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EXPLICIT FORMULAS FOR LOCAL FACTORS: ADDENDA AND ERRATA

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Introduction

In [3], the author studied certain local integrals derived from Fourier coefficient computations on Eisenstein series. Members of a family of Dirichlet series were characterized as a product of an explicit term with a mysterious polynomial factor. In a recent letter to the author, Professor Shoyu Nagaoka asked specific questions concerning the polynomial factor. Several of these questions can be answered by the techniques in [3]. In Part I of that paper, the relevant term is described precisely; however, in Part II, the term is described as a mysterious, albeit finite, sum. The present paper complete [3] by recording what little is known of that sum.

We illustrate our tables by settling one of the questions raised in Professor Nagaoka's letter. Let F is a totally real number field and let K/F be a purely imaginary quadratic extension. Let $\mathfrak{D}$ be the discriminant of K/F, and let $h \in \mathfrak{D}^{-1}$. For a finite prime $\mathcal{P}$ of F,

$$\bar{\alpha}^{(2)}_{\mathcal{P}}\left(S, \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}\right) = (1 - q^{-s})(1 - \psi(\mathcal{P})q^{1-s})(1 - \psi(\mathcal{P})q^{2-s})^{-1}\left(\sum_{j=0}^{b} q^{j(3-s)}\right), \tag{1}$$

where the α-series derives from Eisenstein series for the hermitian modular group of genus 2, ψ is the ideal character of K/F (normalized to be 0 if $\mathcal{P}$ ramifies), $q = N\mathcal{P}$, and $\mathcal{P}^b$ devides the ideal $(h)\mathfrak{D}$ while $\mathcal{P}^{b+1}$ does not.

1. The α-series

Let F be a local of any characteristic except 2 and let R be (a choice of) the ring of integers of F. Let $\mathcal{P}$ be the prime of R, and put

$$q = N\mathcal{P}. \tag{2}$$

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Let A be a semi-simple finite dimensional F-algebra, and let B be the corresponding maximal order of A. For $k \in \mathbf{N}$, let B^k be the right B-module of $k \times 1$ column vectors. For $k, r \in \mathbf{N}$ such that $k \geq r$, an $r \times k$ matrix M with entries in B is called *primitive* if there is a $(k - r) \times k$ matrix N such that

$$\begin{pmatrix} N \\ M \end{pmatrix} \in GL_k(B).$$

If L is a B-module and $K \subseteq L$ is a submodule, let $[L : K]$ be the cardinality of $|L/K|$. If L and K are B-submodules of A^k for some $k \in \mathbf{N}$, then define $[L : K] = [L : L \cap K]/[K : L \cap K]$ if each index on the right is finite. For $k \in \mathbf{N}$, define $v : GL_k(A) \to \mathbf{Q}$ by

(3) $$q^{v(T)} = [B^k : T \cdot B^k].$$

In practice, the function $q^{v(T)}$ is $| dt(T) |_{\mathscr{P}}^{-d}$, where dt is some sort of reduced norm function $GL_k(A) \to F$, $\|_{\mathscr{P}}$ is a normalized valuation at $\mathscr{P}$ and d is a positive constant. In [3] and in what follows, we work with the function v instead of determinants and valuations. For this reason, our α-series differ by a constant exponent from the usual ones, as used in [1] or [4]. We will comment on this later.

Let $k \in \mathbf{N}$. For $T \in M_k(A)$, define $j(T)$ by

(4) $$q^{j(T)} = [TB^k + B^k : B^k].$$

Another interpretation for $j(T)$ is as follows. Express $T = D^{-1}C$ where $(C\, D)$ is a primitive $k \times (2k)$ matrix. Then $j(T) = v(D)$. Again, in other treatments, the j-factor is typically replaced by $| dt(D) |_{\mathscr{P}}^{-1}$.

Fix a non-trivial group character χ from the additive group of F to the unit circle of $\mathbf{C}$. For our present purposes, any character will do. When we refer to Professor Nagaoka's question, we adopt the standard choice. Extend χ to $M_k(A)$ for each $k \in \mathbf{N}$ by composing the original character with the reduced trace, as described in [3].

Let ρ be an involution for A/F. Let $U(\rho)$ be the set of B-units ε such that $\varepsilon\varepsilon^{\rho} = 1$. For $\varepsilon \in U$, a (ρ, ε)-*hermitian lattice* is a free B-module M of finite rank paired with an R-bilinear form $(,) : M \times M \to A$ such that for $x, y \in M$ and $b, c \in B$.

(5.a) $$(bx, cy) = b \cdot (x, y) \cdot c^{\rho},$$

(5.b) $$(x, y) = \varepsilon (y, x)^{\rho}.$$

Let $k \in \mathbf{N}$. For each $\varepsilon \in U$, put

$$\Sigma(k,\,\varepsilon) = \{T \in M_k(A) : T = \varepsilon({}^tT^\rho)\},$$
$$\Sigma(k,\,\varepsilon,\,B) = \Sigma(k,\,\varepsilon) \cap M_k(B), \tag{6}$$

and also

$$\Sigma(k,\,\varepsilon,\,B)\,\# = \{T \in \Sigma(k,\,\varepsilon^\rho) : \chi(T\cdot\Sigma(k,\,\varepsilon,\,B)) = \{1\}\}. \tag{7}$$

The lattice $\Sigma(k,\,\varepsilon)$ obviously corresponds to all $(\rho,\,\varepsilon)$-hermitian forms on B^k. We refer to its members as being $(\rho,\,\varepsilon)$-*hermitian*. The function which takes $T \in \Sigma(k,\,\varepsilon,\,B)\,\#$ to the function $X \to \chi(TX)$ identifies the additive group of $\Sigma(k,\,\varepsilon,\,B)\,\#$ with the character group of $\Sigma(k,\,\varepsilon,\,B)$; for that reason, we refer to the former as the *dual lattice.*

Because most of the work in [3] deals with dual lattices, we set the problem in a manner in which the members of the dual lattice are $(\rho,\,\varepsilon)$-hermitian. For this reason, we set up the α-series as a sum over $\Sigma(k,\,\varepsilon^\rho)$ instead of $\Sigma(k,\,\varepsilon)$.

Let ρ be an involution of A, let $\varepsilon \in U(\rho)$, let $m \in \mathbf{N}$ and let $N \in \Sigma(m,\,\varepsilon^\rho,\,B)\,\#$. Define the α-series for this data by

$$\alpha(N,\,t) = \sum_{x \in \Sigma(m,\varepsilon^\rho)/\Sigma(m,\varepsilon^\rho,B)} \chi(Nx)\cdot t^{j(x)}, \tag{8}$$

where t is a formal variable. This is the correct form of [3; (5.10)], with B playing the role of S. The Dirichlet α-series used by Nagaoka [1] or Shimura [4] have the form

$$\alpha(N,\,s) = \alpha(N,\,q^{-s/d}), \tag{9}$$

where the constant d is the exponent factor characterized by $q^{v(T)} = |\,dt(T)\,|_{\mathscr{P}}^{-d}$ Tautologically, for any $u \in GL_k(B)$, $\alpha(uN\cdot{}^tu^\rho,\,t) = \alpha(N,\,t)$

Analysis of the α-series divides into two cases. First, suppose $A = \Delta \oplus \Delta^\circ$, where Δ is a simple F-algebra and Δ° is its opposite, and ρ is defined by $(b,\,c) \to (c,\,b)$. In this case, the involution ρ and the choice of ε is irrelevant. The α-series (8) can be rephrased as an infinite sum over $M_k(\Delta)$. The reformulation is analyzed in [3; Part I]. The analysis is complete, and we will make no additions to it here.

2. Hermitian lattices

With the split case settled, all other situations reduce to the hypothesis

(10) A is a division F-algebra,
F is the fixed field of ρ on the center of A.

Under assumption (10), we hereafter denote A by Δ and the ring B by S. Fix $\varepsilon \in U(\rho)$. From now on, for T a square matrix, put

$$T^* = {}^tT^\rho, \qquad \text{and} \tag{11}$$
$$T^{-*} = (T^*)^{-1} \quad \text{if } T \text{ is invertible.}$$

For $k \in \mathbf{N}$, $N \in M_k(\Delta)$ and $C \in GL_k(\Delta)$, put $N[C] = C^{-1}NC^{-*}$.

Let $\mathbf{m}$ be the maximal ideal of S, and let π be a generator of $\mathbf{m}$. Define a logarithmic valuation ι on $\Delta^* = \Delta - \{0\}$ by

$$\forall x \in \Delta^*,\ \pi^{-\iota(x)}x \in S - \mathbf{m}. \tag{12}$$

We adopt the convention that $\iota(0) = \infty$. For $X \subseteq \Delta$ a non-empty set, put

$$\iota(X) = \inf\{\iota(x) : x \in X\}.$$

For M a hermitian lattice, define

$$s(M) = \iota(\{(x, y) : x, y \in M\}). \tag{13}$$

For $n \in \mathbf{Z}$, put

$$\Delta_n = \{d \in \Delta : \iota(d) \geq n\}, \tag{14}$$
$$A_n = \{b + \varepsilon b^\rho : b \in \Delta_n\}.$$

Put

$$\mathfrak{D} = \{d \in \Delta : \forall b \in S,\ \chi(bd + b^\rho d^\rho) = 1\}, \tag{15}$$
$$\delta = \iota(\mathfrak{D}).$$

For $n \in \mathbf{Z}$, let $\mathrm{Cat}(\rho, \varepsilon, n)$ be the class of all (ρ, ε)-hermitian lattices M such that

$$s(M) \geq n, \tag{16}$$
$$\forall v \in M, \qquad (v, v) \in A_n.$$

In [3; Section 8], we define a notion of morphism between members of $\mathrm{Cat}(\rho, \varepsilon, n)$, and turn the class into a category. That structure is technical, and is omitted here. Certain lattices in this category have a special property, and are called *n-modular*; again, the precise definition is omitted, and we refer the reader to [3] for proof of the properties of n-modular lattices which we need. The *hyperbolic lattices of denominator* $-n$ are n-modular.

Parameters σ, σ_1, σ_2, σ_3, σ_4 and σ_5 are defined in [3; (5.8) and (5.9)]. Except for σ_2, these are usually trivial to calculate. To get σ_2, use the fact [3; Lemma 5.1]

(17) $$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_4 + \sigma.$$

Depending on these parameters and on n, the category $\mathrm{Cat}(\rho, \varepsilon, n)$ is classified as one of four *types*, in [3; (8.19)]. The category relevant to our calculation is $\mathrm{Cat}(\rho, \varepsilon, \delta)$. It is also a consequence of [3; Lemma 5.1] that, for $k \in \mathbf{N}$, $\sum(k, \varepsilon^\rho, S)\,\#$ is the set of all matrices which correspond to member of $\mathrm{Cat}(\rho, \varepsilon, \delta)$ of rank k.

The function v_1, on square, invertible matrices, is introduced in [3; Definition 7.1]. The only comments that we make here are (a) v_1 depends on ρ and ε, and (b) like v, $v_1(T)$ has the form $|\, dt(T)\,|_{\mathscr{P}}^{-d_1}$ where d_1 is some constant dependent on the raw data.

3. Definite exponents

For the next part of the argument, fix $m \in \mathbf{N}$. Fix $N \in \sum(m, \varepsilon^\rho, S)\,\# \cap GL_m(\Delta)$. Express

(18) $$m = 2g_0 + \lambda_0, \text{ where } g_0 \in \mathbf{Z} \text{ and } \lambda_0 \in \{0,1\}.$$

We now add a parameter not in [3]. Depending on the type of $\mathrm{Cat}(\rho, \varepsilon, \delta)$, define λ_1 as

(19) $$\lambda_1 = \begin{cases} \lambda_0 & \text{for Type I,} \\ 0 & \text{for Type II or IV,} \\ 1 & \text{for Type III.} \end{cases}$$

Let

(20) $$Y(N) = \{C \in GL_m(\Delta) \cap M_m(S) : N[C] \in \textstyle\sum(m, \varepsilon^\rho, S)\,\#\}.$$

Note that $GL_m(S)$ acts on Y on the right, and the quotient $Y(N)/GL_m(S)$ is finite. Following Siegel, our first major result is that $\alpha(N, t)$ is a sum of terms, one for each $C \in Y/GL_m(S)$. The term for C has to do with the structure of $N[C]$ in $\mathrm{Cat}(\rho, \varepsilon, \delta)$.

For

(21) $$g, h \in \mathbf{N} \cup \{0\},\ \lambda, \mu \in \{0,1\} \text{ and } \eta \in \{-1,1\},$$

define a polynomial in the indeterminate t by

(22) $$R(g, h, \lambda, \eta, \mu ; t) = \prod_{j=0}^{g+h+\lambda-1} (1 - q^{j\sigma_3} t^{\sigma_3})$$
$$\times \left\{(1 + \eta(1-\mu) q^{(g+h)\sigma_3+\sigma_1+\sigma_2-\sigma}\, t^{\sigma_3}) \prod_{i=1}^{g+h-1} (1 + q^{i\sigma_3+\sigma_1+\sigma_2-\sigma}\, t^{\sigma_3})\right\},$$

where the bracketed part is set equal to 1 if $g + h = 0$. Equation (22) is the correct form of (9.22) in [3]. We only consider this function when $\mu \leq h$, $\eta = 1$ if $\lambda = 1$, and $\lambda = 0$ if $\mu = 1$.

The significance of (22) is as follows. Let $M \in \sum(m, \varepsilon^{\rho}, S) \# \cap GL_m(\Delta)$. Regard M as a hermitian structure on S^m. Then M is isomorphic to an orthogonal sum $L \perp D$ where L is δ-modular and $s(D) > \delta$. Define $(g, h, \lambda, \eta, \mu) = (g(M), h(M), \lambda(M), \eta(M), \mu(M))$ to be the unique tuple which satisfies (21) and

(23) $\operatorname{rank}(L) = 2g + \lambda$,
$\operatorname{rank}(D) = h$
$\eta = -1$ *if and only if* L has even rank and is not hyperbolic,
μ is the defect of D.

The defect is defined in [3; Definition 8.3], and generalizes the classical notion of defect in quadratic forms over fields of characteristic 2. It occurs only for Type IV situations. Define

$$R(M\,; t) = R(g(M), h(M), \lambda(M), \eta(M), \mu(M)\,; t). \tag{24}$$

Now for $C \in Y(N)/GL_m(S)$, put $R(N, C\,; t) = R(N[C]\,; t)$. Then

$$\alpha(N, t) = \sum_{C \in Y(N)/GL_m(S)} q^{(r-1)v(C)+v_1(C)}\, t^{2v(C)}\, R(N, C\,; t). \tag{25}$$

We shall isolate the greatest common divisor of the summands in (25).

If there is $C \in Y(N)$ such that $N[C]$ is modular, define $\eta_0 = \eta_0(N)$ to be 1 *unless* $N[C]$ has even rank and is not hyperbolic; in the latter case, define $\eta_0 = -1$. If $N[C]$ is *not* modular for any C, put $\eta_0 = 0$. For each $C \in Y(N)$, define

$$\begin{aligned} P(N, C\,; t) &= q^{(r-1)v(C)+v_1(C)}\, t^{2\cdot v(C)} \prod_{j=g_0+\lambda_0}^{g+h+\lambda-1} (1 - q^{j\sigma_3} t^{\sigma_3}) \\ &\times \left\{ \frac{(1 + \eta(1-\mu) q^{(g+h-1)\sigma_3+\sigma_4} t^{\sigma_3})}{(1 + \eta_0 q^{(g_0-1)\sigma_3+\sigma_4} t^{\sigma_3})} \right\} \times \prod_{i=g_0-1}^{g+h-2} (1 + q^{i\sigma_3+\sigma_4} t^{\sigma_3}) \text{ if } \eta_0 \neq 0, \text{ or} \\ P(N, C\,; t) &= q^{(r-1)v(C)+v_1(C)}\, t^{2\cdot v(C)} \prod_{j=g_0+1}^{g+h+\lambda-1} (1 - q^{j\sigma_3} t^{\sigma_3}) \\ &\times \left\{ (1 + \eta(1-\mu) q^{(g+h-1)\sigma_3+\sigma_4} t^{\sigma_3}) \prod_{i=g_0+\lambda_1}^{g+h-2} (1 + q^{i\sigma_3+\sigma_4} t^{\sigma_3}) \right\} \text{ if } \eta_0 = 0, \end{aligned} \tag{26}$$

where, in the second formula, the bracketed expression is 1 if $g + h - 1 < g_0 + \lambda_1$. In fact, $P(N, C\,; t)$ is $R(N, C\,; t)$ divided by the greatest common factor of all polynomials $R(N, C'\,; t)$. Define $P(N\,; t)$ be the sum of $P(N, C\,; t)$ as C

varies over $Y(N)/GL_m(S)$. Essentially, $P(N\,;t)$ is the troublesome generalization of the σ-functions that appear in the Eisenstein series for $SL_2(\mathbf{Q})$.

4. Hermitian matrices of all ranks

Suppose $N_1 \in \Sigma(m, \varepsilon^\rho, S)\#$ has the form

$$N_1 = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}, \tag{27}$$

where $r \in \mathbf{N}$, $N \in \Sigma(r, \varepsilon^\rho, S)\# \cap GL_r(\Delta)$. If $N_1 = 0$, adopt the convention that $r = 0$ and $\alpha(N, t) = 1$; all of the formulas that follow will then be valid. Now

$$a(N_1, t) = F_{m,r}(t)\cdot\alpha(N, q^{m-r}t) \tag{28}$$

where

$$F_{m,r}(t) = \frac{\prod_{i=0}^{m-r-1}(1+q^{i\sigma_3+\sigma_4}t^{\sigma_3})\prod_{i=0}^{m-r-1}(1-q^{i\sigma_3}t^{\sigma_3})}{\prod_{j=0}^{m-r-1}(1-q^{(m-1+j)\sigma_3+\sigma_5}t^{2\sigma_3})}. \tag{29}$$

Define g_0, λ_0, λ_1 and η_0 as in the previous section, for the matrix N. (If $r = 0$, put $g_0 = \lambda_0 = \lambda_1 = 0$ and $\eta_0 = 1$.) Then $\alpha(N_1, t)$ is the product of $P(N, q^{m-r}t)$ times

$$\begin{cases} \left\{\dfrac{\prod_{i=0}^{m-g_0-\lambda_0-2}(1+q^{i\sigma_3+\sigma_4}t^{\sigma_3})\prod_{i=0}^{m-g_0-1}(1-q^{i\sigma_3}t^{\sigma_3})}{\prod_{j=0}^{m-r-1}(1-q^{(m-1+j)\sigma_3+\sigma_5}t^{2\sigma_3})}\right\} \\ \quad\times(1+\eta_0 q^{(m-g_0-\lambda_0-1)\sigma_3+\sigma_4}t^{\sigma_3}) & \text{if } \eta_0 \neq 0 \text{ and } g_0 \neq 0, \\ \left\{\dfrac{\prod_{i=0}^{m-\lambda_0-1}(1+q^{i\sigma_3+\sigma_4}t^{\sigma_3})\prod_{i=0}^{m-1}(1-q^{i\sigma_3}t^{\sigma_3})}{\prod_{j=0}^{m-r-1}(1+q^{(m-1+j)\sigma_3+\sigma_5}t^{2\sigma_3})}\right\} & \text{if } \eta_0 \neq 0 \text{ and } g_0 = 0, \\ \left\{\dfrac{\prod_{i=0}^{m-g_0-\lambda_0+\lambda_1-1}(1+q^{i\sigma_3+\sigma_4}t^{\sigma_3})\prod_{i=0}^{m-g_0-\lambda_0}(1-q^{i\sigma_3}t^{\sigma_3})}{\prod_{j=0}^{m-r-1}(1+q^{(m-1+j)\sigma_3+\sigma_5}t^{2\sigma_3})}\right\} & \text{if } \eta_0 = 0. \end{cases} \tag{30}$$

Table (30) is the correct form of [3; Theorem 5.3].

5. On a question by Professor Nagaoka

Let F_0 be a totally real number field, let K_0/F_0 be a purely imaginary quadratic extension field and let ρ_0 be the Galois involution of K_0/F_0. Let ψ be the ideal character of K_0/F_0. Let $\mathscr{P}$ be a finite prime of F_0, let F be the localization of F_0 at $\mathscr{P}$, and let $K = K_0 \otimes_{F_0} F$ and $\rho = \rho_0 \otimes_{F_0} 1_F$. Let ω be a local generator of $\mathscr{P}$, and put $q = N\mathscr{P}$. To normalize our series, we need to compare $v(\omega)$ with $|\omega|_{\mathscr{P}}^{-1} = q$.

Let p be the rational prime which divides q, and let δ be a generator of the discriminant of $F/\mathbf{Q}_p$. On $\mathbf{Q}_p$, define χ_0 by $\chi_0(t) = e^{2\pi i r}$ for $r \in \mathbf{Q}$ any rational such that $r + t \in \mathbf{Z}_p$. Define χ_F to be the composition of χ_0 with the trace function of $F/\mathbf{Q}_p$. If M is any square matrix over K whose trace t lies in F, define $\chi(M) = \chi_F(t)$.

Let h be a non-zero member of the different of $F/\mathbf{Q}_p$ — that is, the fractional ideal generated by δ^{-1} — and let $b \in \mathbf{N} \cup \{0\}$ such that ω^b divides $h\delta$ while ω^{b+1} does not. We claim that

$$\bar{\alpha}_{\mathscr{P}}^{(2)}\left(s, \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}\right) = (1 - q^{-s})(1 - \psi(\mathscr{P})q^{1-s})(1 - \psi(\mathscr{P})q^{2-s})^{-1}\left(\sum_{j=0}^{b} q^{j(3-s)}\right), \tag{31}$$

where the α-series derives from Eisenstein series for the hermitian modular group of genus 2 as in [1] or [4]. Here, $m = 2$, $r = 1$ and $\varepsilon = \varepsilon^{\rho} = 1$.

The justification of (31) depends on the behavior of $\mathscr{P}$ in K_0. Different factorizations for $\mathscr{P}$ in S require different tables.

Case I: *$\mathscr{P}$ splits.*

This is the situation *not* discussed in the present addendum. Here, $K \cong F \oplus F$, and [3; Part I] applies. Inspection shows that $v(\omega) = 1$, so

$$\bar{\alpha}_{\mathscr{P}}^{(2)}\left(s, \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}\right) = \alpha\left(\begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix} . q^{-s}\right).$$

Although the discriminant is not mentioned by name in [3; Part 1], it is referred to in its role as genarator of the fractional ideal

$$I = \{s \in F : \chi(R \cdot s) = \{1\}\}.$$

The indexing set for the polynomial $p(E, t)$ defined in [3; (2.4)] for the 1×1 matrix (δh) can be represented by $\{\omega^j\}_{j=0}$. Thus,

$$p(\delta h, t) = \sum_{j=0}^{b} t^j. \tag{32}$$

Using [3; (2.6)] for parameters $k = r = 2$, $m = 1$ and σ (as defined in [3; Theorem 2.1]) equal to 1, we get

$$\bar{\alpha}^{(2)}_{\mathscr{P}}\left(s, \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}\right) = (1 - q^{-s})(1 - q^{1-s})(1 - q^{2-s})^{-1}\left(\sum_{j=0}^{b} q^{j(3-s)}\right). \tag{33}$$

Since $\psi(\mathscr{P}) = 1$, (33) is (31).

All remaining cases refer to the new tables. Let us make some general comments.

Hereafter, we assume K is a field extension of F. Let S be its ring of integers, and let δ_K be its discriminant as a $\mathbf{Q}_p$-extension. Let π be a generator of the prime of S.

We begin with a minor issue of normalization. For M a square matrix over K, define $\tau(M)$ to be the image of M's trace under the trace map of K/F. Now [3; Part II] consider matrix characters of the form $\zeta \circ \tau$. The character used in [1] or [4] is *not* $\chi_F \circ \tau$. Because the character is evaluated only on matrices whose trace is in F, there is no need to apply the trace of the extension K/F. However, we can describe this standard character as $\chi' \circ \tau$ where $\chi'(x) = \chi_F(x/2)$! Thus, the series of [3; Part II] do emulate the standard local integrals.

As in Case I, the discriminant δ plays a role. Let $k \in \mathbf{N}$. The dual lattice $\sum(k, \varepsilon^{\rho}, S)\#$ consists of all $k \times k$ $(\rho, 1)$-hermitian matrices whose diagonal entries lie in the fraction F-ideal generated by δ^{-1} and whose off-diagonal entries lie in the fractional K-ideal generated by ${\delta_K}^{-1}$. Again, we fix $h \in \delta^{-1}R$.

The key parameters specialize as

$$\begin{aligned} &\sigma = 2,\ \sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = 2,\ \sigma_4 = 1,\ \sigma_5 = 2, \\ &v(\pi) = 2,\ v_1(\pi) = 2 \qquad \text{if } \mathscr{P} \text{ is unramified}, \\ &\sigma = 2,\ \sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = 1,\ \sigma_4 = 0,\ \sigma_5 = 1, \\ &v(\pi) = 1,\ v_1(\pi) = 1 \qquad \text{if } \mathscr{P} \text{ ramifies}. \end{aligned} \tag{34}$$

The unramified situation will divide into two cases.

Regardless of ramification, $\sigma > \sigma_1 + \sigma_2$. Thus, $\mathrm{Cat}(\rho, 1, \delta)$ is of Type I or Type III. In particular, the defect of any hermitian lattice will be 0. Classically, the defect is a concept related to quadratic forms rather than hermitian forms. Its present irrelevance is not surprising.

Regardless of ramification, $v(\omega) = 2$. This means that we wish to replace the variable t by $q^{-s/2}$ to get the appropriate Dirichlet series. In general, the exponential constant factor will be $1/\sigma$.

We generate the polynomial for the matrix $N = (h)$. In this calculation,

$g_0 = 0$ and $\lambda_0 = 1$. The η term for $N[C]$ will always be 1, while η_0 could be 0 or 1, depending on h.

Case II: *$\mathcal{P}$ is unramified, $b = 2y$ is even.*

The polynomial $P(N\,;t)$ is a sum indexed by matrices $c = (\omega^x)$ for $0 \leq x \leq y$. When $x = y$, $N[c]$ is modular, hence, $\eta_0 = 1$, and $P(N, \omega^y\,; t) = q^{2y}t^{4y}$. For $0 \leq x < y$, the key parameters are $g = 0$, $h = 1$, $\lambda = 0$, $\eta = 1$ and $\mu = 0$, which yields

$$P(N, \omega^x\,; t) = q^{2x}\,t^{4x}(1 + qt^2) = q^{2x}t^{4x} + q^{2x+1}\,t^{2(2x+1)}.$$

Consequently,

$$(35)\qquad \begin{aligned} P(N\,; t) &= \sum_{j=0}^{2y} q^j\,t^{2j}.\\ P(N\,; qt) &= \sum_{j=0}^{2y} q^{3j}\,t^{2j} = \sum_{j=0}^{b}(q^3\,t^2)^j. \end{aligned}$$

The extra factor (30) works out to be

$$\frac{(1 + q\,t^2)(1 - t^2)(1 - q^2\,t^2)}{(1 - q^4\,t^4)} = \frac{(1 + q\,t^2)(1 - t^2)}{(1 + q^2\,t^2)}.$$

Now replace t by $q^{-s/2}$ and combine the terms to get

$$(36)\qquad (1 - q^{-s})(1 + q^{1-s})(1 + q^{2-s})^{-1}\left\{\sum_{j=0}^{b} q^{j(3-s)}\right\}.$$

This is exactly (31) after replacing $\psi(\mathcal{P}) = -1$.

Case III: *$\mathcal{P}$ unramified, $b = 2y + 1$ is odd*

In this case, $\eta_0 = 0$, and we use different formulas. Since the relevant category is Type I or Type III, the parameter λ_1 must be 1. For $0 \leq x \leq y$, the parameters are $g = 0$, $h = 1$, $\lambda = 0$, $\eta = 1$, $\mu = 0$, and

$$P(N, \omega^x\,; t) = q^{2x}\,t^{4x}.$$

The combined factor is

$$(37)\qquad \begin{aligned} &\frac{(1 + q\,t^2)(1 + q^3\,t^2)(1 - t^2)(1 - q^2\,t^2)}{(1 - q^4\,t^4)}\left\{\sum_{j=0}^{y} q^{6j}t^{4j}\right\}\\ &= \left\{\frac{(1 + q\,t^2)(1 - t^2)}{(1 + q^2\,t^2)}\right\}\left\{(1 + q^3\,t^2)\sum_{j=0}^{y} q^{6j}t^{4j}\right\} \end{aligned}$$

$$= \left\{\frac{(1+q\,t^2)(1-t^2)}{(1+q^2\,t^2)}\right\}\left\{\sum_{j=0}^{2y+1} q^{3j} t^{2j}\right\}.$$

Again after substitution $t = q^{-s/2}$, we get (31) with $\psi(\mathscr{P}) = -1$.

Case IV: *$\mathscr{P}$ is ramified*

We may choose that $\omega = \pi\pi^{\rho}$. Since $\mathscr{P}$ is ramified, $\mathrm{Cat}(\rho, 1, \delta)$ is Type III. Thus, $g_0 = 0$, $\lambda_0 = 1$ and $\eta_0 = 0$. For each $0 \leq x \leq b$, $P(N, \pi^x; t)$ has parameters $g = 0$, $h = 1$, $\lambda = 0$, $\eta = 1$ and $\mu = 0$. We get

$$P(N\,;t) = \sum_{j=0}^{b} q^j\,t^{2j}, \text{ and } P(N\,;qt) = \sum_{j=0}^{b} q^{3j}t^{2j}. \tag{38}$$

Happily, the extra factor (30) simplifies:

$$\frac{(1+t)(1+qt)(1-t)(1-qt)}{(1-q^2\,t^2)} = (1-t^2).$$

The net rational factor becomes

$$(1-t^2)\sum_{j=0}^{b} q^{3j}t^{2j}. \tag{39}$$

After substitution $t = q^{-s/2}$, we get (31) with $\psi(\mathscr{P}) = 0$.

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