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OSCILLATION OF MODES OF SOME SEMI-STABLE LÉVY PROCESSES

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§1. Introduction

In this paper it is shown that there is a unimodal Lévy process with oscillating mode. After the author first constructed an example of such a self-decomposable process, Sato pointed out that it belongs to the class of semi-stable processes with $\beta < 0$. We prove that all non-symmetric semi-stable self-decomposable processes with $\beta < 0$ have oscillating modes.

A measure μ on $\mathbf{R}$ is said to be *unimodal* with mode $a \in \mathbf{R}$ if $\mu(dx) = c\,\delta_a(dx) + f(x)dx$, where c is non-negative, δ_a is the delta measure at a and $f(x)$ is non-decreasing on $(-\infty, a)$ and non-increasing on (a, ∞). If a measure μ is unimodal, then either its mode is unique or the set of its modes is a closed interval. Let $\{X_t\}$, $t \in [0, \infty)$, be a Lévy process on $\mathbf{R}$ (that is, a stochastically continuous process with stationary independent increments starting at the origin) and let μ_t be the distribution of X_t. The Lévy process $\{X_t\}$ is said to be unimodal if μ_t is unimodal for each t. When a Lévy process $\{X_t\}$ is unimodal, we denote a mode of μ_t by $a(t)$. In case the set of modes of μ_t is a closed interval, there is freedom of choice of $a(t)$. The Lévy process $\{X_t\}$ is said to be *self-decomposable* if μ_t is an L distribution for each t. A self-decomposable Lévy process is simply called a self-decomposable process. Yamazato proves in the celebrated paper [16] that every self-decomposable process is unimodal. We say that a Lévy process $\{X_t\}$ is semi-stable if there exist real numbers β and γ such that $0 < |\beta| < 1$, $1 < \gamma$, $\gamma = |\beta|^{-\lambda}$ $(0 < \lambda \leqq 2)$ and

$$\hat{\mu}_t(z) = \hat{\mu}_{\gamma t}(\beta z) \tag{1.1}$$

for every $z \in \mathbf{R}$ and every $t \geqq 0$, where

$$\hat{\mu}_t(z) = \int_0^\infty e^{izx}\,\mu_t(dx). \tag{1.2}$$

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Semi-stable processes are introduced by Lévy [2].

Many results on unimodality of Lévy processes are obtained by Medgyessy [3], Sato [4, 5, 6], Sato-Yamazato [7], Steutel-van Harn [8], Watanabe [9, 10, 11, 12, 13], Wolfe [14, 15] and Yamazato [16, 17, 18, 19, 20]. Among these works, only Sato [4, 5, 6] investigates behavior of modes of unimodal Lévy processes. He shows in [4] that if a unimodal Lévy process $\{X_t\}$ has mean $m = EX_1$ $(-\infty \leqq m \leqq \infty)$, then

(1.3) $$\lim_{t\to\infty} t^{-1} a(t) = m.$$

Hence $a(t) \to \infty$ in case $0 < m \leqq \infty$ and $a(t) \to -\infty$ in case $-\infty \leqq m < 0$, as $t \to \infty$. The purpose of this paper is to show that a unimodal Lévy process $\{X_t\}$ can have mode $a(t)$ oscillating as $t \to \infty$ if $m = 0$ or if m does not exist. Namely we shall prove the following theorem.

THEOREM 1. *Let $\{X_t\}$ be a non-symmetric semi-stable self-decomposable process with $-1 < \beta < 0$ and $0 < \lambda < 2$. Then $a(t)$ is unique for each $t \geqq 0$, continuous on $[0, \infty)$ and oscillating as $t \to \infty$ and $t \downarrow 0$:*

(1.4) $$\limsup_{t\to\infty} a(t) = \infty, \quad \liminf_{t\to\infty} a(t) = -\infty.$$
$$\limsup_{t\downarrow 0} \operatorname{sgn} a(t) = 1, \quad \liminf_{t\downarrow 0} \operatorname{sgn} a(t) = -1.$$

Moreover, if $0 < \lambda < 1$, then

(1.5) $$\limsup_{t\to\infty} t^{-1}a(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} t^{-1}a(t) = -\infty.$$

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§2. Restatement of Theorem 1

Let $\{X_t\}$ be a Lévy process on $\mathbf{R}$. Then the characteristic function of X_t is expressed as

(2.1) $$E \exp(izX_t) = \exp(t\psi(z)),$$

(2.2) $$\psi(z) = ibz - 2^{-1}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx(1 + x^2)^{-1})\, \nu(dx),$$

where $b \in \mathbf{R}$, $\sigma^2 \geqq 0$ and ν is a measure on $\mathbf{R}$ with $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} x^2(1+x^2)^{-1}\nu(dx) < \infty$, called the Lévy measure of $\{X_t\}$. We define $k(x)$ by $\nu(dx) = |x|^{-1}k(x)dx$, if ν is absolutely continuous. A necessary and sufficient condition for a Lévy process $\{X_t\}$ to be self-decomposable is that ν is absolutely continuous and $k(x)$ is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, \infty)$.

Let $\{X_t\}$ be a semi-stable Lévy process with $-1 < \beta < 0$ and $0 < \lambda < 2$. Then ν is given by

$$\int_{-\infty}^{u-} \nu(dx) = |u|^{-\lambda}\xi(\log|u|) \text{ for } u < 0, \tag{2.3}$$

$$\int_{u+}^{\infty} \nu(dx) = u^{-\lambda}\xi(\log u - \log|\beta|) \text{ for } u > 0,$$

where $\xi(x)$ is a positive right-continuous periodic function on R with period $-2\log|\beta|$. Further $\psi(z)$ defined in (2.1) is represented as follows:

$$\psi(z) = \int_{-\infty}^{\infty} (e^{izx} - 1)\nu(dx) \tag{2.4}$$

for $0 < \lambda < 1$,

$$\psi(z) = \int_{-\infty}^{\infty} (e^{izx} - 1 - izx)\nu(dx) \tag{2.5}$$

for $1 < \lambda < 2$, and

$$\psi(z) = ibz + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx(1+x^2)^{-1})\,\nu(dx) \tag{2.6}$$

with

$$2b + \int_{-\infty}^{\infty} \frac{(1-\beta^2)x^3}{(1+x^2)(1+\beta^2x^2)}\,\nu(dx) = 0 \tag{2.7}$$

for $\lambda = 1$. Conversely these are sufficient conditions for a Lévy process $\{X_t\}$ to be semi-stable with $-1 < \beta < 0$ and $0 < \lambda < 2$. This is easily proved by using the discussion of Kagan-Linnik-Rao [1]. Note that $E|X_1| = \infty$ for $0 < \lambda \leqq 1$ and $EX_1 = 0$ for $1 < \lambda < 2$. Thus a Lévy process $\{X_t\}$ is self-decomposable and semi-stable with $-1 < \beta < 0$ and $0 < \lambda < 2$ if and only if the following conditions are satisfied:

(S.1) ν is represented as

(2.8) $$\nu(dx) = |x|^{-\lambda-1}\eta(\log|x|)dx \quad \text{for } x<0,$$
$$= x^{-\lambda-1}\eta(\log x - \log|\beta|)dx \quad \text{for } x>0,$$

where $\eta(x)$ is a positive right-continuous periodic function on $\mathbf{R}$ with period $-2\log|\beta|$.

(S.2) $\exp(-\lambda x)\eta(x)$ is non-increasing on $\mathbf{R}$.

(S.3) The equation (2.4), (2.5), or (2.6) with (2.7) holds according as $0<\lambda<1$, $1<\lambda<2$, or $\lambda=1$.

In general there are two possible cases for a unimodal Lévy process $\{X_t\}$:

Case 1. For each t zero is a mode of μ_t.

Case 2. For some t_0 zero is not a mode of μ_{t_0}.

Let $\{X_t\}$ be a semi-stable self-decomposable process with $-1<\beta<0$ and $0<\lambda<2$. Since $\{X_t\}$ is self-decomposable, μ_t is absolutely continuous and unimodal for each $t>0$. Let $\mu_t(dx)=f_t(x)dx$ for $t>0$. We find from the representation (2.8) of ν that $a(t)$ is unique for each $t \geqq 0$ by Theorem 1.3 of Sato-Yamazato [7] and hence $a(t)$ is continuous on $[0, \infty)$ by Lemma 2.1 of Sato [5]. We see from semi-stability that

(2.9) $$f_{\gamma t}(x) = |\beta| f_t(\beta x),$$

which implies that

(2.10) $$a(\gamma t) = \beta^{-1}a(t).$$

Repeating this procedure, we find that

(2.11) $$a(\gamma^n t) = \beta^{-n}a(t)$$

for every integer n. Hence if $\{X_t\}$ is in Case 2, then $a(\gamma^n t_0)$ is oscillating as $n\to\infty$ and $\operatorname{sgn} a(\gamma^n t_0)$ is oscillating as $n\to-\infty$ and satisfies (1.4). That is, $a(t)$ is continuous on $[0, \infty)$ and oscillating as $t\to\infty$ and $\operatorname{sgn} a(t)$ is oscillating as $t\downarrow 0$. Moreover, if $0<\lambda<1$, then

(2.12) $$\frac{a(\gamma^n t_0)}{\gamma^n t_0} = \frac{a(t_0)}{t_0(\gamma\beta)^n}$$

with $|\beta\gamma| = |\beta|^{1-\lambda} < 1$ and hence $t^{-1}a(t)$ is oscillating as $t\to\infty$ and satisfies (1.5). Thus if we show the following theorem, then Theorem 1 is true.

THEOREM 1′. *Let* $\{X_t\}$ *be a semi-stable self-decomposable process with* $-1 < \beta < 0$ *and* $0 < \lambda < 2$. *If* $\{X_t\}$ *is non-symmetric, then it is in Case* 2.

Let us denote by $\mathrm{Re}\, w$ and $\mathrm{Im}\, w$ the real part and the imaginary part of a complex number w, respectively.

We see from (1.1) and (2.1) that every non-symmetric semi-stable process with $-1 < \beta < 0$ satisfies the following balancing condition:

(B) There exist positive numbers θ_1 and θ_2 such that $\theta_2 > \theta_1$, $\mathrm{Im}\, \psi(\theta_1) \neq 0$ and $\mathrm{Im}\, \psi(\theta_2) = 0$.

In fact, there exists $\theta_1 > 0$ such that $\mathrm{Im}\, \psi(\theta_1) \neq 0$, since the process is non-symmetric. Note that $\mathrm{Im}\, \psi(z)$ is a continuous odd function. Hence, from semi-stability with $-1 < \beta < 0$, $\mathrm{Im}\, \psi(|\beta|^{-1} \theta_1) = -\gamma\, \mathrm{Im}\, \psi(\theta_1)$, which yields the existence of θ_2 such that $|\beta|^{-1} \theta_1 > \theta_2 > \theta_1$ and $\mathrm{Im}\, \psi(\theta_2) = 0$.

In Section 3 we shall prove the following theorem, which is a generalization of Theorem 1′.

THEOREM 2. *Let* $\{X_t\}$ *be a self-decomposable process satisfying* (B). *Then* $\{X_t\}$ *is in Case* 2.

§3. Proof of Theorem 2

In order to prove Theorem 2, we need several lemmas. A Lévy process is said to be non-deterministic, if it is not a deterministic motion.

LEMMA 3.1. *Let* $\{X_t\}$ *be a non-deterministic self-decomposable process. Then we have*

(i) $\mathrm{Re}\, \psi(z)$ *is a continuous even function on* $\mathbf{R}$ *and* $-\mathrm{Re}\, \psi(z)$ *is positive and increasing on* $(0, \infty)$ *satisfying* $\mathrm{Re}\, \psi(0) = 0$ *and* $\lim_{z\to\infty} -\mathrm{Re}\, \psi(z) = \infty$.

(ii) $\mathrm{Im}\, \psi(z)$ *is a continuous odd function on* $\mathbf{R}$.

Proof. We shall only prove that $-\mathrm{Re}\, \psi(z)$ is increasing on $(0, \infty)$, since the other assertions are trivial. We obtain from (2.2) that

$$-\mathrm{Re}\, \psi(z) = 2^{-1} \sigma^2 z^2 + \int_0^\infty (1 - \cos zx) x^{-1} h(x) dx, \tag{3.1}$$

where $h(x) = k(x) + k(-x)$ is non-increasing on $(0, \infty)$ by self-decomposability. Let $0 < z_1 < z_2$. We have

$$(3.2)\qquad -\operatorname{Re}\psi(z_2)+\operatorname{Re}\psi(z_1)$$
$$=2^{-1}\sigma^2(z_2^2-z_1^2)+\int_0^\infty(1-\cos u)u^{-1}\Big(h\Big(\frac{u}{z_2}\Big)-h\Big(\frac{u}{z_1}\Big)\Big)du\geqq 0.$$

In (3.2) the equality "$=0$" holds if and only if

$$(3.3)\qquad \sigma=0 \text{ and } h\Big(\frac{x}{z_2}\Big)=h\Big(\frac{x}{z_1}\Big) \text{ for every } x>0,$$

since we can assume that $h(x)$ is right-continuous on $(0,\infty)$. The condition (3.3) shows that, for every $x>0$,

$$(3.4)\qquad h(x)=h\Big(\Big(\frac{z_2}{z_1}\Big)^n x\Big)\to 0$$

as $n\to\infty$, which yields $\nu=0$. Therefore, the equality "$=0$" in (3.2) does not hold, since $\{X_t\}$ is non-deterministic. Thus we have proved Lemma 3.1.

LEMMA 3.2. *Let $\{X_t\}$ be a non-deterministic self-decomposable process. Then, for every $z_1\in\mathbf{R}$, there exist positive numbers $c(z_1)$ and $\delta(z_1)$ such that*

$$(3.5)\qquad |\operatorname{Re}\psi(z)-\operatorname{Re}\psi(z_1)|\geqq c(z_1)|z-z_1|^3$$

for all z satisfying $|z-z_1|\leqq\delta(z_1)$.

Proof. Suppose that $\sigma^2>0$. Then we find from (3.2) that

$$(3.6)\qquad |\operatorname{Re}\psi(z)-\operatorname{Re}\psi(z_1)|\geqq 2^{-1}\sigma^2|z^2-z_1^2|$$

for every z_1 and z. Setting $c(0)=2^{-1}\sigma^2$, $\delta(0)=1$ and, for $z_1\neq 0$, $c(z_1)=4^{-1}\sigma^2|z_1|$ and $\delta(z_1)=(2^{-1}|z_1|)\wedge 1$, we get (3.5). Hence, from now on, we assume that $\sigma=0$. We divide the remaining proof into two cases.

(i) Suppose that $z_1=0$. Then we obtain from (3.1) that

$$(3.7)\qquad -\operatorname{Re}\psi(z)=I_1(z)+I_2(z),$$

where

$$I_1(z)=\int_0^\varepsilon(1-\cos zx)x^{-1}h(x)dx$$

and

$$I_2(z)=\int_\varepsilon^\infty(1-\cos zx)x^{-1}h(x)dx$$

for $0 < \varepsilon < \infty$. Noting that $I_2(z) \geqq 0$, we see that

$$\lim_{z\to 0}\frac{-\operatorname{Re}\psi(z)}{z^2} \geqq \lim_{z\to 0}\frac{I_1(z)}{z^2} = \int_0^{\varepsilon} 2^{-1} xh(x)dx > 0, \tag{3.8}$$

which implies (3.5) for sufficiently small positive numbers $c(0)$ and $\delta(0)$.

(ii) Suppose that $z_1 \neq 0$. Without loss of generality, we can assume $z_1 > 0$. Define $h_1(x) = h(x) - h(x)\wedge\varepsilon$ and $h_2(x) = h(x)\wedge\varepsilon$ for sufficiently small $\varepsilon > 0$ so that $h_1(x)$ does not identically vanish. Then (3.1) is expressed as

$$-\operatorname{Re}\psi(z) = J_1(z) + J_2(z), \tag{3.9}$$

where

$$J_j(z) = \int_0^{\infty}(1-\cos zx)x^{-1}h_j(x)dx$$

for $j = 1,2$. We find from Lemma 3.1 that $J_1(z)$ and $J_2(z)$ are increasing on $(0, \infty)$. Hence

$$|\operatorname{Re}\psi(z) - \operatorname{Re}\psi(z_1)| \geqq |J_1(z) - J_1(z_1)|. \tag{3.10}$$

Differentiating $J_1(z)$, we have

$$\begin{aligned}\frac{d}{dz}J_1(z) &= \int_0^{\infty}(\sin zx)\,h_1(x)dx \\ &= z^{-1}\sum_{n=0}^{\infty}\int_{2n\pi}^{(2n+1)\pi}(\sin u)\Big(h_1\Big(\frac{u}{z}\Big) - h_1\Big(\frac{u+\pi}{z}\Big)\Big\}du \geqq 0\end{aligned} \tag{3.11}$$

for $z > 0$, because $h_1(x)$ is non-increasing on $(0, \infty)$. If $(d/dz)J_1(z_1) > 0$, then (3.5) follows from (3.10) for sufficiently small positive numbers $c(z_1)$ and $\delta(z_1)$. Suppose that $(d/dz)J_1(z_1) = 0$. We find from (3.11) that $(d/dz)J_1(z_1) = 0$ if and only if

$$h_1\Big(\frac{2n\pi}{z_1}+\Big) = h_1\Big(\frac{2(n+1)\pi}{z_1}-\Big) \tag{3.12}$$

for every non-negative integer n, that is, $h_1(x)$ is written as

$$h_1(x) = \sum_{j=1}^{N}\varepsilon_j I_{(0,b_j)}(x), \tag{3.13}$$

for $x > 0$, where N is a positive integer and, for each j, ε_j is a positive number, $b_j = z_1^{-1}2n_j\pi$ for some positive integer n_j and $I_{(0,b_j)}(x)$ is the indicator function of the interval $(0, b_j)$. We obtain from (3.13) that

$$\frac{d}{dz}J_1(z) = \sum_{j=1}^{N} \varepsilon_j z^{-1}(1 - \cos zb_j). \tag{3.14}$$

Differentiating (3.14) and then letting $z = z_1$,

$$\frac{d^2}{dz^2}J_1(z_1) = \sum_{j=1}^{N} \varepsilon_j\{- z_1^{-2}(1 - \cos z_1b_j) + z_1^{-1} b_j \sin z_1b_j\} = 0 \tag{3.15}$$

and

$$\begin{aligned}\frac{d^3}{dz^3}J_1(z_1) &= \sum_{j=1}^{N} \varepsilon_j\{2z_1^{-3}(1 - \cos z_1b_j) - 2z_1^{-2} b_j \sin z_1b_j \\ &\qquad + z_1^{-1} b_j^2 \cos z_1b_j\} \\ &= \sum_{j=1}^{N} \varepsilon_j z_1^{-1} b_j^2 > 0.\end{aligned} \tag{3.16}$$

These show that (3.5) is true for $z_1 > 0$ with sufficiently small positive numbers $c(z_1)$ and $\delta(z_1)$ when $(d/dz)J_1(z_1) = 0$. The proof of Lemma 3.2 is complete.

Let us denote the complex plane by $\mathbf{C}$.

LEMMA 3.3. *Let $\{X_t\}$ be a non-deterministic self-decomposable process. Suppose that $\{X_t\}$ is in Case 1. Let $c_1 = 2/h(0+)$ if $\sigma = 0$ and $0 < h(0+) < \infty$. Let $c_1 = 0$ if $h(0+) = \infty$ or if $\sigma^2 > 0$. Let*

$$D = \{\bigcup_{z\geq 0} L_z\} \cup \{w \in \mathbf{C} : \mathrm{Re}\, w < 0\} \tag{3.17}$$

with $L_z = \{w \in \mathbf{C} : w = -\mathrm{Re}\, \psi(z) + yi,\ |y| > |\mathrm{Im}\, \psi(z)|\}$, that is, D is the connected component containing -1 of the set $\mathbf{C} \cap \{-\psi(z) : z \in \mathbf{R}\}^c$. Then

$$\int_{-\infty}^{\infty} \frac{z\alpha \exp[c\{\alpha + \psi(z)\}]}{\alpha + \psi(z)}\, \mathrm{d}z = 0 \tag{3.18}$$

for every $c > c_1$ and $\alpha \in D$.

Proof. From Lemma 2.4 of Sato-Yamazato [7], we find that $|z\exp(t\psi(z))|$ is integrable on $\mathbf{R}$ with respect to z for $t > c_1$. Hence the density function $f_t(x)$ of $\mu_t(dx)$ is continuously differentiable in x for $t > c_1$. Since $\{X_t\}$ is in Case 1,

$$\frac{d}{dx}f_t(0) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} z\exp(t\psi(z))\, dz = 0 \tag{3.19}$$

for $t > c_1$. We have

$$(3.20)\qquad \int_c^\infty |z\exp[t\{\alpha+\psi(z)\}]|\,dt = -\frac{|z|\exp[c\{\operatorname{Re}\alpha+\operatorname{Re}\psi(z)\}]}{\operatorname{Re}\alpha+\operatorname{Re}\psi(z)},$$

which is integrable on $\mathbf{R}$ with respect to z for $c > c_1$ and $\operatorname{Re}\alpha < 0$. By using Fubini's theorem, we obtain from (3.19) that

$$(3.21)\qquad \begin{aligned} 0 &= \int_c^\infty dt \int_{-\infty}^\infty z\exp[t\{\alpha+\psi(z)\}]\,dz \\ &= -\int_{-\infty}^\infty \frac{z\exp[c\{\alpha+\psi(z)\}]}{\alpha+\psi(z)}\,dz \end{aligned}$$

for $c > c_1$ and $\operatorname{Re}\alpha < 0$. Define

$$(3.22)\qquad F(\alpha) = \int_{-\infty}^\infty \frac{z\exp[c\{\alpha+\psi(z)\}]}{\alpha+\psi(z)}\,dz$$

and

$$(3.23)\qquad F_N(\alpha) = \int_{-N}^N \frac{z\exp[c\{\alpha+\psi(z)\}]}{\alpha+\psi(z)}\,dz$$

for $c > c_1$, $\alpha \in D$ and $N > 0$. We note from Lemma 3.1 that D is a domain in $\mathbf{C}$ containing the left half plane. Because $F_N(\alpha)$ is analytic in D with respect to α and convergent to $F(\alpha)$ uniformly on every compact set in D as $N \to \infty$, $F(\alpha)$ is analytic in D. We see from (3.21) that $F(\alpha) = 0$ for $\operatorname{Re}\alpha < 0$ and hence $F(\alpha) = 0$ in D by the uniqueness principle. Multiplying α to the equation $F(\alpha) = 0$, we get (3.18). Thus we have proved Lemma 3.3.

Proof of Theorem 2. We find from (B) that $\{X_t\}$ is non-symmetric and non-deterministic. Suppose that $\{X_t\}$ is in Case 1. We shall show that this leads to a contradiction. Without loss of generality, we can assume from (B) that there exist real numbers z_1 and z_2 such that $0 \leqq z_1 < z_2$, $\operatorname{Im}\psi(z_1) = \operatorname{Im}\psi(z_2) = 0$ and $\operatorname{Im}\psi(z) < 0$ on (z_1, z_2). Define

$$(3.24)\qquad g(\alpha, c, z) = \frac{z\alpha\exp[c\{\alpha+\psi(z)\}]}{\alpha+\psi(z)}.$$

Let ε and δ be sufficiently small positive numbers. Let

$E(\delta, 1) = \{z \in \mathbf{R} : z_1 - \delta \leqq |z| \leqq z_1 + \delta\}$,
$E(\delta, 2) = \{z \in \mathbf{R} : z_2 - \delta \leqq |z| \leqq z_2 + \delta\}$,
$E(\delta, 3) = \{z \in \mathbf{R} : z_1 + \delta \leqq |z| \leqq z_2 - \delta\}$ and
$E(\delta, 4) = \{z \in \mathbf{R} : |z| \leqq z_1 - \delta \text{ or } |z| \geqq z_2 + \delta\}$. Then we have

(3.25) $$\int_{-\infty}^{\infty} g(\alpha, c, z)dz = \sum_{j=1}^{4} I_j(\alpha, c, \delta),$$

where $I_j(\alpha, c, \delta) = \int_{E(\delta, j)} g(\alpha, c, z)dz$ for $1 \leqq j \leqq 4$. For complex numbers w_1 and w_2 let us denote by $L(w_1, w_2)$ the directed line-segment from w_1 to w_2 in $\mathbf{C}$. Let $K = \sup_{z_1<z<z_2}(-2 \operatorname{Im} \psi(z))$,

$$\begin{aligned}
\Gamma(\varepsilon, 1) &= L(-\psi(z_1) - \varepsilon i, -\psi(z_1) - Ki),\\
\Gamma(\varepsilon, 2) &= L(-\psi(z_1) - Ki, -\psi(z_2) - Ki),\\
\Gamma(\varepsilon, 3) &= L(-\psi(z_2) - Ki, -\psi(z_2) - \varepsilon i),\\
\Gamma(\varepsilon, 4) &= L(-\psi(z_2) + \varepsilon i, -\psi(z_2) + Ki),\\
\Gamma(\varepsilon, 5) &= L(-\psi(z_2) + Ki, -\psi(z_1) + Ki),\\
\Gamma(\varepsilon, 6) &= L(-\psi(z_1) + Ki, -\psi(z_1) + \varepsilon i),
\end{aligned}$$

and let $\Gamma(\varepsilon)$ be the union of the directed line-segments $\Gamma(\varepsilon, j)$, $j = 1,\ldots,6$. In the following, integrals along $\Gamma(\varepsilon, j)$ or $\Gamma(\varepsilon)$ with respect to α are line integrals. Note that $\Gamma(\varepsilon)$ is contained in D by Lemma 3.1. Hence we obtain from (3.18) in Lemma 3.3 that

(3.26) $$\int_{\Gamma(\varepsilon)} d\alpha \int_{-\infty}^{\infty} g(\alpha, c, z)\, dz = 0$$

for $0 < \varepsilon < K$ and for $c > c_1$. Let $A(\varepsilon)$ be the union of the directed line-segments $\Gamma(\varepsilon, j)$, $j = 2,\ldots,5$, and let $B(\varepsilon)$ be the union of $\Gamma(\varepsilon,1)$ and $\Gamma(\varepsilon, 6)$. Let $\tilde{A}(\varepsilon)$ and $\tilde{B}(\varepsilon)$ denote the sets of points on $A(\varepsilon)$ and $B(\varepsilon)$, respectively. By Lemma 3.1, we can choose sufficiently small positive numbers δ_1 and d_1, which do not depend on ε, such that

(3.27) $$|\alpha + \psi(z)| \geqq d_1$$

for $z \in E(\delta_1, 1)$ and $\alpha \in \tilde{A}(\varepsilon)$. Hence we can find $M_1 > 0$, which does not depend on ε, such that

(3.28) $$|g(\alpha, c, z)| \leqq M_1$$

for $z \in E(\delta_1, 1)$ and $\alpha \in \tilde{A}(\varepsilon)$. It follows that

(3.29) $$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{A(\varepsilon)} I_1(\alpha, c, \delta)d\alpha\\
&= \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{E(\delta,1)} dz \int_{A(\varepsilon)} g(\alpha, c, z)d\alpha = 0.
\end{aligned}$$

On the other hand, we can choose $\delta_2 > 0$ and $M_2 > 0$, which do not depend on ε, such that

$$|g(\alpha, c, z)(\alpha + \psi(z))| \leqq M_2 \tag{3.30}$$

for $z \in E(\delta_2, 1)$ and $\alpha \in \tilde{B}(\varepsilon)$. Hence we have, for $0 < \delta < \delta_2$,

$$\left| \int_{B(\varepsilon)} I_1(\alpha, c, \delta) d\alpha \right| \leqq M_2 \int_{E(\delta,1)} dz \int_{B(\varepsilon)} \frac{|d\alpha|}{|\alpha + \psi(z)|}. \tag{3.31}$$

Define $N = \sup_{z \in E(\delta_2,1)} |\mathrm{Im}\, \psi(z)|$, $L = \sup_{z \in E(\delta_2,1)} |\mathrm{Re}\, \psi(z) - \mathrm{Re}\, \psi(z_1)|$ and $a = |\mathrm{Re}\, \psi(z) - \mathrm{Re}\, \psi(z_1)|^{-1}(K + N)$. For $z \in E(\delta_2, 1)$, $z \neq z_1$, we get that

$$\begin{aligned} \int_{B(\varepsilon)} \frac{|d\alpha|}{|\alpha + \psi(z)|} &= \int_{\varepsilon}^{K} [\{(\mathrm{Re}\, \psi(z) - \mathrm{Re}\, \psi(z_1))^2 + (\mathrm{Im}\, \psi(z) - \theta)^2\}^{-1/2} \\ &\qquad + \{(\mathrm{Re}\, \psi(z) - \mathrm{Re}\, \psi(z_1))^2 + (\mathrm{Im}\, \psi(z) + \theta)^2\}^{-1/2}] d\theta \\ &< 8 \int_0^a (1 + u)^{-1} du \\ &\leqq 8 \log (K + N + L) - 8 \log |\mathrm{Re}\, \psi(z) - \mathrm{Re}\, \psi(z_1)|, \end{aligned} \tag{3.32}$$

where we use $(1 + u^2)^{-1/2} \leqq 2(1 + u)^{-1}$ for $u \geqq 0$. Recalling Lemma 3.2, we obtain from (3.31) and (3.32) that

$$\begin{aligned} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{B(\varepsilon)} I_1(\alpha, c, \delta)\, d\alpha \\ = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{E(\delta,1)} dz \int_{B(\varepsilon)} g(\alpha, c, z)\, d\alpha = 0. \end{aligned} \tag{3.33}$$

Hence we find from (3.29) that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\Gamma(\varepsilon)} I_1(\alpha, c, \delta)\, d\alpha = 0. \tag{3.34}$$

Similarly we get that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\Gamma(\varepsilon)} I_2(\alpha, c, \delta)\, d\alpha = 0. \tag{3.35}$$

Making use of Cauchy's integral formula, we have

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\Gamma(\varepsilon)} I_3(\alpha, c, \delta)\, d\alpha \tag{3.36}$$

$$= \lim_{\delta\to 0} 2\pi i \int_{E(\delta,3)} z(-\psi(z))\, dz$$

$$= -2\pi i\Big(\int_{z_1}^{z_2} z\,\psi(z)\,dz + \int_{-z_2}^{-z_1} z\,\psi(z)\,dz\Big)$$

$$= 4\pi \int_{z_1}^{z_2} z \operatorname{Im} \psi(z)\, dz.$$

Since, for $c > c_1$, $I_4(\alpha, c, \delta)$ is analytic with respect to α in the rectangle $\{w : -\psi(z_1) < \operatorname{Re} w < -\psi(z_2), |\operatorname{Im} w| < K\}$, we see by Cauchy's integral theorem that

$$\lim_{\varepsilon\to 0} \int_{\Gamma(\varepsilon)} I_4(\alpha, c, \delta)\, d\alpha = 0 \tag{3.37}$$

for $c > c_1$. Hence we obtain from (3.26), (3.34), (3.35), (3.36) and (3.37) that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \int_{\Gamma(\varepsilon)} d\alpha \int_{-\infty}^{\infty} g(\alpha, c, z)\, dz \\ &= 4\pi \int_{z_1}^{z_2} z \operatorname{Im} \psi(z)\, dz < 0 \end{aligned} \tag{3.38}$$

for $c > c_1$. This is a contradiction. Thus the proof of Theorem 2 is complete.

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