\varphi}_3 \\ &\quad + \tfrac{1}{2}(\varphi_2\hat{\varphi}_2)^3 + 8 \times \tfrac{1}{2}\{\rho_0(2z)^3\rho_0(14z)^3 - \rho_1(2z)^3\rho_1(14z)^3\}\,.\end{aligned}$$

By (T8–10), (T15–16) and (T19), we have

$$\begin{aligned}&(\rho_0(2z)\rho_0(14z))^3 - (\rho_1(2z)\rho_1(14z))^3 \\ &\quad = \tfrac{1}{8}(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 - \varphi_4\hat{\varphi}_4)^2(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 + 2\varphi_4\hat{\varphi}_4)\end{aligned}$$

and then we get easily

$$\begin{aligned}\Theta_m(z) &= (\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^3 - \tfrac{3}{2}(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 - \varphi_4\hat{\varphi}_4)(\varphi_4\hat{\varphi}_4)^2 \\ &= \Theta^{(7)}(2z)^3 - 6\eta_m(z)\end{aligned}$$

where, in the last equality, we used (T15–16) and (A22). Now Theorem 2.1 for $m = 1^3 7^3$ follows from Theorem 2.2.

(5–2) Let $m = 1\cdot 2\cdot 7\cdot 14$. By ($\Xi$) and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{\theta_3(2z)\theta_3(14z) + \theta_4(2z)\theta_4(14z)\}\theta_3(4z)\theta_3(28z) \\ &\quad + \tfrac{1}{2}\theta_2(2z)\theta_2(14z)\theta_3(4z)\theta_3(28z) \\ &\quad + \tfrac{1}{2}\theta_2(4z)\theta_2(28z)\{\theta_3(2z)\theta_3(14z) + \theta_4(2z)\theta_4(14z)\} \\ &\quad + \tfrac{1}{2}\theta_2(2z)\theta_2(4z)\theta_2(14z)\theta_2(28z) \\ &\quad + 4 \times \tfrac{1}{2}\rho_0(4z)\rho_0(28z)\{\rho_0(2z)\rho_0(14z) - \rho_1(2z)\rho_1(14z)\} \\ &= \tfrac{1}{2}\{\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 + \varphi_4\hat{\varphi}_4\}\{\theta_2(4z)\theta_2(28z) + \theta_3(4z)\theta_3(28z)\} \\ &\quad + 2\rho_0(4z)\rho_0(28z)\{\rho_0(2z)\rho_0(14z) - \rho_1(2z)\rho_1(14z)\}\,.\end{aligned}$$

Then, by (T3) and (T19), we get

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 + \varphi_4\hat{\varphi}_4\}\Theta^{(7)}(4z) + \tfrac{1}{4}(\theta_2\hat{\theta}_2)^2 \\ &= \tfrac{1}{2}(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 + \varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3 - \theta_2\hat{\theta}_2)\Theta^{(7)}(4z) + \tfrac{1}{4}(\theta_2\hat{\theta}_2)^2 \\ &= \Theta^{(7)}(2z)\Theta^{(7)}(4z) - \tfrac{1}{2}\theta_2\hat{\theta}_2\varphi_2\hat{\varphi}_2 \\ &= \Theta^{(7)}(2z)\Theta^{(7)}(4z) - 2\eta_m(z)\,.\end{aligned}$$

Then Theorem 2.2 implies that $\Theta_m(z)/\eta_m(z)$ is a generator for 14+.

(5–3) Let $m = 3\cdot 21$. By (Ξ) and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{\theta_3(6z)\theta_3(42z) + \theta_4(6z)\theta_4(42z)\} + \tfrac{1}{2}\theta_2(6z)\theta_2(42z) \\ &\quad + \{\rho_0(6z)\rho_0(42z) - \rho_1(6z)\rho_1(42z)\}\,.\end{aligned}$$

Then, by (T15–16) and (T19), we have

$$\begin{aligned}\Theta_m(z) &= \theta_2(6z)\theta_2(42z) + \theta_3(6z)\theta_3(42z)\\ &= \Theta^{(7)}(6z)\,.\end{aligned}$$

Let $f(z) = \Theta^{(7)}(2z)^3/\eta_n(z)$ $(n = 1^3 7^3)$. Then we have

$$f(3z)^{1/3} = \Theta^{(7)}(6z)/\eta_m(z) = \Theta_m(z)/\eta_m(z) \qquad (m = 3\cdot 21)$$

This means that $\Theta_m(z)/\eta(z)$ is a generator for $21/3+$ (cf. Table 3 of [3]).

(6) Let $m = 1^2 2\cdot 4\cdot 8^2$. By (Ξ) and Table 1, we have

$$\begin{aligned}\Theta_m(z) = {}& \tfrac{1}{2}\{\theta_3(2z)^2 + \theta_4(2z)^2\}\theta_3(4z)\theta_3(8z)\theta_3(16z)^2\\ &+ \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(8z)\theta_3(16z)^2\\ &+ 2\times\tfrac{1}{2}\theta_2(8z)\theta_2(16z)\{\theta_3(2z)^2 + \theta_4(2z)^2\}\theta_3(4z)\theta_3(16z)\\ &+ 2\times\tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(16z)\theta_3(8z)\theta_3(16z)\\ &+ \tfrac{1}{2}\theta_2(16z)^2\{\theta_3(2z)^2 + \theta_4(2z)^2\}\theta_3(4z)\theta_3(8z)\\ &+ \tfrac{1}{2}\theta_2(2z)^2\theta_2(4z)\theta_2(8z)\theta_2(16z)^2\\ &+ 8\times\tfrac{1}{2}\{\rho_0(2z)^2 - \rho_1(2z)^2\}\rho_0(4z)\rho_0(8z)\rho_0(16z)^2\,.\end{aligned}$$

Calculating parts of the summation, we have

(i) (1st term) + (3rd term) + (5th term)
$=\theta_3(4z)^3\{\theta_2(8z)^3 + \theta_3(8z)^3\}$

(ii) (2nd term) + (4th term) + (6th term)
$=\tfrac{1}{4}\theta_2(2z)^2\theta_3(2z)\theta_2(4z)^3$

(iii) (7th term) $= \tfrac{1}{2}\theta_2(2z)^2\theta_3(2z)\theta_2(4z)^3$.

Thus we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(4z)^3\{\theta_2(8z) + \theta_3(8z)\}^3\\ &\quad - 3\theta_3(4z)^3\theta_2(8z)\theta_2(8z)\{\theta_2(8z) + \theta_3(8z)\} + \tfrac{3}{4}\theta_2(2z)^2\theta_3(2z)\theta_2(4z)^3\\ &= \theta_3(2z)^3\theta_3(4z)^3 - \tfrac{3}{2}\theta_2(4z)^2\theta_3(4z)\theta_3(2z)\theta_4(2z)^2\\ &= \theta_3(2z)^3\theta_3(4z)^3 - 6\eta_m(z)\,.\end{aligned}$$

Using (T21), we get

$$\Theta_m(z)/\eta_m(z) = \eta(2z)^8\eta(4z)^8/\eta(z)^8\eta(8z)^8 - 6$$

which is a generator for $8+$ by Table 3 of [3].

(7) Let $m = 1^2 11^2$, Set

$$\varphi_i = \theta_i(2z) \quad\text{and}\quad \hat{\varphi}_i = \theta_i(22z)\,.$$

By (Ξ) and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{(\varphi_3\hat{\varphi}_3)^2 + (\varphi_4\hat{\varphi}_4)^2\} + 2 \times \tfrac{1}{2}\varphi_2\hat{\varphi}_2\varphi_3\hat{\varphi}_3 + \tfrac{1}{2}(\varphi_2\hat{\varphi}_2)^2 \\ &\quad + 4 \times \tfrac{1}{2}\{(\rho_0(2z)\rho_0(22z))^2 - (\rho_1(2z)\rho_1(22z))^2\} \\ &= \tfrac{1}{2}(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^2 + \tfrac{1}{2}\theta_3\hat{\theta}_3\theta_4\hat{\theta}_4 + \tfrac{1}{2}\theta_2\hat{\theta}_2(\theta_3\hat{\theta}_3 - \theta_4\hat{\theta}_4)\end{aligned}$$

where, in the second equality, we used (T7) and (T8–9). Now using

$$(\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^2 = \tfrac{1}{2}\{(\theta_2\hat{\theta}_2)^2 + (\theta_3\hat{\theta}_3)^2 + (\theta_4\hat{\theta}_4)^2\} \qquad (\hat{\theta}_i(z) = \theta_i(11z))$$

which can be easily derived from (T–6), we get

$$\begin{aligned}\Theta_m(z) &= (\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^2 - \tfrac{1}{4}\{(\theta_2\hat{\theta}_2)^2 + (\theta_3\hat{\theta}_3)^2 + (\theta_4\hat{\theta}_4)^2\} \\ &\quad + \tfrac{1}{2}\theta_3\hat{\theta}_3\theta_4\hat{\theta}_4 + \tfrac{1}{2}\theta_2\hat{\theta}_2(\theta_3\hat{\theta}_3 - \theta_4\hat{\theta}_4) \\ &= (\varphi_2\hat{\varphi}_2 + \varphi_3\hat{\varphi}_3)^2 - \tfrac{1}{4}(\theta_2\hat{\theta}_2 - \theta_3\hat{\theta}_3 + \theta_4\hat{\theta}_4)^2 \\ &= \Theta^{(11)}(2z)^2 - 4\eta_m(z)\end{aligned}$$

where we used (T24). Then it follows from Theorem 2.2 that $\Theta_m(z)/\eta_m(z)$ is a generator for 11+.

(8) Let $m = 1\cdot 3\cdot 5\cdot 15$. By ($\Xi$) and Table 1, we have

$$\begin{aligned}\Theta_m(z) &= \tfrac{1}{2}\{\theta_3(2z)\theta_3(6z)\theta_3(10z)\theta_3(30z) + \theta_4(2z)\theta_4(6z)\theta_4(10z)\theta_4(30z)\} \\ &\quad + \tfrac{1}{2}\theta_2(6z)\theta_2(10z)\theta_3(2z)\theta_3(30z) + \tfrac{1}{2}\theta_2(2z)\theta_2(30z)\theta_3(6z)\theta_3(10z) \\ &\quad + \theta_2(2z)\theta_2(6z)\theta_2(10z)\theta_2(30z) + 4 \times \tfrac{1}{2}\{\rho_0(2z)\rho_0(6z)\rho_0(10z)\rho_0(30z) \\ &\quad - \rho_1(2z)\rho_1(6z)\rho_1(10z)\rho_1(30z)\}\,.\end{aligned}$$

Applications of Schröter's formula, which are similar to those in Example A3–4 of Appendix, yield that the last term of the above summation is equal to

$$\theta_2(6z)\theta_2(10z)\theta_3(2z)\theta_3(30z) + \theta_2(2z)\theta_2(30z)\theta_3(6z)\theta_3(10z).$$

Also repeated applications of the formula (T12) yield

$$\begin{aligned}&\theta_4(2z)\theta_4(6z)\theta_4(10z)\theta_4(30z) \\ &\quad = \theta_2(2z)\theta_2(6z)\theta_2(10z)\theta_2(30z) + \theta_3(2z)\theta_3(6z)\theta_3(10z)\theta_3(30z) \\ &\quad - \theta_2(2z)\theta_2(6z)\theta_3(10z)\theta_3(30z) - \theta_2(10z)\theta_2(30z)\theta_3(2z)\theta_3(6z)\,.\end{aligned}$$

Then it is easy to see

$$\Theta_m(z) = \Theta^{(3)}(2z)\Theta^{(3)}(10z) - \tfrac{3}{2}\psi(2z)\psi(6z)$$

where $\psi(z) = \theta_2(z)\theta_3(5z) - \theta_2(5z)\theta_3(z)$.

Using (T25) (and (T1–2)), we get

$$\Theta_m(z) = \Theta^{(3)}(2z)\Theta^{(3)}(10z) \quad - 6\eta_m(z) .$$

Now it follows from Theorem 2.2 that $\Theta_m(z)/\eta_m(z)$ ($m = 1\cdot 3\cdot 5\cdot 15$) is a generator for $15+$.

(9) Let $m = 1\cdot 23$. By (Ξ) and Table 1, we have

$$\Theta_m(z) = \tfrac{1}{2}\{\theta_2(2z)\theta_2(46z) + \theta_3(2z)\theta_3(46z)\} + \tfrac{1}{2}\theta_4(2z)\theta_4(46z) + 2 \times \tfrac{1}{2}\{\rho_0(2z)\rho_0(46z) - \rho_1(2z)\rho_1(46z)\} .$$

Now we want to prove

$$(\sharp) \qquad \Theta_m(z) = \Theta^{(23)}(2z) - 2\eta_m(z) .$$

For that purpose, we have to show

$$(*) \qquad \rho_0(z)\rho_0(23z) - \rho_1(z)\rho_1(23z) - \tfrac{1}{2}\{\theta_2(z)\theta_2(23z) + \theta_3(z)\theta_3(23z) - \theta_4(z)\theta_4(23z)\} = -2\eta(z/2)\eta(\tfrac{23}{2}z)$$

from which ($\sharp$) clearly follows.

Applications of Schröter's formula yield that the left hand side of ($*$) is equal to

$$-2q\sigma(q)\sigma(q^{23})$$

where

$$\sigma(q) = \theta(q^2, q^{24}) + q^5\theta(q^{22}, q^{24}) - q\theta(q^{10}, q^{24}) - q^2\theta(q^{14}, q^{24}) .$$

On the other hand, it is not difficult to see

$$q^{-1/12}\eta(z) = \sum_{n\in \mathbf{Z}} (-1)^n q^{3n^2+n} = \sigma(q^2) .$$

Thus we get ($\sharp$), which, by Theorem 2.2, implies that $\Theta_m(z)/\eta_m(z)$ is a generator for $23+$.

(10) Let $m = 2^{12}$. By (1.25), we have

$$\Theta_m(z) = \tfrac{1}{2}\{\theta_2(2z)^{12} + \theta_3(2z)^{12} + \theta_4(2z)^{12}\} ,$$

As $\theta_2(2z)^4 - \theta_3(2z)^4 + \theta_4(2z)^4 = 0$ by (T11), we see

$$\begin{aligned}\theta_2(2z)^{12} - \theta_3(2z)^{12} + \theta_4(2z)^{12} &= -3(\theta_2(2z)\theta_3(2z)\theta_4(2z))^4 \\ &= -48\eta_m(z) ,\end{aligned}$$

Thus we get

$$\Theta_m(z) = \theta_3(2z)^{12} - 24\eta_m(z)$$

and so

$$\Theta_m(z)/\eta_m(z) = \eta(2z)^{48}/\eta(z)^{24}\eta(4z)^{24} - 24$$

which is a generator for $4+$ by Table 3 of [3].

(11) Let $m = 3^8$. By (Ξ) and Table 1, we have

$$\Theta_m(z) = \tfrac{1}{2}\{\theta_3(6z)^8 + \theta_4(6z)^8\} + 14 \times \tfrac{1}{2}\theta_2(6z)^4\theta_3(6z)^4 + \tfrac{1}{2}\theta_2(6z)^8 + 16 \times \tfrac{1}{2}\{\rho_0(6z)^8 - \rho_1(6z)^8\} .$$

Set $\hat{\theta}_i = \theta_i(3z)$. Then it is easy to see

$$\begin{aligned}\Theta_m(z) &= \hat{\theta}_3^8 - \hat{\theta}_3^4\hat{\theta}_4^4 + \hat{\theta}_4^8 \\ &= \tfrac{1}{2}(\hat{\theta}_3^8 + \hat{\theta}_4^8) + \tfrac{1}{2}(\hat{\theta}_3^4 - \hat{\theta}_4^4)^2 \\ &= \tfrac{1}{2}(\hat{\theta}_2^8 + \hat{\theta}_3^8 + \hat{\theta}_4^8) \\ &= E_4(3z) .\end{aligned}$$

As is well known, $E_4(z)^3/\eta(z)^{24} = j(z) - 720$ is a generator for $1+$ $(=SL(2, \boldsymbol{Z}))$ and so $\Theta_m(z)/\eta_m(z)$ is a generator for $3/3$ (cf. Table 3 of [3]).

(12) Let $m = 2^4 4^4$. Then we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(4z)^4\theta_3(8z)^4 + \{\theta_2(4z)^4\theta_3(8z)^4 + 6\theta_2(8z)^2\theta_3(4z)^4\theta_3(8z)^2\} \\ &\quad + \{\theta_2(8z)^4\theta_3(4z)^4 + 6\theta_2(4z)^4\theta_2(8z)^2\theta_3(8z)^2\} + \theta_2(4z)^4\theta_2(8z)^8 \\ &= (\theta_2(4z)^4 + \theta_3(4z)^4)(\theta_2(8z)^4 + 6\theta_2(8z)^2\theta_3(8z)^2 + \theta_3(8z)^4) \\ &= (\theta_2(4z)^4 + \theta_3(4z)^4)^2 \\ &= \{\tfrac{1}{2}(\theta_3(2z)^4 + \theta_4(2z)^4)\}^2\end{aligned}$$

Let $f(z) = \frac{1}{2}(\theta_3(z)^4 + \theta_4(z)^4)^2/\eta_n(z)$ $(n = 1^8 2^8)$. Then $f(z)$ is a generator for $2+$ by what we have already proved and we have $f(2z)^{1/2} = \Theta_m(z)/\eta_m(z)$ $(m = 2^4 4^4)$. This means that $\Theta_m(z)/\eta_m(z)$ is a generator for $4/2+$ by Table 3 of [3].

(13) Let $m = 4^6$. Then we have

$$\begin{aligned}\Theta(z) &= \theta_3(8z)^6 + 3\theta_2(8z)^2\theta_3(8z)^4 + 3\theta_2(8z)^4\theta_3(8z)^2 + \theta_2(8z)^6 \\ &= (\theta_2(8z)^2 + \theta_3(8z)^2)^3 \\ &= \theta_3(4z)^6 .\end{aligned}$$

So we have

$$\Theta_m(z)/\eta_m(z) = \eta(4z)^{24}/\eta(2z)^{12}\eta(8z)^{12}$$

which is a generator for $8/2+$ by Table 3 of [3].

(14) Let $m = 6^4$. Then we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(12z)^4 + 2\theta_2(12z)^2\theta_3(12z)^2 + \theta_2(12z)^4 \\ &= (\theta_2(12z)^2 + \theta_3(12z)^2)^2 \\ &= \theta_3(6z)^4 .\end{aligned}$$

So we have

$$\Theta_m(z)/\eta_m(z) = \eta(6z)^{16}/\eta(3z)^8\eta(12z)^8$$

which is a generator for 12/3+ by Table 3 of [3].

(15) Let $m = 2^2 10^2$. Then we have

$$\begin{aligned}\Theta_m(z) &= (\theta_3(4z)\theta_3(20z))^2 + 2\theta_2(4z)\theta_2(20z)\theta_3(4z)\theta_3(20z) + (\theta_2(4z)\theta_2(20z))^2 \\ &= (\theta_3(4z)\theta_3(20z) + \theta_2(4z)\theta_2(20z))^2 \\ &= \tfrac{1}{4}(\theta_3(z)\theta_3(5z) + \theta_4(z)\theta_4(5z))^2 .\end{aligned}$$

Set $\hat{\theta}_i = \theta_i(5z)$. Then, by (T25), we have

$$\begin{aligned}\Theta_m(z) + 4\eta_m(z) &= \tfrac{1}{4}(\theta_3\hat{\theta}_3 + \theta_4\hat{\theta}_4)^2 + \tfrac{1}{4}(\theta_3\hat{\theta}_4 - \theta_4\hat{\theta}_3)^2 \\ &= \tfrac{1}{4}(\theta_3^2 + \theta_4^2)(\hat{\theta}_3^2 + \hat{\theta}_4^2) \\ &= \theta_3(2z)^2\theta_3(10z)^2 .\end{aligned}$$

Thus we get

$$\Theta_m(z)/\eta_m(z) = \eta(2z)^8\eta(10z)^8/\eta(z)^4\eta(4z)^4\eta(5z)^4\eta(20z)^4 - 4$$

which is a generator for 20+ by Table 3 of [3].

(16) Let $m = 2\cdot 4\cdot 6\cdot 12$. Then we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(4z)\theta_3(12z)\theta_3(8z)\theta_3(24z) + \theta_2(4z)\theta_2(12z)\theta_3(8z)\theta_3(24z) \\ &\quad + \theta_2(8z)\theta_2(24z)\theta_3(4z)\theta_3(12z) + \theta_2(4z)\theta_2(12z)\theta_2(8z)\theta_2(24z) \\ &= \Theta^{(3)}(4z)\Theta^{(3)}(8z) .\end{aligned}$$

Let $f(z) = (\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2/\eta_n(z)$ $(n = 1^2 2^2 3^2 6^2)$. Then $f(z)$ is a generator for 6+ by what we have already proved and we have $f(2z)^{1/2} = \Theta_m(z)/\eta_m(z)$ $(m = 2\cdot 4\cdot 6\cdot 12)$. This means that $\Theta_m(z)/\eta_m(z)$ is a generator for 12/2+ by Table 3 of [3].

(17) Let $m = 12^2$. Then we have

$$\begin{aligned}\Theta_m(z) &= \theta_3(24z)^2 + \theta_2(24z)^2 \\ &= \theta_3(12z)^2 .\end{aligned}$$

So we get

$$\Theta_m(z)/\eta_m(z) \doteq \eta(12z)^8/\eta(6z)^4\eta(24z)^4$$

which is a generator for 24/6+ by Table 3 of [3].

Now we have proved Theorem 2.1 for all elements of M_{24} except for an element with a cycle decomposition $1^4 5^4$. For such an element we argue as follows.

Let $m = 1^4 5^4$. Firstly we see from (Ξ) and Table 1 in Section 1 that $\Theta_m(z)$ is as in Table 2. Secondly it is not difficult to see that the invariant sublattice Λ_m has a discriminant 5^4 and so $\Theta_m(z)$ is a modular form of level 5 and weight 4 (with a trivial character). Furthermore, it is known that the vector space of such modular forms is 3-dimensional (cf. [5; Theorem 2.23]). Thus the coincidence of the first three Fourier coefficients of two modular forms of level 5 and weight 4 will imply that such two modular forms must be identical. On the other hand, in [4], Koike proved that there exists a modular form $\theta_m(z)$ of level 5 and weight 4 such that $\theta_m(z)/\eta_m(z)$ is a generator for 5+. Then, by direct computations, we see that the first three Fourier coefficients of our $\Theta_m(z)$ and Koike's $\theta_m(z)$ certainly coincide (cf. Table II of [4]). Thus we must have $\Theta_m(z) = \theta_m(z)$. This completes the proof of Theorem 2.1.

Appendix. Schröter's formula

We define

(A1) $$\theta(x, q) = \sum_{n\in Z} x^n q^{n^2} .$$

This power series in q has the convergent radius 1, for any non-zero x. If we put $q = e^{\pi i z}$, we have

(A2) $$\theta_3(z) = \sum q^{n^2} = \theta(1, q) ,$$

(A3) $$\theta_2(z) = \sum q^{(n+1/2)^2} = q^{1/4}\theta(q, q) ,$$

(A4) $$\theta_4(z) = \sum (-1)^n q^{n^2} = \theta(-1, q) .$$

Note that we define $q^{1/\alpha} = e^{\pi i z/\alpha}$ for a natural number α. It is easy to see that

(A5) $$\theta(-q, q) = 0 .$$

Also one can easily represent the functions $\Theta_\alpha(v, z)$ of two variables v and z by the functions $\theta(x, q)$, where $1 \leqq \alpha \leqq 4$. On the other hand, for $q = e^{\pi i z}$, defining

(A6) $$\rho_0(z) = \sum_n q^{(n+1/4)^2} ,$$

(A7) $$\rho_1(z) = \sum_n (-1)^n q^{(n+1/4)^2} ,$$

we have

(A8) $$\rho_0(z) = q^{1/16}\theta(q^{1/2}, q) \quad \text{and} \quad \rho_1(z) = q^{1/16}\theta(-q^{1/2}, q) .$$

From definition, it is clear that

(A9) $$\theta(-x, -q) = \theta(x, q) ,$$

(A10) $$\theta(x^{-1}, q) = \theta(x, q) ,$$

(A11) $$\theta(xq^2, q) = (xq)^{-1}\theta(x, q) ,$$

(A11)′ $$\theta(xq^{-2}, q) = xq^{-1}\theta(x, q) .$$

Note that the formula (A11) (and (A11)′) is derived from the calculus $\sum x^n q^{2n} q^{n^2} = (xq)^{-1} \sum x^{(n+1)} q^{(n+1)^2}$.

We fix a natural number α. In the definition (A1), writing $n = \alpha m + \rho$ $(0 \leqq \rho < \alpha)$, we have the following

LEMMA A.1. *For a natural number α, we have*

(A12) $$\theta(x, q) = \sum_{\rho=0}^{\alpha-1} x^\rho q^{\rho^2} \theta(x^\alpha q^{2\alpha\rho}, q^{\alpha^2}) .$$

EXAMPLE A.1. For $\alpha = 2$, putting $x = \pm 1$ or $\pm q$, we have

$$\theta(1, q) = \theta(1, q^4) + q\theta(q^4, q^4) ,$$
$$\theta(-1, q) = \theta(1, q^4) - q\theta(q^4, q^4) ,$$
$$\theta(q, q) = \theta(q^2, q^4) + q^2\theta(q^6, q^4) = 2\theta(q^2, q^4) ,$$
$$0 = \theta(-q, q) = \theta(q^2, q^4) - q^2\theta(q^6, q^4) .$$

Note that, from (A2), (A3) and (A4), the first two formulas are equivalent to

(T1) $$\theta_3(z) = \theta_3(4z) + \theta_2(4z) .$$

(T2) $$\theta_4(z) = \theta_3(4z) - \theta_2(4z) ,$$

respectively. The third one can be written

(T3) $$2\rho_0(4z) = \theta_2(z) ,$$

using (A8).

The following lemma is refered as "formula of Schröter" in Tannery and Molk's "Elements de la theorie des fonctions elliptiques" (n° 285).

LEMMA A.2. (Schröter). *Let α and β two natural numbers. Then*

$$\text{(A13)}\qquad \theta(x, q^{\alpha})\theta(y, q^{\beta}) = \sum_{\rho=0}^{\alpha+\beta-1} y^{\rho} q^{\beta\rho^2}\theta(xyq^{2\beta\rho}, q^{\alpha+\beta})\theta(x^{-\beta}y^{\alpha}q^{2\alpha\beta\rho}, q^{\alpha\beta(\alpha+\beta)}) ,$$

Proof. In the summation

$$\theta(x, q^{\alpha})\theta(y, q^{\beta}) = \sum_m \sum_n x^m y^n q^{\alpha m^2+\beta n^2} ,$$

we put

$$n = m + (\alpha+\beta)\sigma + \rho \qquad (0 \leqq \rho < \alpha+\beta)$$

where σ runs over Z. Also we put $\mu = m + \beta\sigma$. Then

$$\alpha m^2 + \beta n^2 = (\alpha+\beta)\mu^2 + 2\beta\rho\mu + \alpha\beta(\alpha+\beta)\sigma^2 + 2\alpha\beta\rho\sigma + \beta\rho^2 ,$$

and also we have

$$x^m y^n = (xy)^{\mu}(x^{-\beta}y^{\alpha})^{\sigma}y^{\rho} .$$

Thus it is easy to see that (A13) holds. q.e.d.

EXAMPLE A.2. (Duplication). In (A13), putting $\alpha=\beta=1$ and $y=\pm x$, we have

$$\theta(x, q)^2 = \theta(x^2, q^2)\theta(1, q^2) + xq\theta(x^2q^2, q^2)\theta(q^2, q^2) ,$$
$$\theta(x, q)\theta(-x, q) = \theta(-x^2, q^2)\theta(-1, q^2) .$$

Specializing $x = \pm 1$ or $\pm q$, we have $\theta(1, q)^2 = \theta(1, q^2)^2 + q\theta(q^2, q^2)^2$, $\theta(-1, q)^2 = \theta(1, q^2)^2 - q\theta(q^2, q^2)^2$ and $\theta(q, q)^2 = 2\theta(q^2, q^2)\theta(1, q^2)$, noting that $\theta(q^4, q^2) = q^{-2}\theta(1, q^2)$, for example. Also we have $\theta(1, q)\theta(-1, q) = \theta(-1, q^2)^2$. These are equivalent to

$$\text{(T4)}\qquad \theta_2(z)^2 = 2\theta_2(2z)\theta_3(2z) ,$$

$$\text{(T5)}\qquad \theta_3(z)^2 = \theta_3(2z)^2 + \theta_2(2z)^2 ,$$

$$\text{(T6)}\qquad \theta_4(z)^2 = \theta_3(2z)^2 - \theta_2(2z)^2 ,$$

$$\text{(T7)}\qquad \theta_3(z)\theta_4(z) = \theta_4(2z)^2 .$$

Now putting $x = \delta = \pm 1$ and $y = q$, we have $\theta(\delta, q)\theta(q, q) = 2\theta(\delta q, q^2)^2$. From these, we have

(T8) $$2\rho_0(2z)^2 = \theta_2(z)\theta_3(z)\ ,$$

(T9) $$2\rho_1(2z)^2 = \theta_2(z)\theta_4(z)\ .$$

Also we can derive

(T10) $$\rho_0(2z)\rho_1(2z) = 2^{-1}\theta_2(z)\theta_4(2z)\ .$$

Returning to the first formula in our example, we substitute x by $\pm x$ or $\pm xq$. Then we have

$$\theta(x, q)^4 + x^2q\theta(-xq, q)^4 = \theta(-x, q)^4 + x^2q\theta(xq, q)^4\ .$$

If we put $A = \theta(1, q^2)$, $B = \theta(q^2, q^2)$, $X = \theta(x^2, q^2)$ and $Y = \theta(x^2q^2, q^2)$, this is equivalent to

$$\theta(x, q)^4 + x^2q\theta(-xq, q)^4 = (A^2 + qB^2)(X^2 + x^2qY^2)$$

In the above formula, specializing $x = 1$, we have

(T11) $$\theta_3(z)^4 = \theta_2(z)^4 + \theta_4(z)^4\ .$$

Note that (T11) can be also derived from (T4), (T5) and (T6).

EXAMPLE A.3. In (A13), putting $\alpha = 3$ and $\beta = 1$ and $x = \pm 1$ or $\pm q^3$ and $y = \pm 1$ or $\pm q$, we have

$$\begin{aligned}
&\theta(1, q^3)\theta(1, q)\\
&\qquad = \theta(1, q^4)\theta(1, q^{12}) + q^4\theta(q^4, q^4)\theta(q^{12}, q^{12}) + 2q\theta(q^2, q^4)\theta(q^6, q^{12})\ ,\\
&\theta(-1, q^3)\theta(-1, q)\\
&\qquad = \theta(1, q^4)\theta(1, q^{12}) + q^4\theta(q^4, q^4)\theta(q^{12}, q^{12}) - 2q\theta(q^2, q^4)\theta(q^6, q^{12})\ ,\\
&0 = \theta(-q^3, q^3)\theta(-q, q)\\
&\quad = \theta(q^4, q^4)\theta(1, q^{12}) + q^2\theta(1, q^4)\theta(q^{12}, q^{12}) - 2\theta(q^2, q)\theta(q^6, q^{12})\ ,\\
&\theta(q^3, q^3)\theta(q, q)\\
&\qquad = \theta(q^4, q^4)\theta(1, q^{12}) + q^2\theta(1, q^4)\theta(q^{12}, q^{12}) + 2\theta(q^2, q^4)\theta(q^6, q^{12})\ .
\end{aligned}$$

Thus we have $\theta(q^3, q^3)\theta(q, q) = 4\theta(q^2, q^4)\theta(q^6, q^{12})$, for example. Now it is easy to show that

(T12) $$\theta_3(3z)\theta_3(z) - \theta_4(3z)\theta_4(z) = \theta_2(3z)\theta_2(z)\ ,$$

using (A2), (A3) and (A4). (cf. [7] p. 175). Also we have

(T13) $$\begin{aligned}&\theta_3(3z)\theta_3(z) + \theta_4(3z)\theta_4(z)\\ &\qquad = 2(\theta_3(4z)\theta_3(12z) + \theta_2(4z)\theta_2(12z)) = 2\Theta^{(3)}(4z)\ .\end{aligned}$$

In (A13), now we put $\alpha = \beta = 2$, $x = q$ and $y = -q$. Then we have

$$\theta(q, q^2)\theta(-q, q^2) = \theta(-q^2, q^4)\{\theta(1, q^{16}) - q^4\theta(q^{16}, q^{16})\}$$
$$= \theta(-q^2, q^4)\theta(-1, q^4) .$$

This formula can be written as

(T14) $$\rho_0(2z)\rho_1(2z) = \rho_1(4z)\theta_4(4z) .$$

In this case, the other formulas to be obtained are equivalent to (T8) and (T9).

EXAMPLE A.4. The case $\alpha = 7$ and $\beta = 1$ is quite similar to the case $\alpha = 3$ and $\beta = 1$. Putting $X = \theta(1, q^7)\theta(1, q) + \theta(-1, q^7)\theta(-1, q)$, we see that

$$X = 2\theta(1, q^8)\theta(1, q^{56}) + 2q^{16}\theta(q^8, q^8)\theta(q^{56}, q^{56}) + 4q^4\theta(q^4, q^8)\theta(q^{28}, q^{56})$$

Also we have

$$\theta(q^7, q^7)\theta(q, q) = 2\theta(q^8, q^8)\theta(1, q^{56}) + 2q^{12}\theta(1, q^8)\theta(q^{56}, q^{56})$$
$$+ 4q^2\theta(q^4, q^8)\theta(q^{28}, q^{56}) .$$

Multiplying the latter term by q^2, we have

(T15) $$\theta_3(7z)\theta_3(z) + \theta_4(7z)\theta_4(z) + \theta_2(7z)\theta_2(z)$$
$$= 2\{\theta_3(2z)\theta_3(14z) + \theta_2(2z)\theta_2(14z)\} = 2\Theta^{(7)}(2z) ,$$

(T16) $$\theta_3(7z)\theta_3(z) + \theta_4(7z)\theta_4(z) - \theta_2(7z)\theta_2(z) = 2\theta_4(2z)\theta_4(14z) .$$

Note that, from Lemma A.1, we have

$$\theta(\delta, q^2) = \theta(1, q^8) + \delta q^2\theta(q^8, q^8) ,$$

with $\delta = \pm 1$. Using also the formula

$$\theta(\delta q, q^2) = \theta(q^2, q^8) + \delta q\theta(q^6, q^8) ,$$

and (A8), we can show that

(T17) $$\theta_3(7z)\theta_3(z) - \theta_4(7z)\theta_4(z) + \theta_2(7z)\theta_2(z) = 4\rho_0(2z)\rho_0(14z) ,$$

(T18) $$\theta_3(7z)\theta_3(z) - \theta_4(7z)\theta_4(z) - \theta_2(7z)\theta_2(z) = 4\rho_1(2z)\rho_1(14z) .$$

Thus we have shown that

(T19) $$\rho_0(2z)\rho_0(14z) - \rho_1(2z)\rho_1(14z) = 2^{-1}\theta_2(7z)\theta_2(z) .$$

The case $\alpha = 11$ and $\beta = 1$ is similar to our example. But it is queer that we can not find pretty formulas in the case $\alpha = 5$ and $\beta = 1$.

Jacobi's triple product theorem is described in the following way. The infinite product

$$T(x, q) = \prod_{n=1}^{\infty} (1 - xq^n) \tag{A14}$$

is absolutely convergent for $|q| < 1$ and for any x. As the function of x, $T(x, q)$ has its zeros at $x = q^{-n}$, for all natural number n. It is easy to see that

$$T(x, -q) = T(x, q^2)T(-xq^{-1}, q^2) , \tag{A15}$$

$$T(x, q) = (1 - xq)T(xq, q) . \tag{A16}$$

LEMMA A.3. (Jacobi) *The following triple product theorem holds*:

$$\theta(x, q) = T(1, q^2)T(-xq^{-1}, q^2)T(-x^{-1}q^{-1}, q^2) . \tag{A17}$$

The proof is omitted. In this notation, the Dedekind's eta function is represented as

$$\eta(z) = q^{1/12}T(1, q^2) , \tag{A18}$$

for $q = e^{\pi iz}$. Also our theta functions $\theta_3(z)$, $\theta_4(z)$ and $\theta(z)$ are represented as infinite products, specializing $x = \pm 1$ or q in (A17):

$$\theta_3(z) = T(1, q^2)T(-q^{-1}, q^2)^2 , \tag{A19}$$

$$\theta_4(z) = T(1, q^2)T(q^{-1}, q^2)^2 , \tag{A20}$$

$$\theta_2(z) = 2q^{1/4}T(1, q^2)T(-1, q^2)^2 . \tag{A21}$$

Note that $T(-q^{-1}, q^2) = \prod(1 + q^{2n-1})$, and

$$T(q^{-1}, q^2) = \prod (1 - q^{2n-1}) \text{ and } T(-q^{-2}, q^2) = 2T(-1, q^2) = 2\prod (1 + q^{2n}) .$$

As $\prod (1 + q^{2n}) \times \prod (1 + q^{2n-1}) \times \prod (1 - q^{2n-1}) = 1$, we have

$$\theta_2(z)\theta_3(z)\theta_4(z) = 2q^{1/4}T(1, q^2)^3 = 2\eta(z)^3 . \tag{A22}$$

EXAMPLE A.5. As $\prod (1 - q^{2n-1}) = \prod (1 - q^n)/\prod (1 - q^{2n})$, so that

$$T(q^{-1}, q^2) = T(1, q)T(1, q^2)^{-1} .$$

Also as $\prod (1 + q^{2n}) = \prod (1 - q^{4n})/\prod (1 - q^{2n})$, so that

$$T(-1, q^2) = T(1, q^4)T(1, q^2)^{-1} .$$

Lastly we also have

$$T(-q^{-1}, q^2) = T(1, q)^{-1}T(1, q^2)^2T(1, q^4)^{-1} .$$

These give the following formulas:

(T20) $$\theta_2(z) = 2\eta(2z)^2\eta(z)^{-1} = 2\{1^{-1}2^2\} ,$$

(T21) $$\theta_3(2z) = \eta(2z)^5\eta(z)^{-2}\eta(4z)^{-2} = \{1^{-2}2^54^{-2}\} ,$$

(T22) $$\theta_4(2z) = \eta(z)^2\eta(2z)^{-1} = \{1^22^{-1}\} .$$

We calculate $\theta(-q, q^3)$ by (A17). Then we have

$$\theta(-q, q^3) = T(1, q^6)T(q^{-2}, q^6)T(q^{-4}, q^6) = T(1, q^2) .$$

This shows that

(T23) $$q^{1/12}\theta(-q, q^3) = \eta(z) ,$$

with $q = e^{\pi iz}$. On the other hand, $\theta(-q, q^3) = \sum(-1)^n q^{3n^2+n}$, from definition. (This gives Euler's identity)

EXAMPLE A.6. We consider the case $\alpha = 11$ and $\beta = 1$. Just as in Example A.4, we calculate $X = \theta(1, q^{11})\theta(1, q) - \theta(-1, q^{11})\theta(-1, q)$ and $Y = \theta(q^{11}, q^{11})\theta(q, q)$. From these we have

$$X - q^3Y = 4q\{\theta(q^2, q^{12}) - q^2\theta(q^{10}, q^{12})\} \times \{\theta(q^{22}, q^{132}) - q^{22}\theta(q^{110}, q^{132})\} .$$

On the other hand, to the function $\theta(-q, q^3)$, applying (A12) with $\alpha = 2$, we have

$$\theta(-q, q^3) = \theta(q^2, q^{12}) - q^2\theta(q^{10}, q^{12}) .$$

Thus we have shown that

(T24) $$\theta_3(11z)\theta_3(z) - \theta_4(11z)\theta_4(z) - \theta_2(11z)\theta_2(z) = 4\eta(z)\eta(11z) .$$

The case $\alpha = 5$ and $\beta = 1$ is different from the other cases. Here we calculate

$$\theta(-1, q^5)\theta(1, q) - \theta(1, q^5)\theta(-1, q) = 4q\theta(-q^2, q^6)\theta(-q^{10}, q^{30}) .$$

Using (T23) direclty, we have

(T25) $$\theta_4(5z)\theta_3(z) - \theta_3(5z)\theta_4(z) = 4\eta(2z)\eta(10z) .$$

Finally we make a mention of the formulation of theta formula.

LEMMA A.4. *For the function $\theta(x, q)$, the following "theta formula" holds*:

(A23) $$\theta(e^{\gamma}, e^{\delta}) = \kappa\theta(e^{\alpha}, e^{\beta}) ,$$

(A24) $$\kappa = e^{\alpha^2/4\beta} \times \sqrt{-\beta/\pi} ,$$

where α *and* β *are complex number such that* $\mathrm{Re}(\beta) < 0$, *and* $\beta\delta = \pi^2$ *and* $\alpha^2\delta + \gamma^2\beta = 0$. *That is,* $\delta = \pi^2/\beta$ *and* $\gamma = \pi i\alpha/\beta$ *(or* $\gamma = -\pi i\alpha/\beta$*). Note also that we assume* $\mathrm{Re}(\sqrt{-\beta/\pi}) > 0$.

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