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INTEGRAL GEOMETRY UNDER CUT LOCI IN COMPACT SYMMETRIC SPACES

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Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

Introduction

The theory of integral geometry has mainly treated identities between integral invariants of submanifolds in Riemannian homogeneous spaces like as $\int_G I(M \cap gN)$ $d\mu_G(g)$, where M and N are submanifolds in a Riemannian homogeneous spaces of a Lie group G and $I(M \cap gN)$ is an integral invariant of $M \cap gN$. For example Poincaré's formula is one of typical identities in integral geometry, which is as follows. We denote by $M(\mathbf{R}^2)$ the identity component of the group of isometries of the plane $\mathbf{R}^2$ with a suitable invariant measure $\mu_{M(\mathbf{R}^2)}$. The Poincaré's formula for two curves c_1 and c_2 in R^2 is given by

$$\int_{M(\mathrm{R}^2)} \#\,(c_1 \cap gc_2)\, d\mu_{M(\mathrm{R}^2)} = 2L(c_1)L(c_2),$$

where $\#\,(X)$ denotes the number of the points of X and $L(c)$ denotes the length of c. See I.7.2 Poincaré's formula in [15] for more information about it. Chern [3], Kurita [9], Brothers [2] and Howard [7] extended this formula to the case of Riemann homogeneous spaces. We use the notation in Howard [7]. Let M and N be submanifolds of finite volume in a Riemannian homogeneous space G/K of a Lie group G which satisfy $\dim(G/K) \leq \dim M + \dim N$. Then

$$\int_G \mathrm{vol}(M \cap gN)\, d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M,\ T_y^\perp N)\, d\mu_{M\times N}(x,\ y).$$

σ_K is an integral invariant, which is defined in Section 1. In the case of $G/K = \mathbf{R}^2$, σ_K is constant and the above formula implies the Poincaré's one. More generally

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in the case that G/K is of constant sectional curvature σ_K is constant and we can get a similar formula as the Poincaré's one. But in general σ_K is complicated and it is difficult to understand its geometric meanings.

In this paper we shall estimate σ_K from above instead of exloring its geometric meanings in detail and the volumes of submanifolds from below by the integral of their intersection numbers with the cut loci in a compact simply connected irreducible symmetric space. We shall show the following theorem in Section 3.

THEOREM 3.1. *Let $M = G/K$ be a compact simply connected irreducible symmetric space, p be the dimension of a Helgason sphere in M and N be a p-dimensional submanifold of finite volume in M. If the condition* (1) *is satisfied, then we obtain*

$$\int_G \#(N \cap gC_o(M))\,d\mu_G(g) \leq C\,\mathrm{vol}(N)\mathrm{vol}(C_0(M)).$$

The above equality holds if for each point x in N there is a Helgason sphere tangent to N at x. If the conditions (1)-(3) *are satisfied, then*

$$\mathrm{vol}(S) \leq \frac{1}{C\,\mathrm{vol}(C_o(M))}\int_G \#(N \cap gC_o(M))\,d\mu_G(g) \leq \mathrm{vol}(N)$$

for a Helgason sphere S in M and N whose inclusion map is not null homotopic. In particular, a Helgason sphere is volume minimizing in the class of submanifolds of dimension p whose inclusion maps are not null homotopic.

After Theorem 3.1 we shall apply it to compact symmetric spaces of rank one in Theorem 3.2, compact Hermitian symmetric spaces in Theorem 3.4 and quaternionic Grassmann manifolds in Theorem 5.1. Lê [10] has also applied the method of integral geometry to some estimates of the volume of submanifolds and explicit estimates mainly in Grassmann manifolds.

Concerning Helgason spheres Helgason [5] established these spheres in compact irreducible symmetric spaces. Ohnita [12] proved that Helgason spheres in compact simply connected irreducible symmetric spaces are stable as minimal submanifolds. Lê has proved the volume minimizing property of the Helgason spheres in all compact simply connected irreducible symmetric spaces by a different method, that is, *geodesic defect* in [11].

I would like to thank Professor Yoshihiro Ohnita for directing my attention to integral geometry and fruitful discussion on the subjects treated in this paper.

1. Integral geometry in homogeneous spaces

In this section we shall mention some definitions and fundamental properties of integral geometry in homogeneous spaces.

Before we mention integral geometry in homogeneous spaces, we give some definitions according to Howard [7]. Let E be a finite-dimensional real vector space with an inner product $\langle\ ,\ \rangle$. This inner product naturally induces an inner product on the exterior algebra $\wedge^k E$ of degree k and its norm on $\wedge^k E$ is denoted by $|\cdot|$. If $e_1, \ldots, e_n$ is an orthonormal basis of E, then

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \quad (1 \leq i_1 < \cdots < i_k \leq n)$$

is an orthonormal basis of $\wedge^k E$. For vector subspaces V and W with orthonormal bases $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ respectively, we define $\sigma(V, W)$ by

$$\sigma(V, W) = |v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q|.$$

This is independent of the choice of orthonormal bases.

LEMMA 1.1. *Let V and W be vector subspaces of a real vector space E with an inner product such that* $\dim E = \dim V + \dim W$. *Then we obtain*

$$\sigma(V, W) = \sigma(V^{\perp}, W^{\perp}).$$

Proof. Let $\langle\ ,\ \rangle$ be the inner product and $p = \dim V$, $q = \dim W$. At first we consider the case $V \cap W \neq \{0\}$. In this case $\sigma(V, W) = 0$. Since $(V + W)^{\perp} \neq \{0\}$ and $(V + W)^{\perp} \subset V^{\perp} \cap W^{\perp}$, we obtain $\sigma(V^{\perp}, W^{\perp}) = 0$, thus

$$\sigma(V, W) = \sigma(V^{\perp}, W^{\perp}).$$

Next we consider the case $V \cap W = \{0\}$. In this case there is a linear map φ from $W^{\perp}$ to W such that

$$V = \{x + \varphi(x) \mid x \in W^{\perp}\}.$$

Then we obtain

$$V^{\perp} = \{w - {}^t\varphi(w) \mid w \in W\}.$$

Define a symmetric bilinear form B on $W^{\perp}$ by

$$B(x, y) = \langle \varphi(x), \varphi(y) \rangle \quad \text{for } x, y \in W^{\perp}.$$

We choose an orthonormal basis $x_1, \ldots, x_p$ of $W^{\perp}$ which diagonalizes B. By this choice

$$\frac{x_1+\varphi(x_1)}{\sqrt{1+|\varphi(x_1)|^2}},\ldots,\frac{x_p+\varphi(x_p)}{\sqrt{1+|\varphi(x_p)|^2}}$$

is an orthonormal basis of V and we obtain

$$\sigma(V, W)=\prod_{i=1}^{p}\frac{1}{\sqrt{1+|\varphi(x_i)|^2}}.$$

Here

$$1+|\varphi(x_i)|^2=\langle(1_{W^\perp}+{}^t\varphi\varphi)(x_i), x_i\rangle,$$

and

$$\prod_{i=1}^{p}(1+|\varphi(x_i)|^2)=\det(1_{W^\perp}+{}^t\varphi\varphi).$$

Therefore we obtain

$$\sigma(V, W)=\det(1_{W^\perp}+{}^t\varphi\varphi)^{-1/2}.$$

Similarly

$$\sigma(V^\perp, W^\perp)=\det(1_W+\varphi{}^t\varphi)^{-1/2}.$$

Some calculations on linear algebra imply the identity

$$\det(1_{W^\perp}+{}^t\varphi\varphi)=\det(1_W+\varphi{}^t\varphi)$$

and we obtain

$$\sigma(V, W)=\sigma(V^\perp, W^\perp).$$

Let G be a Lie group and K be a closed subgroup of G. We assume that the image under the linear isotropy representation of K at the origin o of the homogeneous space G/K is compact. Then G has a left invariant Riemannian metric $\langle\, ,\rangle$ which is also invariant under the right multiplication by all elements of K. Fix such a Riemannian metric $\langle\, ,\rangle$ on G. It induces a Riemannian metric on G/K which is invariant under the left action of G on G/K. Let X be a Riemannian manifold. We denote by μ_X the Riemannian measure induced by the Riemannian metric of X. We regard K as a Riemannian manifold with the induced Riemannian metric. For g, h in G, a vector subspace V of $T_{go}(G/K)$ and a vector subspace W of $T_{ho}(G/K)$, we define $\sigma_K(V, W)$ by

$$\sigma_K(V, W) = \int_K \sigma(dg_o^{-1}V, dk_o^{-1}dh_o^{-1}W)d\mu_K(k). \tag{1.1}$$

This is independent of the choice of g and h. The following theorem is a special case of Poincaré's formula (3.4) in [7].

THEOREM 1.2. *Let M and N be submanifolds of finite volume in G/K. Assume that* $\dim M + \dim N = \dim(G/K)$ *and that G is unimodular. Then*

$$\int_G \#(M \cap gN)d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N)d\mu_{M\times N}(x, y),$$

where the symbol $\# S$ means the number of all elements of a set S.

In Section 3 we shall use this formula when G/K is a compact symmetric space and N is the cut locus at the origin of G/K.

2. Cut loci and Helgason spheres

We shall review the notion of the cut locus of a point in a compact connected Riemannian manifold and some results on the structure of cut loci in a compact symmetric spaces. Using these we shall consider a relation between cut loci and Helgason spheres in a compact symmetric space.

Let M be a compact connected Riemannian manifold and Exp_x denote the exponential map at x. For a unit vector X in T_xM, $\gamma_X(t) = \mathrm{Exp}_x(tX)$ $(t \in \mathbf{R})$ is a geodesic parameterized by arc length starting from x with the initial direction X. If t is small, γ_X is a minimizing geodesic joining x and $\gamma_X(t)$. If $\gamma_X|_{[0,t_0(X)]}$ is minimizing and if $\gamma_X|_{[0,t]}$ is not minimizing for $t > t_0(X)$, then $t_0(X)X$ is called a *tangent cut point* of x along γ_X and $\mathrm{Exp}(t_0(X)X)$ is called a *cut point* of x along γ_X. The *tangent cut locus* $\tilde{C}_x(M)$ of x is defined by

$$\tilde{C}_x(M) = \bigcup_{\substack{X\in T_xM \\ |X|=1}} t_0(X)X,$$

which is homeomorphic to a hypersphere in T_xM. See Kobayashi [8]. The *cut locus* $C_x(M)$ of x is defined by

$$C_x(M) = \mathrm{Exp}_x(\tilde{C}_x(M)).$$

Put

$$\tilde{B}_x(M) = \bigcup_{\substack{X\in T_xM \\ |X|=1}} \{tX \mid 0 \le t < t_0(X)\},$$

which is homeomorphic to an open ball in T_xM and

$$B_x(M) = M - C_x(M).$$

Then Exp_x induces a diffeomorphism from $\tilde{B}_x(M)$ onto $B_x(M)$. Therefore $B_x(M)$ is homeomorphic to an open ball in $T_x(M)$. Thus we obtain a decomposition of M to a disjoint union of an open cell $B_x(M)$ and a compact subset $C_x(M)$. By this decomposition we obtain the following lemma.

LEMMA 2.1. *Let M be a compact connected Riemannian manifold and S be a subset of M whose inclusion map is not null homotopic. Then $N \cap C_x(M)$ is not empty for any x in M.*

We shall review the notion of compact symmetric spaces and some results on the structure of their cut loci obtained by Sakai [14] and Takeuchi [16]. Let G be a compact connected Lie group and θ be an involutive automorphism of G. Put

$$G_\theta = \{g \in G \mid \theta(g) = g\}.$$

For a closed subgroup K of G which lies between G_θ and the identify component of G_θ, (G, K) is a compact symmetric pair. Since G is compact, there is a bi-invariant Riemannian metric $(\ ,\ )$ on G and it induces a G-invariant Riemannian metric on the homogeneous space $M = G/K$, which is also denoted by $(\ ,\ )$. Consequently M is a compact symmetric space. Conversely any compact symmetric space is constructed in such a way.

We shall describe the cut locus $C_o(M)$ of the origin o in a compact symmetric space $M = G/K$ by a Lie-group theoretical method. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. θ induces an involutive automorphism of $\mathfrak{g}$, which is also denoted by θ. Then we obtain

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Denote

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

We have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

of $\mathfrak{g}$. Take a maximal Abelian subspace $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{p}$ and a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ including $\mathfrak{a}_\mathfrak{p}$. The complexification $\mathfrak{h} = \mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. For each element α in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

An element α in $\mathfrak{h}^*$ is called a *root* of $\mathfrak{g}^C$ wigh respect to $\mathfrak{h}$ if $\mathfrak{g}_\alpha \neq \{0\}$. Let Δ denote the set of all nonzero zoots of $\mathfrak{g}^C$. Then we obtain the root space decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

Let $\Delta(\mathfrak{p})$ denote the set of nonzero roots which do not vanish identically on $\mathfrak{a}_\mathfrak{p}$. Since G is compact, $\alpha(\mathfrak{t}) \subset \sqrt{-1}\mathbf{R}$ for each root α in Δ, so α is real valued on $\sqrt{-1}\mathfrak{t}$. We can choose compatible lexicographic orderings on the dual spaces of $\sqrt{-1}\mathfrak{t}$ and $\sqrt{-1}\mathfrak{a}_\mathfrak{p}$, which means that a real valued form on $\sqrt{-1}\mathfrak{t}$ whose restriction to $\sqrt{-1}\mathfrak{a}_\mathfrak{p}$ is positive is positive. Denote by $\bar{\cdot}$ the restriction of forms on $\mathfrak{t}^C$ to $\mathfrak{a}_\mathfrak{p}^C$ and

$$\Sigma = \{\bar{\alpha} \mid \alpha \in \Delta\} - \{0\},$$

which is called the *root system* of the symmetric space M with respect to $\mathfrak{a}_\mathfrak{p}$. Now we assume that M is irreducible and simply connected. Then Σ is an irreducible root system. Let γ_0 be the highest root in Σ and $\{\gamma_1, \dots, \gamma_r\}$ be the fundamental root system of Σ, where r is the rank of the symmetric space M. Set

$$\mathfrak{a}_s = \sqrt{-1}\{H \in \sqrt{-1}\mathfrak{a}_\mathfrak{p} \mid \gamma_0(H) = \pi,\ \gamma_i(H) \geq 0 \text{ for } 1 \leq i \leq r\},$$

$$\mathfrak{a}_s^0 = \sqrt{-1}\{H \in \sqrt{-1}\mathfrak{a}_\mathfrak{p} \mid \gamma_0(H) = \pi,\ \gamma_i(H) > 0 \text{ for } 1 \leq i \leq r\}.$$

LEMMA 2.2. *Let $M = G/K$ be a compact simply connected irreducible symmetric space. Then the cut locus $C_o(M)$ of the origin o in M is described as follows:*

$$C_o(M) = \bigcup_{k\in K} k\,\mathrm{Exp}_o(\mathfrak{a}_s).$$

The subset $C_o^0(M) = \cup_{k\in K} k\,\mathrm{Exp}_o(\mathfrak{a}_s^0)$ is a submanifold of M and its complement in $C_o(M)$ is stratified by finitely many submanifolds whose dimensions are less than $\dim C_o^0(M)$.

For the proof of this lemma, see Section 3 of Chapter VII in [6], Theorem 5.3 in [14] or Corollary 3 of Theorem 1.1 in [16]. We can regard the Riemannian measure on $C_o^0(M)$ of the induced Riemannian metric as a measure on $C_o(M)$ with respect to which the measure of $C_o(M) - C_o^0(M)$ is zero.

In order to describe the tangent spaces to $C_o^0(M)$ we shall define root spaces of the root system Σ. For each γ in Σ, put

$$\tilde{\mathfrak{g}}_\gamma = \{X \in \mathfrak{g} \mid [H, X] = \gamma(H)X \text{ for all } H \in \mathfrak{a}_\mathfrak{p}^{\mathbf{C}}\}.$$

Then we obtain

$$\tilde{\mathfrak{g}}_\gamma = \sum_{\substack{\alpha\in\Delta \\ \bar{\alpha}=\gamma}} \mathfrak{g}_\alpha.$$

Denote by Σ^+ the set of all positive roots in Σ. For each γ in Σ^+, put

$$\mathfrak{p}_\gamma = \mathfrak{p} \cap (\tilde{\mathfrak{g}}_\gamma + \tilde{\mathfrak{g}}_{-\gamma}).$$

We get an orthogonal direct sum decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{a}_\mathfrak{p} + \sum_{\gamma\in\Sigma^+} \mathfrak{p}_\gamma.$$

We have the following lemma from the proof of Theorem 3 in [18].

LEMMA 2.3. *For k in K and H_0 in $\overset{0}{\mathfrak{a}}_s$, $k\,\mathrm{Exp}_o H_0$ is contained in $C_o^0(M)$ and*

$$T_{k\mathrm{Exp}_o H_0}(C_o^0(M)) = d(k \exp H_0)_o \Big(\{H \in \mathfrak{a}_\mathfrak{p} \mid \gamma_0(H) = 0\} + \sum_{\gamma\in\Sigma^+-\{\gamma_0\}} \mathfrak{p}_\gamma\Big).$$

We shall define a Helgason sphere and mention some fundamental properties of it. Helgason proved the following theorem in [5].

THEOREM 2.4. *Let M be a compact irreducible symmetric space except a real projective space and κ be the maximum of the sectional curvatures of M. Then there exists a totally geodesic sphere of constant sectional curvature κ. Any two such sphere of the same dimension are conjugate under the identity component of the group of all isometries of M.*

DEFINITION 2.5. We call a maximal dimensional sphere mentioned in Theorem 2.4 a *Helgason sphere* in M.

Since the symmetric space M is irreducible, $\mathfrak{g}$ is semisimple, so the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$ is nondegenerate. For each γ in Σ, A_γ, denotes the element in $\mathfrak{a}_\mathfrak{p}^C$ satisfying

$$\langle H, A_\gamma\rangle = \gamma(H) \quad \text{for all } H \in \mathfrak{a}_\mathfrak{p}^{\mathbf{C}}.$$

We can choose such an A_γ, because $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{a}_\mathfrak{p}^{\mathbf{C}}$. γ is real valued on $\sqrt{-1}\mathfrak{a}_\mathfrak{p}$, so $\sqrt{-1}A_\gamma$, is contained in $\mathfrak{a}_\mathfrak{p}$. We denote

$$S = \mathrm{Exp}_o(\mathbf{R}\sqrt{-1}A_{\gamma_0} + \mathfrak{p}_{\gamma_0}).$$

Then S is a Helgason sphere in M and

$$T_o(S) = \mathbf{R}\sqrt{-1}A_{\gamma_0} + \mathfrak{p}_{\gamma_0}. \tag{2.1}$$

By Lemma 2.3 and (2.1) we obtain an orthogonal direct sum decomposition of $T_o(M)$:

$$T_o(M) = T_o(S) + d(k \exp H)_o^{-1} T_{k\mathrm{Exp}H}(C_o^0(M)), \tag{2.2}$$

where $k \in K$ and $H \in \mathfrak{a}_s^0$. In particular,

$$\dim M = \dim S + \dim C_o^0(M).$$

3. Integral geometry under cut loci

We shall show an inequality which estimates the volumes of submanifolds of the same dimension as that of a Helgason sphere from below in a compact simply connected irreducible symmetric space which satisfies certain conditions. Integral geometry under cut loci plays an important role in this estimate.

Let $M = G/K$ be a compact simply connected irreducible symmetric space and p be the dimension of a Helgason sphere in M. We use the notation in Section 2. We set

$$\mathfrak{s} = \mathbf{R}\sqrt{-1}A_{\gamma_0} + \mathfrak{p}_{\gamma_0}$$

and denote by P_S the orthogonal projection from $\mathfrak{p}$ to $\mathfrak{s}$.

Now we consider the following four conditions concerning the cut locus and Helgason sphere in M.

(0) There exists a positive constant C such that

$$\int_K | P_S(\mathrm{Ad}(k)v) |^p d\mu_K(k) = C | v |^p \quad \text{for all } v \in \mathfrak{p}.$$

(1) There exists a positive constant C such that

$$\int_K | P_S(\mathrm{Ad}(k)v) |^p d\mu_K(k) \leq C | v |^p \quad \text{for all } v \in \mathfrak{p}.$$

The equality folds for $v \in \mathfrak{s}$.

(2) For any k in K, the map

$$P_S\mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$$

is conformal, that is, there is a nonnegative constant C_k which satisfies

$$(P_S(\mathrm{Ad}(k)u,\ P_S\mathrm{Ad}(k)v) = C_k(u,\ v) \quad \text{for all } u,\ v \in \mathfrak{s}.$$

(3) For a Helgason sphere S in M,

$$\#\,(S \cap gC_o(M)) = 1 \quad \text{for almost all } g \in G.$$

We note that the condition (0) implies (1).

THEOREM 3.1. *Let $M = G/K$ be a compact simply connected irreducible symmetric space, p be the dimension of a Helgason sphere in M and N be a p-dimensional submanifold of finite volume in M. If the condition* (1) *is satisfied, then we obtain*

$$\int_G \#\,(N \cap gC_o(M))\,d\mu_G(g) \leq C\,\mathrm{vol}(N)\mathrm{vol}(C_o(M)).$$

The above equality holds if for each point x in N there is a Helgason sphere tangent to N at x. If the conditions (1)-(3) *are satisfied, then*

$$\mathrm{vol}(S) \leq \frac{1}{C\,\mathrm{vol}(C_o(M))}\int_G \#\,(N \cap gC_o(M))\,d\mu_G(g) \leq \mathrm{vol}(N)$$

for a Helgason sphere S in M and N whose inclusion map is not null homotopic. In particular, a Helgason sphere is volume minimizing in the class of submanifolds of dimension p whose inclusion maps are not null homotopic.

Proof. For simplicity we denote C_o instead of $C_o(M)$ in this proof. Applying Theorem 1.2 to N and C_o, we obtain

$$\int_G \#\,(N \cap gC_o)\,d\mu_G(g) = \int_{N\times C_o} \sigma_K(T_x^\perp(N),\ T_y^\perp(C_o))\,d\mu_{N\times C_o}(x,\ y).$$

We shall investigate $\sigma_K(T_x^\perp(N),\ T_y^\perp(C_o))$ in detail. By the definition (1.1) of σ_K it is sufficient to consider σ_K at the origin. For a suitable g in G, we obtain $dg_o^{-1}T_y^\perp(C_o) = \mathfrak{s}$ by the orthogonal direct sum decomposition (2.2). Let V be a p-dimensional vector subspace of $\mathfrak{p}$. By Lemma 1.1 $\sigma(V^\perp,\ \mathfrak{s}) = \sigma(V,\ \mathfrak{s}^\perp)$, so $\sigma_K(V^\perp,\ \mathfrak{s}) = \sigma_K(V,\ \mathfrak{s}^\perp)$. We shall estimate the value $\sigma_K(V,\ \mathfrak{s}^\perp)$. Take an orthonormal basis $v_1,\ldots,v_p$ of V. For any k in K

$$\sigma(\mathrm{Ad}(k)V,\ \mathfrak{s}^\perp) = |\,P_S\mathrm{Ad}(k)v_1 \wedge \cdots \wedge P_S\mathrm{Ad}(k)v_p\,|,$$

so we obtain

$$\sigma_K(V,\ \mathfrak{s}^\perp) = \int_K |\,P_S\mathrm{Ad}(k)v_1 \wedge \cdots \wedge P_S\mathrm{Ad}(k)v_p\,|\,d\mu_K(k)$$

$$\leq \int_K |P_S \mathrm{Ad}(k)v_1| \cdots |P_S \mathrm{Ad}(k)v_p| \, d\mu_K(k)$$
$$\leq \prod_{i=1}^{p} \left(\int_K |P_S \mathrm{Ad}(k)v_i|^p \, d\mu_K(k)\right)^{1/p}$$

by Hölder's inequality. If the condition (1) is satisfied, then the last term is less than or equal to the constant C. Therefore

$$\int_G \#(N \cap gC_o)\, d\mu_G(g) \leq C \,\mathrm{vol}(N)\mathrm{vol}(C_o).$$

From now on we suppose that the conditions (1)-(3) are satisfied. In the case that $V = \mathfrak{s}$, we obtain

$$|P_S \mathrm{Ad}(k)v_1 \wedge \cdots \wedge P_S \mathrm{Ad}(k)v_p| = |P_S \mathrm{Ad}(k)v_1| \cdots |P_S \mathrm{Ad}(k)v_p|$$

for any k in K by (2). Moreover all of the functions $|P_S \mathrm{Ad}(k)v_i|$ coincide, so

$$\int_K |P_S \mathrm{Ad}(k)v_1| \cdots |P_S \mathrm{Ad}(k)v_p| \, d\mu_K(k) = C.$$

If S is a Helgason sphere in M, then we obtain

$$\int_G \#(S \cap gC_o)\, d\mu_G(g) = C \,\mathrm{vol}(S)\mathrm{vol}(C_o).$$

It is equal to $\mathrm{vol}(G)$ by (3), thus

$$\mathrm{vol}(S) = \frac{\mathrm{vol}(G)}{C\,\mathrm{vol}(C_o)}.$$

Now we assume that the inclusion map of N to M is not null homotopic. It follows from Lemma 2.1 that N has an intersection with gC_o for any g in G and

$$\mathrm{vol}(G) \leq \int_G \#(N \cap gC_o)\, d\mu_G(g).$$

Therefore we obtain

$$\mathrm{vol}(S) \leq \frac{1}{C\,\mathrm{vol}(C_o)} \int_G \#(N \cap gC_o)\, d\mu_G(g) \leq \mathrm{vol}(N).$$

Since the inclusion map of S is not null homotopic by [11], S is volume minimizing in the class of submanifolds of dimension p whose inclusion maps are not null homotopic.

THEOREM 3.2. *Let M be one of the complex and quaternionic projective spaces and the Cayley projective plane, p be the dimension of a Helgason sphere in M and N be a p-dimensional submanifold of finite volume in M. Then we obtain*

$$\int_G \#(N \cap gC_o(M))\, d\mu_G(g) \leq C\,\mathrm{vol}(N)\mathrm{vol}(C_o(M)).$$

The above equality holds if for each point x in N there is a Helgason sphere tangent to N at x. Moreover

$$\mathrm{vol}(S) \leq \frac{1}{C\,\mathrm{vol}(C_o(M))}\int_G \#(N \cap gC_o(M))\, d\mu_G(g) \leq \mathrm{vol}(N)$$

holds for a Helgason sphere S in M and N whose inclusion map is not null homotopic.

Remark. The estimate in Theorem 3.2 for the case of the complex projective spaces is obtained by Lê in Proposition 2.11 of [10]. She also obtained an estimate of the volume of submanifolds of even dimensions in the complex projective spaces in Proposition 3.10 of [10].

Proof. It is known that M is a compact simply connected symmetric space of rank one. A Helgason sphere in M is a projective line in each case. Since the linear isotropy action of K on the unit sphere in $\mathfrak{p}$ is transitive, (0) is satisfied. If M is a complex (quaternionic) projective space, the map $P_S\mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$ is a complex (quaternionic) linear map for any k in K, so it satisfies (2). In the case that M is the Cayley projective plane, (2) is proved in the proof of Théorème 6.6 in [1]. The condition (3) follows from projective geometry. A cell decomposition of M implies that the homology class represented by a projective line generates $H_p(M\,;\mathbf{Z})$, so the inclusion map of a projective line is not null homotopic. Thus we have proved the theorem by Theorem 3.1.

LEMMA 3.3. *Let M be a compact simply connected irreducible symmetric space whose Helgason sphere is of demension* 2. *Then* (0) *is satisfied.*

Proof. We define a symmetric bilinear form B on $\mathfrak{p}$ by

$$B(u, v) = \int_K (P_s\mathrm{Ad}(k)u,\ P_S\mathrm{Ad}(k)v)\, d\mu_K(k) \quad \text{for } u, v \in \mathfrak{p}.$$

The linear isotropy representation of K on $\mathfrak{p}$ is irreducible and B is invariant under this representation. By Schur's lemma there is a positive constant C such that

$$B(u, v) = C(u, v) \quad \text{for } u, v \in \mathfrak{p}.$$

Therefore

$$\int_K |P_S \mathrm{Ad}(k) v|^2 d\mu_K(k) = C|v|^2 \quad \text{for } v \in \mathfrak{p}.$$

which is the condition (0).

THEOREM 3.4. *Let M be a compact irreducible Hermitian symmetric space and N be a 2-dimensional submanifold of finite volume in M. Then we obtain*

$$\int_G \#(N \cap gC_o(M)) d\mu_G(g) \leq C \operatorname{vol}(N) \operatorname{vol}(C_o(M)).$$

The above equality holds if for each point p in N there is a Helgason sphere tangent to N at p. Moreover

$$\operatorname{vol}(S) \leq \frac{1}{C \operatorname{vol}(C_o(M))} \int_G \#(N \cap gC_o(M)) d\mu_G(g) \leq \operatorname{vol}(N)$$

holds for a Helgason sphere S in M and N whose inclusion map is not null homotopic.

Remark. The estimate in Theorem 3.4 is a generalization of that in Theorem 3.2 for the case of the complex projective spaces.

Proof. It is known that M is simply connected, so we can apply the results obtained above to M. By Example 5.11 in [17] a Helgason sphere S in M is a complex submanifold and isometric to a complex projective line. Thus we can apply Lemma 3.3 to M and (0) is satisfied. The map $P_S \mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$ is a complex linear map for any k in K, so (2) is satisfied.

A certain irreducible unitary representation of G with representation space V induces a canonical imbedding M into the complex projective space $P(V)$, which consists of all complex lines through 0 in V. According to Example 5.11 in [17] the image of S under this imbedding is a complex projective line in $P(V)$, that is, $P(W)$ for a 2-dimensional complex vector subspace W in V. By Corollary 8 in [19] $C_x(M) = M \cap C_x(P(V))$ for x in M, so $\#(S \cap C_x(M)) = 1$ for x in $M \cap P(W^{\perp})$. Therefore

$$\#(S \cap C_x(M)) = 1 \quad \text{for almost all } x \in M,$$

and (3) is satisfied. Thus we have proved the theorem by Theorem 3.1.

4. Cut loci in Grassmann manifolds

We shall show that the condition (3) mentioned in Section 3 holds for the real, complex and quaternionic Grassmann manifolds in this section.

Let $\mathbf{K}$ be one of the fields of complex and quaternionic numbers. The $\mathbf{K}$-Grassmann manifolds $\mathrm{Gr}_m(\mathbf{K}^{m+n})$ consists of all subspaces of $\mathbf{K}$-dimension m in $\mathbf{K}^{m+n}$. As is well known, $\mathrm{Gr}_m(\mathbf{K}^{m+n})$ has a Riemannian metric with respect to which it is a compact symmetric space. It is simply connected.

PROPOSITION 4.1. *The condition* (3) *is satisfied for the* $\mathbf{K}$-*Grassmann manifold* $\mathrm{Gr}_m(\mathbf{K}^{m+n})$.

Proof. Take and fix an orthonormal basis $e_1, \ldots, e_{m+n}$ of $\mathbf{K}^{m+n}$

$$S = \{W \in \mathrm{Gr}_m(\mathbf{K}^{m+n}) \mid \langle e_1, \ldots, e_{m-1}\rangle \subset W \subset \langle e_1, \ldots, e_{m+1}\rangle\}$$

is a Helgason sphere of $\mathrm{Gr}_m(\mathbf{K}^{m+n})$. The cut locus of V in $\mathrm{Gr}_m(\mathbf{K}^{m+n})$ is given by

$$C_V = \{W \in M \mid \dim(W \cap V^{\perp}) \geq 1\}.$$

See [20] and [21], or [13]. Since

$$\dim\langle e_1, \ldots, e_{m+1}\rangle = m+1, \ \dim V^{\perp} = n,$$

the following inequality holds:

$$\dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) \geq 1.$$

We shall investigate the number of the points of $C_V \cap S$ separating the cases dependently on $\dim(\langle e_1, \ldots, e_{p-1}\rangle \cap V^{\perp})$.

We first assume

$$\dim\langle e_1, \ldots, e_{p+1}\rangle \cap V^{\perp}) = 0.$$

We denote by p the orthogonal projection to $\langle e_m, e_{m+1}\rangle$ whose domain is restricted to $\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}$.

$$\begin{aligned} 2 &\geq \dim(\mathrm{Im} p) \\ &= \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) - \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) \\ &= \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) \end{aligned}$$

holds by our assumption, so

$$1 \leq \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) \leq 2.$$

If $\dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) = 1$ holds, then

$$C_V \cap S = \{\langle e_1, \ldots, e_{m-1}\rangle + \langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}\}$$

and thus $\#(C_V \cap S) = 1$. If $\dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) = 2$ holds, then for any W in S the dimension of

$$W \cap (\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) = W \cap V^{\perp}$$

is greater than 0, so W belongs to C_V. Thus we get $S \subset C_V$.

We next assume

$$\dim(\langle e_1, \ldots, e_{m-1}\rangle \cap V^{\perp}) \geq 1.$$

For any W in S, $\dim(W \cap V^{\perp}) \geq 1$ and W belongs to C_V. Thus we get $S \subset C_V$.

By the above argument we get the following equality between two subsets of $\mathrm{Gr}_m(\mathbf{K}^{m+n})$:

$$\begin{aligned} &\{V \mid \#(C_V \cap S) = 1\} \\ &= \{V \mid \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) = 0, \dim(\langle e_1, \ldots, e_{m+1}\rangle \cap V^{\perp}) = 1\}, \end{aligned}$$

which is open and dense in $\mathrm{Gr}_m(\mathbf{K}^{m+n})$. Therefore the condition (3) is satisfied.

5. Quaternionic Grassmann manifolds

We shall show that Theorem 3.1 holds for the quaternionic Grassmann manifolds $M = Sp(m+n)/Sp(m) \times Sp(n)$ in this section. We assume that $m \leq n$. A 4-dimensional sphere $S = Sp(2)/Sp(1) \times Sp(1)$ naturally embedded in M is a Helgason sphere. We set $G = Sp(m+n)$, $K = Sp(m) \times Sp(n)$ in this section.

THEOREM 5.1. *Let M be a quaternionic Grassmann manifold and N be a 4-dimensional submanifold of finite volume in M. Then we obtain*

$$\int_G \#(N \cap gC_o(M))\, d\mu_G(g) \leq C \operatorname{vol}(N) \operatorname{vol}(C_o(M)).$$

The above equality holds if and only if N is a union of some pieces of Helgason spheres in M. Moreover

$$\operatorname{vol}(S) \leq \frac{1}{C \operatorname{vol}(C_o(M))} \int_G \#(N \cap gC_o(M))\, d\mu_G(g) \leq \operatorname{vol}(N)$$

holds for a Helgason sphere S in M and N whose inclusion map is not null homotopic

Proof. We have already proved the condition (3) in the Section 4. So we shall show the conditions (1) and (2) for the quaternionic symmetric space M.

We set

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \middle| A \in Sp(m),\ B \in Sp(n) \right\},$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & X \\ -{}^t\bar{X} & 0 \end{bmatrix} \middle| X \in M(m, n; \mathbf{H}) \right\}.$$

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a canonical direct sum decomposition for the symmetric space M. Since $\mathfrak{p}$ is isomorphic to $M(m, n; \mathbf{H})$, we identify them. The action of K on $\mathfrak{p}$ through this identification is

$$\mathrm{Ad} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} (X) = AX\,{}^t\bar{B} \quad (A \in Sp(m),\ B \in Sp(n),\ X \in M(m, n; \mathbf{H})).$$

The tangent space $\mathfrak{s}$ of the Helgason sphere S is

$$\mathfrak{s} = \left\{ \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} \middle| z \in \mathbf{H} \right\} \subset \mathfrak{p}.$$

We identify $\mathfrak{s}$ to the quaternionic numbers $\mathbf{H}$. For $A \in Sp(m)$, $B \in Sp(n)$ we obtain

$$P_S\left(A \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} {}^t\bar{B} \right) = a_{11} z \bar{b}_{11}.$$

Therefore the maps $P_S\mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$ for $k \in K$ are the actions of $Sp(1) \times Sp(1)$ on $\mathbf{H}$ multiplied by real numbers and conformal. Thus we have proved the condition (2). Moreover the determinant of the map $P_S\mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$ is nonnegative, which will be used in the proof of Corollary 5.2.

Now we consider the condition (1). Let

$$\mathfrak{a}_{\mathfrak{p}} = \left\{ \begin{bmatrix} t_1 & & & 0 \\ & \ddots & & \vdots \\ & & t_m & 0 \end{bmatrix} \middle| t_i \in \mathbf{R} \right\}.$$

Then $\mathfrak{a}_{\mathfrak{p}}$ is a maximal Abelian subspace in $\mathfrak{p}$. Since $\int_K | P_S\mathrm{Ad}(k) v |^4 d\mu_K(k)$ is invariant under the action of K on $\mathfrak{p}$, it is sufficient to consider its value on $\mathfrak{a}_{\mathfrak{p}}$.

We take

$$v = \begin{bmatrix} t_1 & & & 0 \\ & \ddots & & \vdots \\ & & t_m & 0 \end{bmatrix} \in \mathfrak{a}_\mathfrak{p}$$

and investigate $\int_K |P_S \mathrm{Ad}(k)v|^4 d\mu_K(k)$. Set $k = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Since

$$P_S(\mathrm{Ad}(k)v) = P_S\left(A \begin{bmatrix} t_1 & & & 0 \\ & \ddots & & \vdots \\ & & t_m & 0 \end{bmatrix} {}^t\bar{B} \right) = \sum_{i=1}^m a_{1i} t_i \bar{b}_{1i},$$

we get

$$|P_S(\mathrm{Ad}(k)v)|^4 = \sum_{i,j,k,l} t_i t_j t_k t_l a_{1i}\bar{b}_{1i} b_{1j}\bar{a}_{1j} a_{1l}\bar{b}_{1l} b_{1k}\bar{a}_{1k}$$

and

$$\int_K |P_S(\mathrm{Ad}(k)v)|^4 d\mu_K(k) = \sum_{i,j,k,l} t_i t_j t_k t_l \int_K a_{1i}\bar{b}_{1i} b_{1j}\bar{a}_{1j} a_{1l}\bar{b}_{1l} b_{1k}\bar{a}_{1k} d\mu_K(k).$$

This polynomial of t_i is invariant under the action of the Weyl group of the symmetric space M, it is a linear combination of $\sum_i t_i^4$ and $\sum_{i<j} t_i^2 t_j^2$. Its coefficient of t_1^4 is

$$\int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k).$$

The integrand of its coefficient of $t_1^2 t_2^2$ is

$$4|a_{11}|^2|b_{11}|^2|a_{12}|^2|b_{12}|^2 + 2\Re((a_{11}\bar{b}_{11}b_{12}\bar{a}_{12})^2),$$

where $\Re z$ is the real part of z. The equation $\Re(z^2) = 2(\Re z)^2 - |z|^2$ holds for $z \in \mathbf{H}$, the above integrand is

$$2|a_{11}|^2|b_{11}|^2|a_{12}|^2|b_{12}|^2 + 4(\Re(a_{11}\bar{b}_{11}b_{12}\bar{a}_{12}))^2.$$

We have $\Re(a_{11}\bar{b}_{11}b_{12}\bar{a}_{12}) = \Re((\bar{a}_{12}a_{11}\bar{b}_{11}b_{12})$. We can change the variables to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}\mathbf{i} \\ a_{21} & a_{22}\mathbf{i} \end{bmatrix}, \tag{5.1}$$

because the measure we use is invariant. We can also do that for $\mathbf{j}$, $\mathbf{k}$ and get

$$\begin{aligned}
&\int_K 4(\Re((a_{11}\bar{b}_{11}b_{12}\bar{a}_{12}))^2 d\mu_K(k) \\
&= \int_K ((\Re((\bar{a}_{12}a_{11}\bar{b}_{11}b_{12}))^2 + (\Re((-\mathbf{i})\bar{a}_{12}a_{11}\bar{b}_{11}b_{12}))^2 \\
&\qquad + (\Re((-\mathbf{j})\bar{a}_{12}a_{11}\bar{b}_{11}b_{12}))^2 + (\Re((-\mathbf{k})\bar{a}_{12}a_{11}\bar{b}_{11}b_{12}))^2) d\mu_K(k) \\
&= \int_K |\bar{a}_{12}a_{11}\bar{b}_{11}b_{12}|^2 d\mu_K(k) \\
&= \int_K |a_{12}|^2 |a_{11}|^2 |b_{11}|^2 |b_{12}|^2 d\mu_K(k).
\end{aligned}$$

Therefore the coefficient of $t_1^2 t_2^2$ is

$$3\int_K |a_{11}|^2 |a_{12}|^2 |b_{11}|^2 |b_{12}|^2 d\mu_K(k) = 3\int_{Sp(m)} |a_{11}|^2 |a_{12}|^2 \int_{Sp(n)} |b_{11}|^2 |b_{12}|^2.$$

We change the variables to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_{11}+a_{12} & a_{11}-a_{12} \\ a_{21}+a_{22} & a_{21}-a_{22} \end{bmatrix}$$

and get

$$\int_{Sp(m)} |a_{11}|^2 |a_{12}|^2 = \frac{1}{4}\int_{Sp(m)} (|a_{11}|^4 + 2|a_{11}|^2 |a_{12}|^2 + |a_{12}|^4 - 4(\Re(a_{11}\bar{a}_{12}))^2).$$

Thus we obtain

$$\int_{Sp(m)} |a_{11}|^2 |a_{12}|^2 = \frac{1}{2}\int_{Sp(m)} (|a_{11}|^4 + |a_{12}|^4 - 4(\Re(a_{11}\bar{a}_{12}))^2).$$

Using (5.1) and get

$$\int_{Sp(m)} 4(\Re(a_{11}\bar{a}_{12}))^2 = \int_{Sp(m)} |a_{11}|^2 |a_{12}|^2$$

and

$$\int_{Sp(m)} |a_{11}|^2 |a_{12}|^2 = \frac{1}{3}\int_{Sp(m)} (|a_{11}|^4 + |a_{12}|^4).$$

By the change of the variables:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{bmatrix}$$

we get

$$\int_{Sp(m)} |a_{11}|^4 = \int_{Sp(m)} |a_{12}|^4$$

and thus

$$\int_{Sp(m)} |a_{11}|^2 |a_{12}|^2 = \frac{2}{3} \int_{Sp(m)} |a_{11}|^4.$$

Similarly

$$\int_{Sp(n)} |b_{11}|^2 |b_{12}|^2 = \frac{2}{3} \int_{Sp(n)} |b_{11}|^4.$$

Thus we have got that the coefficient of $t_1^2 t_2^2$ is

$$\frac{4}{3} \int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k).$$

By the coefficients obtained above we get

$$\begin{aligned}\int_K |P_S(\mathrm{Ad}(k)v)|^4 d\mu_K(k) &= \int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k) \left(\sum_i t_i^4 + \frac{4}{3} \sum_{i<j} t_i^2 t_j^2\right) \\ &\leq \int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k) \left(\sum_i t_i^2\right)^2 \\ &= \int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k) \, |v|^4.\end{aligned}$$

Set

$$C = \int_K |a_{11}|^4 |b_{11}|^4 d\mu_K(k).$$

Then we get

$$\int_K |P_S(\mathrm{Ad}(k)v)|^4 d\mu_K(k) \leq C |v|^4.$$

The above equality holds if and only if all t_i except one are 0, which means that v belong to $\mathrm{Ad}(k)\mathfrak{S}$ for some k in K. Thus we have proved the condition (1).

In the estimate of $\int_K |P_S \mathrm{Ad}(k) v_1 \wedge \cdots \wedge P_S \mathrm{Ad}(k) v_p| \, d\mu_K(k)$ mentioned in the proof of Theorem 3.1 for $p = 4$, if all of v_i do not belong to a same $\mathrm{Ad}(k)\mathfrak{S}$ for some k in K, the inequality

$$\int_K |P_S \mathrm{Ad}(k) v_1 \wedge \cdots \wedge P_S \mathrm{Ad}(k) v_p| \, d\mu_K(k) < C$$

holds. Therefore the equality

$$\int_G \#(N \cap gC_o) d\mu_G(g) = C \operatorname{vol}(N) \operatorname{vol}(C_o)$$

holds if and only if for each point x in N there is a Helgason sphere tangent to N at x. By Proposition 5.1 in Ohnita [12] the above equality holds if and only if N is a union of some pieces of Helgason spheres in M.

COROLLARY 5.2. *We choose and fix an orientation of the Helgason sphere S in the quaternionic Grassmann manifold M. We denote by $\vec{\mathfrak{s}}$ the wedge product of a positively ordered orthonormal basis of $\mathfrak{s}$ and define a 4-form ω on $\mathfrak{p}$ by*

$$\omega(X_1, X_2, X_3, X_4) = \frac{1}{C} \int_K \langle \mathrm{Ad}(k)(X_1 \wedge X_2 \wedge X_3 \wedge X_4), \vec{\mathfrak{s}} \rangle d\mu_K(k)$$

for $X_i \in \mathfrak{p}$. Then ω can be extended to a parallel 4-form on M, denoted the same symbol ω, and ω is a calibration. ω calibrates the Helgason sphere and $*\omega$ calibrates the cut locus in M. Both of the Helgason sphere and the cut locus are volume minimizing in their homology classes of real coefficient.

Proof. By the definition of ω, it is invariant under the action of K. So we can extend ω to a parallel form on M, because M is a symmetric space. In particular the extended ω is closed. Let X_1, X_2, X_3, X_4 be orthonormal vectors in $\mathfrak{p}$.

$$\begin{aligned}
\omega(X_1, X_2, X_3, X_4) &\leq \frac{1}{C} \left| \int_K \langle \mathrm{Ad}(k)(X_1 \wedge X_2 \wedge X_3 \wedge X_4), \vec{\mathfrak{s}} \rangle d\mu_K(k) \right| \\
&\leq \frac{1}{C} \int_K |\langle \mathrm{Ad}(k)(X_1 \wedge X_2 \wedge X_3 \wedge X_4), \vec{\mathfrak{s}} \rangle| \, d\mu_K(k) \\
&\leq \frac{1}{C} \int_K |P_S \mathrm{Ad}(k)(X_1 \wedge X_2 \wedge X_3 \wedge X_4)| \, d\mu_K(k),
\end{aligned}$$

which is already estimated in the proof of Theorem 5.1. Combining this with the fact that the determinant of the map $P_S \mathrm{Ad}(k) : \mathfrak{s} \to \mathfrak{s}$ is nonnegative, which is also mentioned in the proof of Theorem 5.1, we can see that ω is a calibration and calibrates the Helgason sphere S. By (2.2) $*\omega$ calibrates the cut locus in M. The volume minimizing property of the Helgason sphere and the cut locus is a consequence of a general theory of calibrations. See Harvey-Laswon [4].

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