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ON DIFFERENTIAL POLYNOMIALS I

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Abstract. The content of Part I is nothing else than, the theory of binomial polynomial sequences in infinite variables $(u^{(1)}, u^{(2)}, u^{(3)}, \ldots)$ with weight $u^{(l)} = l$. However, sometimes we are concerned with specialization $u^{(l)} \to (\frac{d}{dn})^l u$, therefore, we call the elements in $K[u^{(1)}, u^{(2)}, u^{(3)}, \ldots]$ differential polynomials. As analogies of special polynomials with binomial property, we may construct special differential polynomials with binomial property.

§1. Differential polynomial sequences

The theory on differential polynomial sequences, is formally nothing else than the theory on polynomial sequences in a system of infinite variables,

$$u = (u^{(1)}, u^{(2)}, u^{(3)}, \ldots)$$

with weight

$$\text{weight}\, u^{(l)} = l \qquad (l \geq 1).$$

However sometimes we are concerned with specializations,

$$u^{(l)} \longrightarrow \left(\frac{d}{ds}\right)^l f(s) \qquad (l \geq 1),$$

therefore we call the elements in $K[u^{(1)}, u^{(2)}, u^{(3)}, \ldots]$ differential polynomials instead of polynomials in $(u^{(l)})_{l \geq 1}$. The main result in this paragraph is the expansion formula for binomial differential polynomials sequences.

1.1. Binomial differential polynomial sequences

We shall first recollect the definition of binomial polynomial sequences, given by R. Mullin and G. C. Rota in [2], and shall generalize it slightly, so that the set of binomial polynomial sequences in wide sense has a module structure with respect to infinite triangular matrices.

DEFINITION 1.1. (R. Mullin-G. C. Rota)
A polynomial sequence $(p_n(x))_{n \geq 0}$ in a polynomial algebra $K[x]$, is called to be binomial, if it satisfies

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i) $p_0(x) \equiv 1$

ii) $\deg p_n(x) = n$

iii) $p_n(x+y) = \sum_{l=0}^{n} \binom{n}{k} p_{n-l}(x)\, p_l(y) \qquad (n \geq 0)$

where K means a field of characteristic zero. The condition

iv) $p_n(0) = 0 \qquad (n \geq 1)$

is a consequence of i) and iii).

DEFINITION 1.2. Replacing ii) by a weaker condition

ii*) $\deg p_n(x) \leq n \qquad (n \geq 1)$

we define binomial polynomial sequence in wide sense.

For each polynomial sequence $(p_n(x))_{n\geq 0}$ we associate its generating functions

$$\Phi_p(x \mid t) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} \tag{1.1}$$

which is a formal power series in t. By means of generating functions, the condition iii) is equivalent to

iii*) $\Phi_p(x+y \mid t) = \Phi_p(x \mid t)\Phi_p(y \mid t)$

PROPOSITION 1.1. *The set $P(K[x])$ of binomial polynomial sequences in wide sense in $K[x]$, coincides with the set of polynomial sequences*

$$\{(p_{\alpha,n}(x))_{n\geq 0} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots),\ \alpha_j \in K\},$$

which are defined by means of generating functions,

$$\Phi_{p_\alpha}(x \mid t) = \exp\left[x \sum_{j=1}^{\infty} \alpha_j \frac{t^j}{j!}\right] = \sum_{n=0}^{\infty} p_{\alpha,n}(x) \frac{t^n}{n!} \tag{1.2}$$

Proof. Since $p_0(x) \equiv 1$, we may put

$$\log \Phi_p(x \mid t) = \log\left(1 + \sum_{j=1}^{\infty} p_j(x) \frac{t^j}{j!}\right) = \sum_{j=1}^{\infty} \varphi_j(x) \frac{t^j}{j!}$$

with polynomial $\varphi_j(x)$ $(j \geq 1)$ in $K[x]$. Then condition $\Phi_p(x+y \mid t) = \Phi_p(x \mid t)\Phi_p(y \mid t)$ is equivalent to $\varphi_j(x+y) = \varphi_j(x) + \varphi_j(y)$ $(j \geq 1)$ and this is also equivalent to $\varphi_j(x) = \alpha_j x$ with constants α_j in K. This means

$$\Phi_p(x \mid t) = \exp\left[x \sum_{j=1}^{\infty} \alpha_j \frac{t^j}{j!}\right]$$

PROPOSITION 1.2.

$$p_{\alpha,n}(x) = \sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} n! \left(\prod_j \frac{1}{l_j!}\left(\frac{\alpha_j}{j!}\right)^{l_j}\right) x^m. \tag{1.3}$$

Proof. From the definition of $(p_{\alpha,n})_{n\geq 0}$ it follows,

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{\alpha,n}(x)\frac{t^n}{n!} &= \exp\left[x \sum_{j=1}^{\infty} \alpha_j \left(\frac{t^j}{j!}\right)^{l_j}\right] \\
&= \prod_j \exp\left[x\alpha_j \frac{t^j}{j!}\right] \\
&= \prod_j \left(\sum_{l_j} \frac{1}{l_j!} x^{l_j} \left(\frac{\alpha_j}{j!}\right)^{l_j} t^{jl_j}\right) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} n! \left(\prod_j \frac{1}{l_j!}\left(\frac{\alpha_j}{j!}\right)^{l_j}\right) x^m\right)
\end{aligned}$$

Similarly to polynomial sequences, we define binomial differential polynomial sequences and these in wide sense.

DEFINITION 1.3. A differential polynomial sequence $(p_n(u))_{n\geq 0}$ in $K[u] = K[u^{(1)}, u^{(2)}, u^{(3)}, \ldots]$ is called to be binomial, if it satisfies

i) $p_0(u) \equiv 1$

ii) $\text{weight}\, p_n(u) = n$

iii) $p_n(u+v) = \sum_{l=0}^{n} \binom{n}{l} p_{n-l}(u)\, p_l(v) \qquad (n \geq 0).$

The condition

iv) $p_n(0) = 0 \qquad (n \geq 1).$

is a consequence of i) and iii).

DEFINITION 1.4. Replacing ii) by a weaker condition

ii*) $\text{weight}\, p_n(u) \leq n \qquad (n \geq 1),$

we define binomial differential polynomial sequences in wide sense.

By means of generation functions, condition iii) is equivalent to

iii*) $\Phi_p(u + v \mid t) = \Phi_p(u \mid t)\Phi_p(v \mid t).$

PROPOSITION 1.3. *The set $DP(K[u])$ of binomial polynomial sequences in wide sense in $K[u]$ coincides with the set of differential polynomial sequences,*

$$\{(p_{\alpha,n}(u))_{n\geq 0} \mid \alpha = (\alpha_{ij})_{1\leq i\leq j};\ \alpha_{ij} \in K\}$$

which are given by means of generating functions,

$$\Phi_{p_\alpha}(u \mid t) = \exp\left[\sum_{1\leq i\leq j} \alpha_{ij} u^{(i)} \frac{t^j}{j!}\right] = \sum_{n=0}^{\infty} p_{\alpha,n}{}^{(u)} \frac{t^n}{n!}. \tag{1.4}$$

Proof. Since $p_0(u) \equiv 1$, we may put

$$\log \Phi_p(u \mid t) = \log\left(1 + \sum_{j=1}^{\infty} p_j(u) \frac{t^j}{j!}\right) = \sum_{j=1}^{\infty} \varphi_j(u) \frac{t^j}{j!}$$

with polynomials $\varphi_j(u)$ of weight at most j $(j \geq 1)$ in $K[u]$. Then the condition $\Phi_p(u + v \mid t) = \Phi_p(u \mid t)\Phi_p(v \mid t)$ is equivalent to $\varphi_j(u + v) = \varphi_j(u) + \varphi_j(v)$ $(j \geq 1)$, and this is also equivalent to

$$\varphi_j(u) = \sum_{i=1}^{j} \alpha_{ij} u^{(i)} \qquad (j \geq 1)$$

with constants α_{ij} in K. This means

$$\Phi_j(u \mid t) = \exp\left[\sum_{j=1}^{\infty} \sum_{i=1}^{j} \alpha_{ij} u^{(i)} \frac{t^j}{j!}\right].$$

From the expansion of $\exp[\sum_{j=1}^{\infty} u^{(j)} t^j/j!]$, we obtain the standard binomial differential polynomial sequence $(p_{\mathrm{I},n}(u))_{n\geq 0}$, which corresponds to the standard binomial polynomial sequence $(x^n)_{n\geq 0}$. The relation between $(u^{(1)}, u^{(2)}, u^{(3)}, \ldots)$ and $(p_{\mathrm{I},1}, p_{\mathrm{I},2}, p_{\mathrm{I},3}, \ldots)$ is very important in this article.

PROPOSITION 1.4.

$$\exp\left[\sum_{j=1}^{\infty} u^{(j)}\frac{t^j}{j!}\right] = \sum_{n=0}^{\infty} p_{\mathrm{I},n}(u)\frac{t^n}{n!} \tag{1.5}$$

$$p_{\mathrm{I},n}(u) = \sum_{\sum jl_j=n} n!\prod_j \frac{1}{l_j!}\left(\frac{u^{(i)}}{j!}\right)^{l_j} \tag{1.6}$$

$$u^{(n)} = \sum_{m=1}^{n} (-1)^{m-1}(m-1)!p_{m,n}\left(p_{\mathrm{I},1},\dots,p_{\mathrm{I},n}\right) \tag{1.7}$$

$$p_{m,n}(p_{\mathrm{I},1},\dots,p_{\mathrm{I},n}) = \sum_{\substack{\sum jl_j=n \\ \sum l_j=m}} n!\left(\prod_j \frac{1}{l_j!}\left(\frac{p_{\mathrm{I},j}}{j!}\right)^{l_j}\right) \quad (1\le m\le n). \tag{1.8}$$

Proof. By calculation we have

$$\exp\left[\sum_{j=1}^{\infty} u^{(j)}\frac{t^j}{j!}\right] = \prod_j \exp\left[u^{(j)}\frac{t^j}{j!}\right] = \prod_j\left(\sum_{l_j=0}^{\infty}\frac{1}{l_j!}\left(\frac{u^{(j)}t^j}{j!}\right)^{l_j}\right)$$
$$= \sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{\sum jl_j=n} n!\prod_j\frac{1}{l_j!}\left(\frac{u^n}{j!}\right)^{l_j}.$$

From Tayler expansion

$$\log(1+x) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}x^m$$

we have

$$\sum_{j=1}^{\infty} u^{(j)}\frac{t^j}{j!} = \log\left(1+\sum_{j=1}^{\infty}p_{\mathrm{I},j}(u)\frac{t^j}{j!}\right)$$
$$= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\sum_{j=1}^{\infty}p_{\mathrm{I},j}(u)\frac{t^j}{j!}\right)^m$$
$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m}\sum_{\substack{\sum jl_j=n \\ \sum l_j=m}} n!m!\prod_j\frac{1}{l_j!}\left(\frac{p_{\mathrm{I},j}(u)}{j1}\right)^{l_j}\right)$$
$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{m=1}^{n}(-1)^{m-1}(m-1)!p_{m,n}(p_{\mathrm{I},1}(u),\dots,p_{\mathrm{I},n}(u)).$$

PROPOSITION 1.5.

$$p_{\mathrm{I},n+1}(u)=\sum_{l=0}^{n}\binom{n}{l}u^{(l+1)}p_{\mathrm{I},n-l}(u)\qquad(n\geq 1). \tag{1.9}$$

Proof. Applying d/dt on the both sides of

$$\sum_{n=0}^{\infty}p_{\mathrm{I},n}(u)\frac{t^n}{n!}=\exp\left[\sum_{j=1}^{\infty}u^{(j)}\frac{t^j}{j!}\right],$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}p_{\mathrm{I},n+1}(u)\frac{t^n}{n!}&=\left(\sum_{j=0}^{\infty}\frac{u^{(j+1)}}{j!}t^j\right)\exp\left[\sum_{j=1}^{\infty}u^{(j)}\frac{t^j}{j!}\right]\\&=\left(\sum_{j=0}^{\infty}\frac{u^{(j+1)}}{j!}t^j\right)\left(\sum_{n=0}^{\infty}p_{\mathrm{I},n}(u)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\sum_{l=0}^{\infty}\binom{n}{l}u^{(l+1)}p_{\mathrm{I},n-l}(u)\right).\end{aligned}$$

The next statement is one of the evidence of the standardness of the differential polynomial sequence $(p_{\mathrm{I},n}(u))_{n\geq 0}$.

PROPOSITION 1.6. *Putting* $Z(s)=\exp[u(s)]$, *we have relations between the derivates;*

$$\frac{Z^{(n)}}{Z(s)}=p_{\mathrm{I},n}(u(s))=\sum_{\sum jl_j=n}n!\prod_j\frac{1}{l_j!}\left(\frac{u^{(\imath)}(s)}{j!}\right)^{l_j}, \tag{1.10}$$

where $Z^{(n)}(s)=\left(\frac{d}{ds}\right)^nZ(s)$ *and* $u^{(j)}(s)=\left(\frac{d}{ds}\right)^ju(s)$.

Proof. From Tayler expansion of $u(s+t)$ it follows,

$$\begin{aligned}\frac{Z(s+t)}{Z(s)}&=\exp[u(s+t)-u(s)]=\exp\left[\sum_{j=1}^{\infty}u^{(j)}(s)\frac{t^j}{j!}\right]\\&=\sum_{n=0}^{\infty}p_{\mathrm{I},n}(u(s))\frac{t^n}{n!}.\end{aligned}$$

1.2. Homomorphisms of $DP(K[u])$ onto $P(K[x])$

$R_\infty(K)$ means the K-algebra of triangular matrices $(a_{ij})_{1\le i\le j}$ with coefficients in K, and $G_\infty(K)$ means the group of triangular matrices $(\gamma_{i,j})_{i\le j}$ with $\gamma_{jj\neq 0}$ $(j\ge 1)$. By means of generating functions, the natural $R_\infty(K)$ module structure on $DP(K[U])$ and $P(K[x])$ are defined as follows,

$$\begin{aligned}\Phi_{p_\alpha}(u\mid t)^\lambda\Phi_{p_\beta}(u\mid t)^\mu &= \Phi_{p_{\lambda\alpha+\mu\beta}}(u\mid t)\\ \Phi_{p_\alpha}(u\mid t)^\gamma &= \Phi_{p_{\alpha\gamma}}(u\mid t) \qquad (\alpha,\beta,\gamma\in R_\infty(K);\ \lambda,\mu\in K),\end{aligned}$$

$$\begin{aligned}\Phi_{p_\alpha}(x\mid t)^\lambda\Phi_{p_\beta}(x\mid t)^\mu &= \Phi_{p_{\lambda\alpha+\mu\beta}}(x\mid t)\\ \Phi_{p_\alpha}(x\mid t)^\gamma &= \Phi_{p_{\alpha\gamma}}(x\mid t) \qquad (\alpha=(\alpha_n)_{n\ge1},\beta=(\beta_n)_{n\ge1};\\ &\qquad\qquad\qquad\qquad \lambda,\mu\in K;\ \gamma\in R_\infty(K)).\end{aligned}$$

PROPOSITION 1.7. *To each formal power serious $f(s)$ without constant term, we associate a mapping ρ_f of $DP(K[u])$ into $P(K[x])$;*

$$(1.11)\quad \rho_f\left(\exp\left[\sum_{j=1}^{\infty}\sum_{i=1}^{j}\alpha_{ij}u^{(i)}\frac{t^j}{j!}\right]\right)=\exp\left[x\sum_{j=1}^{\infty}\left(\sum_{i=1}^{j}\alpha_{ij}f^{(i)}(0)\right)\frac{t^j}{j!}\right],$$

them ρ_f is an $R_\infty(K)$-module homomorphism.

This is a direct consequence of the definitions of $R_\infty(K)$-module structures on $DP(K[u])$ and $P(K[x])$.

PROPOSITION 1.8. *The mapping ρ_∞ defined by*

$$(1.12)\qquad \rho_\infty\left(\exp\left[\sum_{j=1}^{\infty}\sum_{i=1}^{j}\alpha_{ij}u^{(i)}\frac{t^j}{j!}\right]\right)=\exp\left[x\sum_{j=1}^{\infty}\sum_{i=1}^{j}\alpha_{ij}\frac{t^j}{j!}\right]$$

is a $R_\infty(K)$-module homomorphism of $DP(K[u])$ onto $P(K[x])$ such that ρ_∞ induces a vector space isomorphism from the vector subspace

$$W=\left\{(p_{\alpha,n}(u))_{n\ge0}\mid \Phi_{p_\alpha}(u\mid t)=\exp\left[\sum_{j=1}^{\infty}\alpha_j u^{(j)}\frac{t^j}{j!}\right],\ \alpha_j\in K\right\}$$

onto the vector space $P(K[x])$.

Proof. Putting $f(s)=\sum_{j=1}^{\infty}s^j/j!$ and $\rho_\infty=\rho_f$, we observe that ρ_∞ is an $R_\infty(K)$-module homomorphism of $DP(K[u])$ onto $P(K[x])$ satisfying the condition in the proposition.

There exists a very simple and concrete cross section of $P(K[x])$ into $DP(K[u])$ which is unfortunately not a vector space homomorphism.

PROPOSITION 1.9. *Let ν_0 be the mapping of $P(K[x])$ defined by*

$$\nu_0(p_n(x)) = \frac{p_n(D)\exp[u]}{\exp[u]} \tag{1.13}$$

then ν_0 is a cross section of $P(K[x])$ into $DP(K[u])$ such that

i) $\nu_0(x^n) = p_{\mathrm{I},n}(u)$

ii) $\rho_0\nu_0 = \mathrm{id}_{P(K[x])}$

where $D^n u = u^{(n)}$ $(n \geq 1)$ and $\rho_0 = \rho_f$, $f(s) = s$.

Proof. Let y be a variable independent over $K[x]$ and let D' be the derivation acting on a variable v independent over $K[x]$ such that

$$D'^n v = v^{(n)} \qquad (n \geq 1)$$

and

$$(\nu_0(p_n(y)) = \frac{p_n(D')\exp(v)}{\exp(v)}.$$

Since $DD' = D'D$ and $Dv = D'u = 0$, for each element $(p_n(x))_{n\geq 0}$ in $P(K[x])$ we have

$$\begin{aligned}\nu_0(p_n(x+y)) &= \frac{p_n(D+D')\exp[u+v]}{\exp[u+v]}\\ &= \sum_{l=0}^{n}\frac{\binom{n}{l}p_{n-l}(D)p_l(D')(\exp[u]\exp[v])}{\exp[u]\exp[v]}\end{aligned}$$

This means ν_0 maps $P(K[x])$ into $DP(K[u])$. On the other hand, putting $z(s) = \exp[u(s)]$ for a generic function $z(s)$ and $D = \frac{d}{ds}$, by virtue of Proposition 1.6 we have

$$\frac{D^n\exp[u(s)]}{\exp[u(s)]} = \frac{Z^{(n)}(s)}{Z(s)} = p_{\mathrm{I},n}(u(s)) \qquad (n \geq 1),$$

hence

$$\nu_0(x^n) = \frac{D^n\exp[u]}{\exp[u]} = p_{\mathrm{I},n}(u) \qquad (n \geq 1)$$

Since ρ_0 means the specialization

$$u^{(1)} \longrightarrow x, \quad u^{(j)} \longrightarrow 0 \qquad (j \geq 2),$$

this means

$$\rho_0(\nu_0(x^n)) = \rho_0\left(\sum_{\sum jl_j=n} n!\prod_j \frac{1}{l_j!}\left(\frac{u^{(j)}}{j!}\right)^{l_j}\right) = x^n.$$

1.3. Expansion formulas

For each $(p_n(u))_{n\geq 0}$ in $DP(K[u])$ the vector subspace spanned by $p_n(u)$ $(n \geq 1)$ is very thin in $K[u]$, hence in order to treat expansion formulas, it is necessary to introduce a suitable equivalence relation in $DP(K[u])$.

DEFINITION 1.5. Two elements $(p_n(u))_{n\geq 0}$ and $(q_n(u))_{n\geq 0}$ in $DP(K[u])$ are called to be similar each other, if there exist two systems of constant $(\lambda_{m,n})_{1\leq m\leq n}$ and $(\mu_{m,n})_{1\leq m\leq n}$ in K such that

$$q_n(u) = \sum_{m=1}^{n} p_m(u)\lambda_{m,n}, \quad p_n(u) = \sum_{m=1}^{n} q_m(u)\mu_{m,n} \qquad (n \geq 1)$$

THEOREM 1.1. (Expansion theorem) *Let* $(p_n(u))_{n\geq 0}$ *and* $(q_n(u))_{n\geq 0}$ *be binomial differential polynomial sequences. Then* $(p_n(u))_{n\geq 0}$ *and* $(q_n(u))_{n\geq 0}$ *are similar each other, if and only if there exists a system of constants* $(\lambda_j)_{j\geq 1}$ *such that* $\lambda_1 \neq 0$ *and*

$$q_n(u) = \sum_{m=1}^{n} p_m(u)\left(\sum_{\substack{\sum jl_j=n\\ \sum l_j=m}} n!\prod_j \frac{1}{l_j!}\left(\frac{\lambda_j}{j!}\right)^{l_j}\right) \qquad (n \geq 1) \tag{1.14}$$

Condition (1.14) *is equivalent to*

$$\Phi_q(u \mid t) = \Phi_p\left(u \mid \sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!}\right) \tag{1.15}$$

Proof. Let $(p_n(u))_{n\geq 0}$ and $(q_n(u))_{n\geq 0}$ be similar binomial differential polynomial sequences and put

$$q_n(u) = \sum_{m=0}^{n} p_m(u)\lambda_{m,n}$$

with $\lambda_{m,n}$ in K. They $\lambda_{0,0} = 1$, $\lambda_{0,n} = 0$ $(n \geq 1)$ and

$$\begin{aligned} q_n(u+v) &= \sum_{m=0}^{n} p_m(u+v)\lambda_{m,n} \\ &= \sum_{m=0}^{n} \left(\sum_{h=0}^{m} \binom{m}{h} p_{m-h}(u)p_h(v) \right) \lambda_{m,n} \\ &= \sum_{l=0}^{n} \binom{n}{l} q_{n-l}(u)q_l(v) \\ &= \sum_{l=0}^{n} \binom{n}{l} \left(\sum_{a} p_a(u)\lambda_{a,n-l} \right) \left(\sum_{b} p_b(v)\lambda_{b,l} \right) \end{aligned}$$

Comparing the coefficients of $p_{m-h}(u)p_h(v)$ in the both sides of

$$\begin{aligned} &\sum_{m=0}^{n} \left(\sum_{h=0}^{m} \binom{m}{h} p_{m-h}(u)p_h(v) \right) \lambda_{m,n} \\ &\qquad = \sum_{l=0}^{n} \binom{n}{l} \left(\sum_{a} p_a(u)\lambda_{a,n-l} \right) \left(\sum_{b} p_b(v)\lambda_{b,l} \right), \end{aligned}$$

we have

$$\begin{aligned} \binom{m}{h} \lambda_{m,n} &= \sum_{l=0}^{n} \binom{n}{l} \lambda_{m-h,n-l}\lambda_{h,l}, \\ \frac{m!}{n!}\lambda_{m,n} &= \sum_{l} \frac{(m-h)!}{(n-l)!}\lambda_{m-h,n-l}\frac{h!}{l!}\lambda_{h,l} \qquad (0 \leq h \leq m \leq n). \end{aligned}$$

Using this relation, we obtain a nice relation on the power series

$$f_m(t) = \sum_{n=m}^{\infty} \frac{m!}{n!}\lambda_{m,n}t^n \qquad (m \geq 1),$$

as follows

$$\begin{aligned} f_m(t) &= \sum_{n=m}^{\infty} \frac{m!}{n!}\lambda_{m,n}t^n \\ &= \left(\sum_{a=m-h}^{\infty} \frac{(m-h)!}{a!}\lambda_{m-h,a}t^a \right) \left(\sum_{b=h}^{\infty} \frac{h!}{b!}\lambda_{h,b}t^b \right) \\ &= f_{m-h}(t)f_h(t) \qquad (1 \leq h \leq m). \end{aligned}$$

This means

$$f_m(t) = f_1(t)^m = \left(\sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!}\right)^m,$$

where $\lambda_j = \lambda_{1,j}$ $(j \geq 1)$. Hence we have

$$\lambda_{m,n} = \sum_{\substack{\sum j l_j = n \\ \sum l_j = m}} n! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j}{j!}\right)^{l_j}.$$

Moreover

$$\begin{aligned}
\Phi_q(u \mid t) &= \sum_{n=0}^{\infty} q_n(u) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{\substack{\sum j l_j = n \\ \sum l_j = m}} n! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j}{j!}\right)^{l_j} p_m(u) \right) \\
&= \sum_{m=0}^{\infty} \frac{p_m(u)}{m!} \left(\sum_{\sum l_j = m} m! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j t^j}{j!}\right)^{l_j} \right) \\
&= \sum_{m=0}^{\infty} \frac{p_m(u)}{m!} \left(\sum_{j=1}^{\infty} \frac{\lambda_j t^j}{j!}\right)^m = \Phi_p\left(u \mid \sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!}\right)
\end{aligned}$$

For its sake of the invertiblity, we observe $\lambda_1 \neq 0$.

Remark. A variable transformation $t \to \sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!}$ $(\lambda_1 \neq 0)$ induces triangular matrix:

$$\sigma(\lambda) = (\sigma_{m,n}(\lambda))$$

$$\sigma_{m,n}(\lambda) = \begin{cases} 0 & (m > n) \\ p_{m,n}(\lambda) = \displaystyle\sum_{\substack{\sum j l_j = n \\ \sum l_j = m}} n! \prod_j \frac{1}{l_j! \left(\frac{\lambda_j}{j!}\right)^{l_j}} & (m \leq n) \end{cases}$$

such that

$$\sum_{j=1}^{\infty} u^{(j)} \frac{1}{j!} \left(\sum_{h=1}^{\infty} \lambda_h \frac{t^h}{h!} \right)^j = \sum_{J=1}^{\infty} \left(\sum_{i=1}^{j} u^{(i)} \sigma_{i,j}(\lambda) \right) \frac{t^j}{j!},$$

$$1 + \sum_{n=1}^{\infty} p_{\mathrm{I},n}(u) \frac{1}{n!} \left(\sum_{h=1}^{\infty} \lambda_h \frac{t^h}{h!} \right)^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} p_{\mathrm{I},m}(u) \sigma_{m,n}(\lambda) \right) \frac{t^n}{n!}.$$

$\sigma(\lambda)$ is given concretely as follows,

$$\sigma(\lambda) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \dots \\ 0 & {\lambda_1}^2 & 3\lambda_1\lambda_2 & 4\lambda_1\lambda_3 + 3{\lambda_2}^2 & 5\lambda_1\lambda_2 + 10\lambda_2\lambda_3 & \\ 0 & 0 & {\lambda_1}^3 & 6{\lambda_1}^2\lambda_2 & 10{\lambda_1}^2\lambda_3 + 15\lambda_1{\lambda_2}^2 & \\ 0 & 0 & 0 & {\lambda_1}^4 & 10{\lambda_1}^3\lambda_2 & \\ 0 & 0 & 0 & 0 & {\lambda_1}^5 & \\ \vdots & & & & & \end{pmatrix}.$$

1.4. Multi-binomial differential polynomials sequences

We choose r infinite variable vectors

$$u = (u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, \dots), \ \dots, \ u_r = (u_r^{(1)}, u_r^{(2)}, u_r^{(3)}, \dots)$$

with weight

$$\text{weight}\, u_1^{(l)} = \dots = \text{weight}\, u_r^{(l)} = l \qquad (l \geq 1).$$

DEFINITION 1.6. A differential polynomial sequence $(p_n(u_1, \dots, u_r))_{n \geq 0}$ in $K[u_1, \dots, u_r]$, is called to be multi-binomial, if it satisfies

i) $p_0(u_1, \dots, u_r) \equiv 1$

ii) $\text{weight}\, p_n(u_1, \dots, u_r) = n$, $\text{weight}\, p_n(u_1, \dots, u_r) = n \qquad (1 \leq k \leq r)$,

iii) $p_n(u_1 + v_1, \dots, u_r + v_r) = \displaystyle\sum_{\sum a_j = n} \binom{n}{a_1, a_2, \dots, a_{2^r}} \prod_{j=1}^{2^r} p_{a_j}(w_{j,1}, \dots, w_{j,r})$,

where $(w_{j,1}, \dots, w_{j,r})$ runs over all the vectors such that

$$w_{j,k} = u_k \text{ or } v_k \qquad (1 \leq k \leq r,\ 1 \leq j \leq 2^r).$$

The condition

iv) $p_n(0, \dots, 0) = 0 \qquad (n \geq 1)$

is a consequence of i) and iii).

DEFINITION 1.7. Replacing ii) by a weaker condition

ii*) weight $p_n(u_1, \ldots, u_r) \le n \qquad (n \ge 1)$,

we define multi-binomial differential polynomial sequences in wide sense.

By means of generating functions, condition iii) is equivalent to

$$\text{iii*)} \quad \Phi_p(u_1 + v_1, \ldots, u_r + v_r \mid t) = \prod_{j=1}^{2^r} \Phi_p(w_{j,1}, \ldots, w_{j,k} \mid t),$$

where $(w_{j,1}, \ldots, w_{j,k})$ runs over all the vectors such that $w_{j,k} = u_k$ or v_k $(1 \le k \le r;\ 1 \le j \le 2^r)$.

PROPOSITION 1.10. *The set $DP(K[u_1, \ldots, u_k])$ of multi-binomial differential polynomial sequences in $K[u_1, \ldots, u_r]$ in wide sense, coincides with the set of differential polynomial sequences*

$$\left\{(p_{\alpha,n}(u_1, \ldots, u_r))_{n \ge 0} \mid \alpha = (\alpha_{i_1,\ldots,i_r;\ j})_{i_1+\ldots+i_r \le j, \alpha_{j_1,\ldots,i_r;\ j} \in K}\right\},$$

which are defined by

$$(1.16) \qquad \Phi_{p_\alpha}(u_1, \ldots, u_r \mid t) = \exp\left[\sum_{j=1}^{\infty} \sum_{j_1+\ldots+j_r \le j} \alpha_{i_1,\ldots,i_r;\ j} u_1^{(i_1)}, \ldots, u_r^{(i_r)} \frac{t^j}{j!}\right]$$
$$= \sum_{n=0}^{\infty} p_{\alpha,n}(u_1, \ldots, u_r) \frac{t^n}{n!}.$$

Proof. Since $p_0(u_1, \ldots, u_r) \equiv 1$, we may put

$$\log \Phi_p(u_1, \ldots, u_r) = \log\left(1 + \sum_{j=1}^{\infty} p_j(u_1, \ldots, u_r) \frac{t^j}{j!}\right)$$
$$= \sum_{j=1}^{\infty} \varphi_j(u_1, \ldots, u_r \mid t) \frac{t^j}{j!}$$

with polynomials $\varphi_j(u_1, \ldots, u_r)$ of weight at most j $(j \ge 1)$ in $K[u_1, \ldots, u_r]$. Then the condition

$$\Phi_p(u_1 + v_1, \ldots, u_r + v_r \mid t) = \prod_{h=1}^{2^r} \Phi_P(w_{h,1}, \ldots, w_{h,r} \mid t)$$

is equivalent to

$$\varphi_j(u_1+v_1,\ldots,u_r+v_r)=\sum_{h=1}^{2^r}\varphi_j(w_{j,1},\ldots,w_{j,r}).$$

This is also equivalent to $\varphi_j(u_1,\ldots,u_r)$ are liner homogeneous in $u_1,\ldots,u_r$, i.e. there exists a system of constants $\alpha_{i,1},\ldots,_{i_r ij}$ in K such that

$$\varphi_j(u_1,\ldots,u_r)=\sum_{i_1+\ldots+i_r\leq j}\alpha_{i_1,\ldots,i_r;\ j}u_1^{(i_1)}\ldots u_r^{(i_r)}\qquad(j\geq 1),$$

i.e.

$$\Phi_p(u_1,\ldots,u_r\mid t)=\exp\left[\sum_{j=1}^{\infty}\left(\sum_{i_1+\cdots+i_r\leq n}\alpha_{i_1,\ldots,i_r;\ j}u_1^{i_1},\ldots,u_r^{(i_r)}\right)\frac{t^j}{j!}\right].$$

Two multi-binomial differential polynomial sequences $(p_n(u_1,\ldots,u_r))_{n\geq 0}$ and $(q_n(u_1,\ldots,u_r))_{n\geq 0}$ are called to be similar each other, if there exist two system of constants in K $(\lambda_{m,n})_{1\geq m\geq n}$ and $(\mu_{m,n})_{1\geq m\geq n}$ such that

$$q_n(u_1,\ldots,u_r)=\sum_{m=1}^{n}p_m(u_1,\ldots,u_r)\lambda_{m,n},$$
$$p_n(u_1,\ldots,u_r)=\sum_{m=1}^{n}q_m(u_1,\ldots,u_r)_{\mu_{m,n}}\qquad(1\leq m\leq n)$$

THEOREM 1.2. (Expansion Theorem) *Multi-binomial differential polynomial sequences* $(p_n(u_1,\ldots,u_r))_{n\geq 0}$ *and* $(q_n(u_1,\ldots,u_r))_{n\geq 0}$ *in* $K[u_1,\ldots,u_r]$ *are similar each other, if and only if there exists a system of constants* $(\lambda_j)_{j\geq 1}$ *in* K *such that* $\lambda_1\neq 0$ *and*

$$(1.17)\quad q_n(u_1,\ldots,u_r)=\sum_{m=1}^{n}p_m(u_1,\ldots,u_r)\left(\sum_{\substack{\sum jl_j=n\\ \sum l_j=m}}n!\prod_j\frac{1}{l_j!}\left(\frac{\lambda_j}{j!}\right)^{l_j}\right)\quad(n\geq 1),$$

condition (1.17) *is equivalent to*

$$(1.18)\qquad \Phi_q(u_1,\ldots,u_r\mid y)=\Phi_p\left(u_1,\ldots,u_r\mid\sum_{J=1}^{\infty}\lambda_j\frac{t^i}{j!}\right).$$

Proof. Assume $q_n(u_1, \dots, u_r) = \sum_{m=1}^{n} p_m(u_1, \dots, u_r)\, \lambda_{m,n}$ $(n \geq 1)$. We fix $w_2, \dots, w_r$ and consider $(p_n(u_1, w_2, \dots, w_r))_{n \geq 0}$ and $(q_n(u_1, w_2, \dots, w_r))_{n \geq 0}$ as differential polynomial sequences in u_1 with coefficients in $K[w_2, \dots, w_r]$, then they are binomial differential polynomial sequences similar each other. Hence by virtue of Theorem 1.1 there exists a system of elements in $K[w_2, \dots, w_r]$ $(\lambda_j(w))_{j \geq 1}$ such that $\lambda_1(w) \neq 0$ and

$$\Phi_q(u_1, w_2, \dots, w_r \mid y) = \Phi_p\left(u_1, w_2, \dots, w_r \mid \sum_{J=1}^{\infty} \lambda_j(w) \frac{t^i}{j!}\right).$$

It is enough to show $\lambda_j(w)$ $(j \geq 1)$ belong to K. Since $p_n(u_1, w_2, \dots, w_r)$ $(n \geq 1)$ are linearly independent over $K[w_2, \dots, w_r]$, this means

$$\lambda_{m,n} = \sum_{\substack{\sum j l_j = n \\ \sum lj = m}} n! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j(w)}{j!}\right)^{l_j} \qquad (1 \leq m \leq n)$$

On the other hand by virtue of Proposition 1.4, using $\nu_n = \sum_{m=1}^{n} \lambda_{m,n}$ $(n \geq 1)$, we have

$$\nu_n = p_{\mathrm{I},m}(\lambda_1(w), \dots, \lambda_n(w)) \qquad (n \geq 1)$$

$$\lambda_j(w) = \sum_{m=1}^{n} (-1)^{m-1}(m-1)! \left(\sum_{\substack{\sum j l_j = n \\ \sum l_j = m}} m! \prod_j \frac{1}{l_j!} \left(\frac{\nu_j}{j!}\right)^{l_J} \right).$$

This proves $\lambda_j(w)$ $(j \geq 1)$ belong to K.

1.5. Binomial partial differential polynomials sequences

We shall use the following multi-indexed notations:

$$n = (n_1, \dots, n_r), \qquad j! = j_1! \dots j_r!, \qquad \binom{n}{j} = \binom{n_1}{j_1} \cdots \binom{u_r}{j_r},$$

$$u^{(n)} = u^{(n_1, \dots, n_r)}, \qquad t^j = t_1^{j_1}, \dots, t_r^{j_r}, \qquad \frac{t^j}{j!} = \frac{t_1^{j_1}}{j_1!} \cdots \frac{t_r^{j_r}}{j_r!},$$

$$\left(\frac{u^{(j)}}{j!}\right)^{l_j} = \left(\frac{u^{(j_1, \dots, j_r)}}{j_1!, \dots, j_r!}\right)^{l_j},$$

$$\gamma = (\gamma_{i,j})_{i \leq j} = \big(\gamma_{(i_1, \dots, i_r), (j_1, \dots, j_r)}\big)_{(i_1, \dots, i_r) \leq (j_1, \dots, j_r)},$$

$$\alpha = (\alpha_n)_{n > 0} = \big(\alpha_{(n_1, \dots, n_r)}\big)_{(n_1, \dots, n_r) > 0},$$

$$\sum j l_j = n = (n_1, \dots, n_r) = \sum (j_1, \dots, j_r) l_{(j_1, \dots, j_r)},$$

where $(u^{(n)})_{n>0}$ means a system of variables with a vector valued weight:

$$\text{weight}\, u^{(n_1,\ldots,n_r)} = (n_1, \ldots, n_r).$$

Replacing the notations in 1.1, 1.2, 1.3, and 1.4 by the above multi-indexed notations, we observe that almost all statements and formulas hold by the same expressions.

DEFINITION 1.8. A partial differential polynomial sequence in $K[u]$

$$(p_n(u))_{n=(n_1,\ldots,n_r)>0}$$

is called to be binomial, if it satisfies

i) $p_0(u) \equiv 1$,

ii) $\text{weight}\, p_n(u) = n$,

iii) $p_n(u+v) = \sum_{0\geq l\geq m} \binom{n}{l} p_{n-l}(u)p_l(v) \ (n \geq 0)$.

By induction on $n = (n_1, \ldots, n_r)$, i) and ii) implies

iv) $p_n(0) = 0 \ (n = (n_1, \ldots, n_r) \geq 0)$.

Using the generating function

$$\Phi_p(u \mid t) = \sum_{n\geq 0} p_n(u)\frac{t^n}{n!}, \tag{1.19}$$

we can express iii) by the equivalent condition,

iii*) $\Phi_p(u+v \mid t) = \Phi_p(u \mid t)\Phi_p(v \mid t)$.

DEFINITION 1.9. Replacing ii) by a weaker condition

$$\text{weight}\, p_n(u) \leq n \qquad (n = (n_1, \ldots, n_r) \geq 0),$$

we define binomial partial differential polynomial sequences in wide sense.

PROPOSITION 1.11. *The set $DP_r(K[u])$ of binomial partial differential polynomial sequences in wide sense in $K[u]$, coincides with the set of partial differential polynomial sequences*

$$\{(p_{\alpha,n}(u))_{n\geq 0} \mid \alpha = (\alpha_{i,j})_{0<i\leq j};\ \alpha_{i^\sigma,j^\sigma} = \alpha_{i,j}\ (\sigma \in S_r), \alpha_{i,j} \in K\},$$

which are given by

$$\Phi_\alpha(u \mid t) = \exp\left[\sum_{i\geq j} \alpha_{i,j}\frac{u^{(i)}}{i!}t^j\right] = \sum_{n\geq 0} p_{\alpha,n}(u)\frac{t^n}{n!} \tag{1.20}$$

where S_r means the symmetric group of degree r.

Proof. For each $(p_n(u))_{n\geq 0}$ in $DP_r(K[u])$ we may put,

$$\log(\Phi_p(u \mid t)) = \log\left(1 + \sum_{j>0} p_j(u)\frac{t^j}{j!}\right) = \sum_{j\geq 0} \varphi_j(u)\frac{t^j}{j!}$$

with a unique system of partial differential polynomials $(\varphi_j(u))_{j\geq 0}$ such that $\varphi_j(u)$ is of weight at most j. From multiplicative property $\Phi_p(u+v \mid t) = \Phi_p(u \mid t)\Phi_p(v \mid t)$ we obtain $\varphi_j(u+v) = \varphi_j(u) + \varphi_j(v)$, i.e. $\varphi_j(u)$ are linear in $u^{(i)}$ $(0 < i \leq j)$. This means there exists a unique system of bisymmetric contains $\alpha_{i,j}$ in K such that

$$\Phi_p(u \mid t) = \exp\left[\sum_{o\leq i\leq j} \alpha_{i,j} u^{(i)}\frac{t^j}{j!}\right]$$

We obtain also the standard binomial partial differential polynomial sequences as follows;

PROPOSITION 1.12.

$$\exp\left[\sum_{0\leq i\leq j} u^{(j)}\frac{t^j}{j!}\right] = \sum_{n\geq 0} p_{\mathrm{I},n}(u)\frac{t^n}{n!} \tag{1.21}$$

$$p_{\mathrm{I},n}(u) = \sum_{n\geq 0} n! \prod_j \frac{1}{l_j!}\left(\frac{u^{(j)}}{j!}\right)^{l_j} \tag{1.22}$$

$$u^{(n)} = \sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} (-1)^{m-1}(m-1)!n! \prod_j \frac{1}{l_j}\left(\frac{p_{\mathrm{I},j}(u)}{j!}\right)^{l_j} \qquad (n \geq 0) \tag{1.23}$$

Proof. By direct calculation we have,

$$\begin{aligned}
\sum_{j\geq 0} u^{(j)}\frac{t^j}{j!} &= \log\left(1 + \sum_{j\geq 0} p_{\mathrm{I},j}(u)\frac{t^j}{j!}\right) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\left(\sum_{j\geq 0} p_{\mathrm{I},j}(u)\frac{t^j}{j!}\right)^m \\
&= \sum_{m\geq 0} \frac{t^n}{n!}\left(\sum_{m=1} \frac{(-1)^{m-1}}{m} \sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} n!m! \prod_j \frac{1}{l_j!}\left(\frac{p_{\mathrm{I},j}(u)}{j!}\right)^{l_j}\right)
\end{aligned}$$

$$= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\substack{\sum jl_j=n \\ \sum l_j=m}} (-1)^{m-1}(m-1)!n! \prod_j \frac{1}{l_j!} \left(\frac{p_{\mathrm{I},j}(u)}{j!}\right)^{l_j}$$

PROPOSITION 1.13. *Putting* $z(s) = \exp[u(s)]$, *we obtain the relation between the partial derivatives;*

$$\frac{z^{(n)}(s)}{z(s)} = p_{\mathrm{I},n}(u(s)) = \sum_{\sum jl_j=n} n! \prod_j \frac{1}{l_j} \left(\frac{u^{(j)}(s)}{j!}\right)^{l_J} \qquad (n \geq 0), \tag{1.24}$$

where $z(s) = z(s_1, \dots, s_r)$, $u(s) = u(s_1, \dots, s_r)$, $z^{(n)}(s) = \left(\frac{\partial}{\partial s}\right)^n z(s)$ *and* $u^{(j)}(s) = \left(\frac{\alpha}{\alpha s}\right)^j u(s)$.

Proof. From Tayler expansion of $u(s+t)$ we have

$$\begin{aligned}\frac{z(s+t)}{z(s)} &= \exp\left[u(s+t) - u(s)\right] = \exp\left[\sum_{j\geq 0} u^{(j)}(s)\frac{t^j}{j!}\right] \\ &= \sum_{n\geq 0} p_{\mathrm{I},n}(u(s))\frac{t^n}{n!}\end{aligned}$$

Two binomial partial differential polynomial sequences $(p_n(u))_{n\geq 0}$ and $(q_n(\mu))_{n\geq 0}$ are called to be similar, if there exist two siptems of constant $(\lambda_{m,n})_{0\geq m\geq n}$ and $(\mu_{m,n})_{0\geq m\geq n}$ such that

$$q_n(u) = \sum_{0<m\leq n} p_m(u)\lambda_{m,n}, \quad p_n(u) = \sum_{0<m\leq n} q_m(u)\mu_{m,n}.$$

THEOREM 1.3. (Expansion Theorem) *Two binomial partial differential polynomial sequences* $(p_n(u))_{n\geq 0}$ *and* $(q_n(u))_{n\geq 0}$ *are similar, if and only if there exists systems of constants in* K

$$(\lambda^{(1)}{}_j)_{j\geq 0}, \dots, (\lambda^{(r)}{}_j)_{j\geq 0}$$

such that

$$\det \begin{pmatrix} \lambda^{(1)}_{e_1} & \lambda^{(1)}_{e_2} & \dots & \lambda^{(1)}_{e_r} \\ \lambda^{(2)}_{e_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda^{(r)}_{e_r} & \dots & \dots & \lambda^{(r)}_{e_r} \end{pmatrix} \neq 0 \tag{1.25}$$

and

$$(1.26)\quad q_n(u) = \sum_{0<m\le n} p_m(u) \left[\sum_{\substack{\sum_k \sum_{j(k)} j^{(k)} l^{(k)}{}_{j(k)} = n \\ \left(\sum_{j(1)} l^{(1)}{}_{j(1)}, \ldots, \sum_{j(r)} l^{(r)}{}_{j(r)}\right) = m}} n! \prod_{k=1}^{r} \prod_{j(k)} \frac{1}{l^{(k)}_{j(k)}!} \left(\frac{\lambda^{(k)}{}_{j(k)}}{j^{(k)}} \right)^{l^{(k)}_{j(k)}} \right] \quad (n \ge 0)$$

where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_r = (0, \ldots, 0, 1)$. *Moreover* (1.25) *is equivalent to*

$$(1.27)\qquad \Phi_q(u \mid t) = \Phi_p\left(u \mid \sum_{j\ge 0} \lambda_j^{(1)} \frac{t^j}{j!}, \ldots, \sum_{j \ge 0} \lambda_J^{(r)} \frac{t^j}{j!} \right).$$

Proof. Putting $q_n(u) = \sum_{0<m\le n} p_m(n)\lambda_{m,n}$, we have

$$\lambda_{0,0} = 1, \qquad \lambda_{0,n} = 0 \qquad (n > 0)$$

and two way expression of $q_n(u+v)$:

$$\begin{aligned} q_n(u+v) &= \sum_{0\le m\le n} p_m(u+v)\lambda_{m,n} \\ &= \sum_{0\le m \le n} \left(\sum_{0 \le h \le m} \binom{m}{h} p_{m-h}(u) p_h(v) \right) \lambda_{m,n} \end{aligned}$$

$$\sum_{0\le l\le n} \binom{n}{l} q_{n-l}^{(u)} q_l^{(v)} = \sum_{0 \le l \le n} \binom{n}{l} \left(\sum_a p_a(u) \lambda_{a,n-a} \right) \left(\sum_b p_b(v) \lambda_{b,l} \right)$$

Comparing the coefficients of $p_{m-h}(u)p_h(v)$ in the both sides, we have

$$\binom{m}{h} \lambda_{m,n} = \sum_{0\le l\le n} \binom{n}{l} \lambda_{m-h,n-h} \lambda_{h,l},$$

$$\frac{m!}{n!} \lambda_{m,n} = \sum_l \frac{(m-h)!}{(n-l)!} \lambda_{m-h,n-l} \frac{h!}{l!} \lambda_{h,l}.$$

Hence, putting

$$f_m(t) = \sum_{m\geq n} \frac{m!}{n!}\lambda_{m,n}t^n \qquad (m>0)$$

we obtain the key relation,

$$\begin{aligned} f_m(t) &= \sum_{m\geq n} \frac{m!}{n!}\lambda_{m,n}t^n = \left(\sum_{m-h\geq a} \frac{(m-h)!}{a!}\lambda_{m-h,a}t^a\right)\left(\sum_{h\geq b}\frac{h!}{b!}\lambda_{h,b}t^b\right) \\ &= f_{m-h}(t)f_h(t) = \prod_{k=1}^{r} f_{e_k}(t)^{m_k} = \prod_{k=1}^{r}\left(\sum_{j\geq 0}\lambda_j^{(k)}\frac{t^j}{j!}\right)^{m_k}, \end{aligned}$$

where $\lambda_j^{(k)} = \lambda_{e_k,j}$ $(j \geq 0)$. This means,

$$\lambda_{m,n} = \sum_{\substack{\sum_k \sum_{j_{(k)}} j^{(k)} l^{(k)}{}_{j_{(k)}} = n \\ \left(\sum_{j^{(1)}} l^{(1)}{}_{j^{(1)}},\ldots,\sum_{j^{(r)}} l^{(r)}{}_{j^{(r)}}\right) = (m_1,\ldots,m_r)}} n! \prod_{k=1}^{r}\prod_{j^{(k)}} \frac{1}{l^{(k)}_{j^{(k)}}!}\left(\frac{\lambda^{(k)}{}_{j^{(k)}}}{j^{(k)}}\right)^{l^{(k)}_{j^{(k)}}}$$

and

$$\begin{aligned} &\Phi_q(u \mid t) \\ &= \sum_{n\geq 0} q_n(u)\frac{t^n}{n!} \\ &= \sum_{n\geq 0}\frac{t^n}{n!}\left(\sum_{\substack{\sum_k \sum_{j_{(k)}} j^{(k)} l^{(k)}_{j_{(k)}} = n \\ \left(\sum_{j^{(1)}} l^{(1)}{}_{j^{(1)}},\ldots,\sum_{j^{(r)}} l^{(r)}_{j^{(r)}}\right) = m}} n!\prod_{k=1}^{r}\prod_{j_{(k)}}\frac{1}{j^{(k)}!}\left(\frac{\lambda^{(k)}{}_{j^{(k)}}}{j^{(k)}}\right)^{l^{(k)}_{j_{(k)}}} p_m(u)\right) \\ &= \sum_{m\geq 0}\frac{1}{m!}\prod_{k=1}^{r}\prod_{j^{(k)}}\frac{1}{j^{(k)}}\left(\frac{\lambda^{(k)}_{j^{(k)}}t^{j^{(k)}}}{j^{(k)}!}\right)^{l^{(k)}_{j^{(k)}}} p_m(u) \\ &= \sum_{m_1!,\ldots,m_r\geq 0}\frac{1}{m_1!,\ldots,m_r!}\prod_{k=1}^{r}\left(\sum\frac{\lambda_{j^k_{(k)}}t^{j^{(k)}}}{j^{(k)}!}\right)^{m_j} p_{m_1,\ldots,m_r}(u) \end{aligned}$$

$$= \Phi_p \left(u \mid \sum_{j \geq 0} \lambda_j^{(1)} \frac{t^j}{j!}, \ldots, \sum_{j \geq 0} \lambda_j^{(r)} \frac{t^j}{j!} \right).$$

1.6. q-binomial differential polynomial sequences

We choose a quantity q in K which is transendental over rational number field Q, and denote briefly

$$(n)_q = \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1}, \qquad (0)_q = 1$$

$$(n)_q! = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q)}{(1-q)^n}$$

$$= (1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1}), \quad (0)_q! = 1,$$

$$\binom{n}{l} = \frac{(n)_q!}{(n-l)_q!(l)_q!}$$

Replacing binomial coefficients with q binomial coefficients $\binom{n}{l}_q$ $(0 \leq l \leq n)$, we can easily define binomial differential polynomial sequences. We introduce two types of infinite variables;

$$\hat{u} = (\hat{u}^{(1)}, \hat{u}^{(2)}, \hat{u}^{(3)}, \ldots), \quad u = (u^{(1)}, u^{(2)}, u^{(3)}, \ldots)$$

with commutation relation

$$\begin{aligned} \hat{u}^{(i)}\hat{u}^{(j)} &= \hat{u}^{(j)}\hat{u}^{(i)}, \qquad u^{(i)}u^{(j)} = u^{(j)}u^{(i)} \\ \hat{u}^{(i)}u^{(j)} &= qu^{(j)}\hat{u}^{(i)} \qquad (i, j \geq 1). \end{aligned} \tag{1.28}$$

DEFINITION 1.10. A differential polynomial sequence $(p_n(u))_{n\geq 0}$ is called to be q-binomial, if it satisfies

i) $p_0(u) \equiv 0$,

ii) weight $p_n(u) = n$,

iii) $p_n(\hat{u} + u) = \sum_{l=0}^{n} \binom{n}{l}_q p_{n-l}(u) p_l(\hat{u}) \qquad (n \geq 1)$.

The condition

iv) $p_n(0) = 0$ $(n \geq 1)$

is a consequence of i) and iii).

DEFINITION 1.11. Replacing ii) by a weaker condition

ii*) weight $p_n(u) \leq n$ $(n \geq 1)$,

we define q-binomial differential polynomial seqences in wide sense.

By means of generating function

$$\Phi_p^{(q)}(u \mid t) = \sum_{n=0}^{\infty} p_n(u) \frac{t^n}{(n)_q!} \tag{1.29}$$

condition iii) is equivalent to

iii*) $\Phi_p^q(u + \hat{u} \mid t) = \Phi_p^{(q)}(u \mid t)\Phi_p(\hat{u} \mid t)$,

where t is commutative with $\hat{u}^{(i)}, u^{(i)}$ $(i, j \geq 1)$.

Since the commantation relation $\hat{x}x = qx\hat{x}$ implies

$$(\hat{x} + x)^n = \sum_{l=0}^{n} \binom{n}{l}_q x^{n-l}\hat{x}^l, \tag{1.30}$$

q-exponential function

$$\exp^{(q)}(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n)_q!} \tag{1.31}$$

satisfies

$$\exp^{(q)}(\hat{x}x) = \exp^{(q)}(\hat{x}) \exp^{(q)}(x). \tag{1.32}$$

q-log function $\log^q(1+t)$ is the formal power series in t which is the inverse function of $\exp^q(t)$, i.e.,

$$\log^q[\exp^q[t]] = t.$$

We briefly denote

$$p_m^{(q)}(n) = \sum_{\substack{\sum il_j = n \\ \sum l_j = m}} (n)_q! \left(\prod_j l_j!((j)_q!)^{l_j} \right)^{-1}. \tag{1.33}$$

PROPOSITION 1.14.

(1.34) $\log^{(q)}(1 + t)$

$$= t + \sum_{n=2}^{\infty} \left[\sum_{r=1}^{n-1} (-1)^r \sum_{\substack{1 < m_1 < m_2 < \\ \dots < m_{r-1} < n}} p_1^{(q)}(m_1) p_{m_1}^{(q)}(m_2) \dots p_{m_{r-1}}^{(q)}(n) \right] \frac{t^n}{n!}$$

Proof. We denote

$$\lambda_j = \frac{j!}{(j)_q!}$$

$$1 + s = \exp^{(q)}[t] = 1 + \sum_{j=1}^{\infty} \frac{t^i}{(j)_q!} = 1 + \sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!}$$

and

$$\log^{(q)}(1 + s) = \sum_{m=1}^{\infty} \alpha_m \frac{s^m}{m!}$$

then

$$\begin{aligned} t = \log^{(q)}(\exp^{(q)}[t]) &= \sum_{m=1}^{\infty} \frac{\alpha_m}{m!} \left(\sum_{j=1}^{\infty} \lambda_j \frac{t^j}{j!} \right)^m \\ &= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{n} \alpha_m \sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} n! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j}{j!} \right)^{l_j} \right] \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{n} \alpha_m p_{m,n}(\lambda) \right] \frac{t^n}{n!}. \end{aligned}$$

This means

$$\alpha_1 p_{1,1}(\lambda) = 1, \qquad \sum_{m=1}^{n} \alpha_m p_{m,n}(\lambda) = 0 \qquad (n \geq 2).$$

On the other hand

$$p_{n,n}(\lambda) = n! \frac{n!}{(n)_q!} \cdot \frac{1}{n!} = \frac{n!}{(n)_q!}, \qquad p_{1,1}(\lambda) = 1$$

$$\frac{p_{m,n}(\lambda)}{p_{n,n}(\lambda)} = \frac{(n)_q!}{n!} \sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} n! \prod_j \frac{1}{l_j!} \left(\frac{\lambda_j}{j!} \right)^{l_j}$$

$$\sum_{\substack{\sum jl_j = n \\ \sum l_j = m}} (n)_q! \left(\prod_j l_j! \left((j)_q!\right)^{l_j} \right)^{-1} = p_m^{(q)}(n),$$

hence

$$\alpha_1 = 1$$

$$\begin{aligned}\alpha_n &= -\sum_{m=1}^{n-1} \alpha_m \frac{p_{m,n}(\lambda)}{p_{n,n}(\lambda)} \\ &= -\sum_{m=1}^{n-1} \alpha_m p_m^{(q)}(n) \\ &= \sum_{r=1}^{n-1} (-1)^r \sum_{\substack{1<m_1<m_2< \\ \dots<m_{r-1}<n}} p_1^{(q)}(m_1) p_{m_1}^{(q)}(m_2) \dots p_{m_{r-1}}^{(q)}(n),\end{aligned}$$

$$\begin{aligned}&\log^{(q)}(1+t) \\ &= t + \sum_{n=2}^{\infty} \left[\sum_{r=1}^{n-1} (-1)^r \sum_{\substack{1<m_1<m_2< \\ \dots<m_{r-1}<n}} p_1^{(q)}(m_1) p_{m_1}^{(q)}(m_2) \dots p_{m_{r-1}}^{(q)}(n) \right] \frac{t^n}{n!}.\end{aligned}$$

PROPOSITION 1.15. *The set $DP^{(q)}(K[u])$ of q-binomial differential polynomial sequences in wide sense in $K[u]$ coinicides with the set of differential polynomial sequences*

$$\{(p_{\alpha,n}(u))_{n\geq 0} \mid \alpha = (\alpha_{ij})_{i\geq j}, \alpha_{ij} \in K\},$$

which are given by menas of generating functions as follows

$$\Phi_{p_\alpha}^{(q)}(u \mid t) = \exp^{(q)} \left[\sum_{1\leq i\leq j} \alpha_{ij} u^{(i)} \frac{t^j}{(j)_q!} \right]. \tag{1.35}$$

Proof. For an element $(p_n(u))$ in $DP^{(q)}(K[u])$ we put

$$\log^{(q)}(\Phi_p^{(q)}(u \mid t)) = \log^{(q)} \left[1 + \sum_{j=1}^{\infty} p_j(u) \frac{t^j}{(j)_q!} \right] = \sum_{j=1}^{\infty} \varphi_j(u) \frac{t^j}{(j)_q!}$$

with polynomials $\varphi_j(u)$ of weight at most j in $K[u]$. Let us prove

$$\varphi(\hat{u} + u) = \varphi_j(\hat{u}) + \varphi_j(u) \qquad (j \geq 1)$$

Since

$$\lim_{q\to 1} \frac{1}{(n)_q!} = \frac{1}{n!}, \qquad \lim_{q\to 1} \exp^{(q)}[t] = \exp[t]$$

we have

$$\begin{aligned}\exp\left[\sum_{j=1}^{\infty}\varphi_j(\hat{u}+u)\frac{t^j}{j!}\right] &= \lim_{q\to 1}\exp^{(q)}\left[\sum_{j=1}^{\infty}\varphi_j(\hat{u}+u)\frac{t^j}{(j)_q!}\right]\\ &= \lim_{q\to 1}\Phi_p^{(q)}(\hat{u}+u\mid t) = \lim_{q\to 1}\left(\Phi_p^{(q)}(u\mid t)\Phi_p^{(q)}(\hat{u}\mid t)\right)\\ &= \lim_{q\to 1}\Phi_p^{(q)}(u\mid t)\lim_{q\to 1}\Phi_p^{(q)}(\hat{u}\mid t)\\ &= \lim_{q\to 1}\exp^{(q)}\left[\sum_{j=1}^{\infty}\varphi_j(u)\frac{t^j}{(j)_q!}\sum_{j=1}^{\infty}\varphi_j(\hat{u})\frac{t^j}{(j)_q!}\right]\\ &= \exp\left[\sum_{j=1}^{\infty}\varphi_j(u)\frac{t^j}{j!}\right]\left[\sum_{j=1}^{\infty}\varphi_j(\hat{u})\frac{t^j}{j!}\right]\\ &= \exp\left[\sum_{j=1}^{\infty}\left(\varphi_j(\hat{u})+\varphi(u)\right)\frac{t^j}{j!}\right]\end{aligned}$$

This means $\varphi_j(\hat{u}+u) = \varphi_j(\hat{u}) + \varphi_j(u)$ $(j \geq 1)$, i.e., $\varphi_j(u)$ $(j \geq 1)$ are liner forms. The converse is obviously true.

THEOREM 1.4. *Two q-binomial differential polynomical sequences* $(p_n(u))_{n\geq 0}$ *and* $(r_n(u))_{n\geq 0}$ *are similar each other, if and only if there exists a system of constans* $(\lambda_j)_{j\geq 1}$ *in* K *such that* $\lambda_1 \neq 0$ *and*

$$r_n(u) = \sum_{m=1}^{n}\left[\frac{m!}{(m)_q!}p_m(u)\sum_{\substack{\sum jl_j=n\\ \sum l_j=m}}(n)_q!\prod_j\frac{1}{l_j!}\left(\frac{\lambda_j}{(j)_q!}\right)^{l_j}\right] \tag{1.36}$$

Condition (1.36) *is equivalent to*

$$\Phi_r^{(q)}(u\mid t) = \Phi_p^{(q)}\left(u \;\Big|\; \sum_{j=1}^{\infty}\lambda_j\frac{t^j}{(j)_q!}\right) \tag{1.37}$$

The proof of this theorem is completely same as that of Theorem 1.1.

Appendix A. Central moments of entropy

1. Using the standard binomial differential polynomials, we an express the n - th central moment of entropy concretsly.

A distribution function means a positive real value function in $s > 0$ which is given by an integral

$$z(s) = \int_{\Omega} \exp[-sf(x)]\mu(dx),$$

where $f(x)$ is a non-negative real value function on a measurable space (Ω, μ) and we assume that d/ds and integration are always commutative. Entropy of the distribution function $z(s)$ is defined by

$$\begin{aligned} E(z(s)) &= \int_{\Omega} \left(-log\left[\frac{\exp[-sf(x)]}{z(s)}\right]\right) \frac{\exp[-sf(x)]}{z(s)}\mu(dx) \\ &= \int_{\Omega} \left(-log\left[\frac{\exp[-sf(x)]}{z(s)}\right]\right) \frac{\exp[-sf(x)]}{z(s)}\mu(dx) \\ &= -s\frac{z^{(1)}(s)}{z(s)} + \log z(s). \end{aligned}$$

The n-th central moment of entropy $E(z(s))$ is defined by

$$M_n(z(s)) = \int_{\Omega} \left(-\log\left[\frac{\exp[-sf(x)]}{z(s)}\right] - E(z(s))\right)^n \mu(dx) \qquad (n \geq 0).$$

Putting $z(s) = \exp[u^{(0)}(s)]$, from Proposition 1.6, we have

$$\sum_{n=0}^{\infty} \frac{z^{(n)}(s)}{z(s)}\frac{t^n}{j!} = \exp\left[\sum_{j=1}^{\infty} u^{(j)}(s)\frac{t^j}{j!}\right] = \sum_{n=0}^{\infty} p_{\mathrm{I},n}(u(s))\frac{t^n}{n!}$$

where

$$\begin{aligned} p_{\mathrm{I},n}(u(s)) &= \sum_{\sum jl_i=n} n!\prod_j \frac{1}{l_j!}\left(\frac{u^{(j)}(s)}{j!}\right)^{l_j} \qquad (n \geq 0) \\ u^{(j)}(s) &= \left(\frac{d}{ds}\right)^j u^{(0)}(s) \qquad (j \geq 1) \end{aligned}$$

THEOREM 1.

$$\text{(A.1)} \qquad \begin{aligned} \sum_{n=0}^{\infty} \frac{M_n(z(s))}{n!}t^n &= \exp\left[\sum_{j=2}^{\infty} \frac{u^{(j)}(s)}{j!}(-st)^j\right] \\ &= \exp\left[\exp\left[u(s-st)\right] - u(s) - u^{(1)}(s)\right] \end{aligned}$$

(A.2) $$M_n(z(s)) = (-s)^n \sum_{\sum_{j\geq 2} j l_j = n} n! \prod_j \frac{1}{l_j!}\left(\frac{u^{(j)}(s)}{j!}\right)^{l_j},$$

(A.3) $$M_1(z(s)) = 0,$$
$$M_2(z(s)) = s^2 u^{(2)}(s) = s^2\left[\frac{z^{(2)}(s)}{z(s)} - \left(\frac{z^{(1)}(s)}{z(s)}\right)^2\right],$$

where $u^{(j)}(s) = (d/ds)^{j-2} u^{(2)}(s)$ $(j \geq 2)$.

Proof. By calculation we have

$$\begin{aligned}
M_n(z(s)) &= \int_\Omega \left(-\log\left[\frac{\exp[-sf(x)]}{z(s)}\right] - E(z(s))\right)^n \frac{\exp[-sf(x)]}{z(s)}\mu(dx)\\
&= \int_\Omega \left(sf(x) + s\frac{z^{(1)}(s)}{z(s)}\right)^n \frac{\exp[-sf(x)]}{z(s)}\mu(dx)\\
&= s^n \sum_{l=0}^n \binom{n}{l}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l \frac{1}{z(s)}\int_\Omega f(x)^{n-l}\exp[-sf(x)]\mu(dx)\\
&= s^n \sum_{l=0}^n \binom{n}{l}\frac{1}{z(s)}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l\\
&\qquad \int_\Omega (-1)^{n-l}\left(\frac{d}{ds}\right)^{n-l}\exp[-sf(x)]\mu(dx)\\
&= s^n \sum_{l=0}^n (-1)^{n-l}\binom{n}{l}\frac{1}{z(s)}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l\left(\frac{d}{ds}\right)^{n-l}\\
&\qquad \int_\Omega \exp[-sf(x)]\,\mu(dx)\\
&= s^n \sum_{l=0}^n (-1)^{n-l}\binom{n}{l}\frac{1}{z(s)}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l z^{n-l}(s)\\
&= s^n \sum_{l=0}^n (-1)^{n-l}\binom{n}{l}\frac{z^{n-l}(s)}{z(s)}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l.
\end{aligned}$$

This means

$$\sum_{n=0}^\infty \frac{M_n(z(s))}{n!}t^n = \sum_{n=0}^\infty \left(\sum_{l=0}^n (-1)^l \binom{n}{l}\frac{z^{(n-l)}(s)}{z(s)}\left(\frac{z^{(1)}(s)}{z(s)}\right)^l\right)\frac{(-st)^n}{n!}$$

$$= \left(\sum_{l=0}^{\infty} \frac{Z^{(l)}(s)}{z(s)} \frac{(-st)^l}{l!} \right) \left(\sum_{l=0}^{\infty} \left(\frac{z^{(1)}(s)}{z(s)} \right)^l \frac{(st)^l}{l!} \right)$$

$$= \exp\left[\sum_{j=1}^{\infty} u^{(j)}(s) \frac{(-st)^j}{l!} \right] \exp\left[u^{(1)} st \right]$$

$$= \exp\left[\sum_{j=2}^{\infty} u^{(j)}(s) \frac{(-st)^j}{j!} \right],$$

$$M_n(z(s)) = \sum_{\sum_{j\geq 2} jl_j = n} n! \prod_j \frac{1}{l_j!} \left(\frac{u^{(j)}(s)}{j!} \right)^{l_j},$$

$$M_1(z(s)) = 0, \qquad M_2(z(s)) = s^2 u^{(2)}(s) = s^2 \left[\frac{z^{(2)}(s)}{z(s)} - \left(\frac{z^{(1)}(s)}{z(s)} \right)^2 \right].$$

2. Relations between the contral moments under certain functional equations

THEOREM 2. *Under the assumption*

$$z\left(\frac{-1}{s}\right) = \lambda z(s) \tag{A.4}$$

or

$$z\left(\frac{1}{s}\right) = \lambda z(s) \tag{A.5}$$

with a non-zero constant λ, we obtain the relations between the central moments,

$$K_n\left(\frac{-1}{s}\right) = n! \sum_{0<h+2l\leq n} \binom{n-l-1}{h+l-1} \frac{(-1)^h K_h(s) s^{(l)} u^{(1)}(s)^l}{h!l!}$$

or

$$K_n\left(\frac{1}{s}\right) = n! \sum_{0<h+2l\leq n} \binom{n-l-1}{h+l-1} \frac{(-1)^h K_h(s) s^{(l)} u^{(1)}(s)^l}{h!l!}$$

Proof. Putting $z(s) = \exp[u(s)]$ and $\alpha = \log \lambda$, we have

$$u\left(\frac{-1}{s}\right) = u(s) + \alpha,$$

$$u^{(1)}\Big(\frac{-1}{s}\Big) = \frac{ds}{d\Big(\frac{-1}{s}\Big)}\frac{d}{ds}(u(s)+\alpha) = s^2u^{(1)}(s),$$

or

$$u\Big(\frac{1}{s}\Big) = u(s)+\alpha,$$

$$u^{(1)}\Big(\frac{1}{s}\Big) = \frac{ds}{d\Big(\frac{1}{s}\Big)}\frac{d}{ds}(u(s)+\alpha) = s^2u^{(1)}(s),$$

$$\begin{aligned} K(s,t) &= \sum_{n=0}^{\infty}\frac{K_n(s)}{n!}(-st)^n \\ &= \exp\left[\sum_{j\geq 2}\frac{u^{(j)}(s)}{j!}(-st)^j\right] \\ &= \exp\Big[\exp\Big[u(s(1-t)) - u(s) - u^{(1)}(s)(-st)\Big]\Big], \end{aligned}$$

Appendix B. The inhomogeneous invariant theory

1. The $GL_2(K)$-germ action on the basic formal power series

We choose an element ω in K different from positive integers, and a system of variables

$$\xi = (\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \ldots)$$

with degree, weight and index such that

$$\deg \xi^{(l)}, \quad \text{weight}\, \xi^{(l)} = l, \quad \xi^{(l)} = w - 2l,$$

We introduce the basic formal power series

(B.1) $$f_\omega(\xi \mid t) = \sum_{l=0}^{\infty}(\omega)_l \xi^{(l)}\frac{t^l}{l!}$$

on which the germ of $GL_2(K)$ acts as follows,

(B.2) $$f_\omega\left(\rho\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}\xi \mid t\right) = \sum_{l=0}^{\infty}(\omega)_l\left(\rho\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}\xi\right)^{(l)}\frac{t^l}{l!}$$
$$(\delta+\delta t)^\omega \sum_{l=0}^{\infty}\frac{(\omega)_l}{l!}\xi^{(l)}\left(\frac{\beta+\alpha t}{\delta+\gamma t}^{(l)}\right),$$

where $(\omega)_l = \omega(\omega-1)(\omega-2)\ldots(\omega-l+1)$ and $\binom{\omega}{l} = (\omega)_l/l!$.

(B.2) is equivalent to the realization of the algebra $sl_2(K)$ in $K[\xi]$,

$$\text{(B.3)}\qquad \begin{aligned} D_\omega \xi^{(l)} &= l\xi^{l-1} \qquad\qquad (\xi^{-1}=0)\\ \Delta_\omega \xi^{(l)} &= (\omega - l)\xi^{(l+1)}\\ H_\omega \xi^{(l)} &= (\omega - 2l)\xi^{(l)} \end{aligned}$$

where

$$\text{(B.4)}\qquad \begin{aligned} [D_\omega, \Delta_\omega] &= H_\omega\\ [H_\omega, D_\omega] &= 2D_\omega\\ [H_\omega, \Delta_\omega] &= -2\Delta_\omega \end{aligned}$$

LEMMA 1.

$$\text{(B.5)}\qquad \begin{aligned} [D_\omega, \Delta_\omega^l] &= -l(l-1)\Delta_\omega^{l-1} + l\Delta_\omega^{l-1}H_\Omega\\ [H_\omega, \Delta_\omega^l] &= -2l\Delta_\omega^l \end{aligned}$$

Proof. Assuming (B.5) for l, we have

$$\begin{aligned} [D_\omega, \Delta_\omega^{l+1}] &= [D_\omega, \Delta_\omega^l]\Delta_\omega + \Delta_\omega^l[D_\omega, \Delta_\omega]\\ &= -l(l-1)\Delta_\omega^l + l\Delta_\omega^l H_\omega \Delta_\omega + \Delta_\omega^l H_\omega + D_\omega^l H_\omega\\ &= -l(l+1)\Delta_\omega^l + l\Delta_\omega^{l-1}[H_\omega, \Delta_\omega] + l\Delta_\omega^l H_\omega + D_\omega^l H_\omega\\ &= -l(l+1)\Delta_\omega^l + (l+1)\Delta_\omega^l H_\omega,\\ [H_\omega, \Delta_\omega^{l+1}] &= [H_\omega, \Delta_\omega^l]\Delta_\omega + \Delta_\omega^l[H_\omega, \Delta_\omega]\\ &= -2l\Delta_\omega^{l+1} - 2\Delta_\omega^{l+1}\\ &= -2(l+1)\Delta_\omega^{l+1} \end{aligned}$$

2. $\langle D_\omega, \Delta_\omega, H_\omega\rangle$-action on the basic inhomogeneous formal power series

We mean by the basic inhomogeneous formal power series the formal power series

$$\text{(B.6)}\qquad 1 + \sum_{l=1}^{\infty} (\omega)_l \frac{\xi^{(l)}}{\xi^{(o)}} \frac{t^l}{l!}$$

Changing variables

$$z^{(l)} = (\omega)_l \xi^{(l)} \qquad (l \geq 0),$$

from (B.3) we have

$$\text{(B.7)}\qquad \begin{aligned} D_\omega z^{(l)} &= l(\omega - l + 1)z^{l-1} \qquad (z^{-1} = 0) \\ \Delta_\omega z^{(l)} &= z^{(l+1)} \\ H_\omega z^{(l)} &= (\omega - 2l)z^{(l)} \end{aligned}$$

Again changing variables $z^{(l)}/z^{(0)}$ $(l \geq 1)$ to $u^{(j)}$ $(j \geq 1)$ by

$$+\sum_{l=1}^{\infty} \frac{z^{(l)}}{z^{(0)}} \frac{t^l}{l!} = \exp\left[\sum_{j=1}^{\infty} u^{(j)} \frac{t^j}{j!}\right],$$

we obtain the following $\langle D_\omega, \Delta_\omega, H_\omega \rangle$-action on $K[u^{(1)}, u^{(2)}, u^{(3)}, \ldots]$;

PROPOSITION 1.

$$\text{(B.8)}\qquad \begin{aligned} D_\omega u^{(j)} &= \begin{cases} \omega & (j = 1) \\ -j(j-1)u^{(j-1)} & (j \geq 2) \end{cases} \\ \Delta_\omega u^{(j)} &= u^{(l+1)} \\ H_\omega u^{(j)} &= -2ju^{(j)} \end{aligned}$$

Proof. We choose a generic analytic function $y(s)$ and $\omega^{(j)}(s) = (d/ds)^j$ $\omega(s)$. Hence by means of differential algebra specialigations

$$\left(\frac{y^{(1)}(s)}{y^{(0)}(s)}, \frac{y^{(2)}(s)}{y^{(0)}(s)} \frac{y^{(3)}(s)}{y^{(0)}(s)}, \ldots;\ \frac{d}{ds}\right) \longrightarrow \left(\frac{z^{(1)}}{z^{(0)}}, \frac{z^{(2)}}{z^{(0)}}, \frac{z^{(3)}}{z^{(0)}}, \ldots, \Delta_\omega\right),$$

$$\left(\omega^{(1)}(s), \omega^{(2)}(s), \omega^{(3)}, \ldots;\ \frac{d}{ds}\right) \longrightarrow \left(u^{(1)}, u^{(2)}, u^{(3)}, \ldots;\ \Delta_\omega\right),$$

we obtain

$$\Delta_\omega^{j-1} u^{(1)} = u^{(j)} \qquad (j \geq 2).$$

From (B.7), denoting $\xi^{(0)} = z^{(0)} = \exp[u^{(0)}]$, we have

$$D_\omega z^{(0)}, H_\omega z^{(0)} = \omega z^{(0)},$$

and

$$\begin{aligned} D_\omega u^{(0)} &= D_\omega(\exp[u^{(0)}])\exp[u^{(0)}]^{-1} = D_\omega z^{(0)} {z^{(0)}}^{-1} = 0 \\ H_\omega u^{(0)} &= H_\omega(\exp[u^{(0)}])\exp[u^{(0)}]^{-1} = \omega z^{(0)} {z^{(0)}}^{-1} = \omega \end{aligned}$$

Hence from $u^{(j)} = \Delta_\omega^{j-1}u^{(1)} = \Delta_\omega^{j-1}\Delta_\omega u^{(0)} = \Delta_\omega^j u^{(0)}$ and Lemma 1 we obtain

$$\begin{aligned} D_\omega u^j &= D_\omega \Delta_\omega^j u^{(0)} = [D_\omega, \Delta_\omega^j]u^{(0)} + \Delta_\omega^j D_\omega u^{(0)} \\ &= [D_\omega, \Delta_\omega^j]u^{(0)} = -j(j-1)\Delta_\omega^{j-1}u^{(0)} + j\Delta_\omega^{j-1}H_\omega u^{(0)} \\ &= \begin{cases} \omega & (j=1) \\ -j(j-1)u^{(j-1)} & (j \geq 2) \end{cases}, \end{aligned}$$

$$\Delta_\omega u^j = u^{(j+1)},$$

$$\begin{aligned} H_\omega u^{(j)} &= [H_\omega, \Delta^j]u^{(0)} + \Delta^j H_\omega u^{(0)} \\ &= -2j\Delta_\omega^j u^{(0)} + u^{(0)} + \Delta^j \omega = -2ju^{(j)} \qquad (j \geq 1), \end{aligned}$$

Now we can conclude as follows:

THEOREM 3. *The invariant theory on the basic inhomogeneous formal power series*

$$1 + \sum_{j=1}^{\infty} u^{(j)} \frac{t^j}{j!} \begin{cases} D_\omega \xi^{(l)} = l\xi^{l-1} \\ \Delta_\omega \xi^{(l)} = (\omega - l)\xi^{(l+1)} \\ H_\omega \xi^{(l)} = (\omega - 2l)\xi^{(l)} \end{cases} \tag{B.9}$$

is equivalent to the invariant theory on the basic inhomogeneous basic form

$$1 + \sum_{j=1}^{\infty} u^{(j)} \frac{t^j}{j!} \tag{B.10}$$

with respect to the realization

$$\begin{aligned} D_\omega u^{(j)} &= \begin{cases} \omega & (j=1) \\ -j(j-1)u^{(j-1)} & (j \geq 2) \end{cases} \\ D_\omega u^{(j)} &= u^{(j+1)} \\ H_\omega u^{(j)} &= -2ju^{(j)} \end{aligned}$$

The structure of the graded algebra Θ of semi incariants in $K[u]$ is very simply expressed as follows:

THEOREM 4. *The isobaric polynomials*

$$\psi_n(u) = \sum_{l=1}^{n} \frac{(n)_l(n-1)_l}{l!} \omega^{n-l} u^{(n-l)} u^{(1)^l} \tag{B.11}$$

are generators of the graded algebra Θ *of semi-invariants, i.e.*

$$\Theta = K[\psi_2(u), \psi_3(u), \psi_4(u), \ldots]$$

Proof. By calculation

$$\begin{aligned} D\psi_n(u) = &\sum_{l=1}^{n} \frac{(n)_l(n-1)_l}{l!} \omega^{n-l} u^{(n-l)} l {u^{(1)}}^{l-1} \omega \\ &- \sum_{l=0}^{n-1} \frac{(n)_l(n-1)_l}{l!} \omega^{n-l} (n-l)(n-l-1) u^{(n-l-1)} {u^{(1)}}^{l} \\ &\sum_{l=1} \frac{(n)_l(n-1)_l}{(l-1)!} \omega^{n-l+1} u^{n-l} {u^{(1)}}^{l-1} \\ &- \sum_{l=0} \frac{(n)_{l+1}(n-1)_{l+1}}{l!} \omega^{n-l} u^{(n-l-1)} {u^{(1)}}^{l} \\ = &\, 0 \end{aligned}$$

On the other hand $K[u^{(1)}, \psi_2(u), \psi_3(u), \psi_4(u), \ldots] = K[u^{(1)}, u^{(2)}, u^{(3)}, \ldots]$ and $u^{(1)}$ is transcendental over $K[\psi_2(u), \psi_3(u), \psi_4(u), \ldots]$, hence $F = \sum_{k=0}^{n} {u^{(1)}}^k g_k(\psi)$ belongs to Θ, if and only if $F = g_0(\psi)$.

A Cashimi operator is a non - zero element in the center of the universal enveloping algebra of $sl_2[K]$, and the next is a generator of Cashimir operators of the realization $\langle D_\omega, \Delta_\omega, H_\omega \rangle$ of $sl_2(K)$,

(B.12) $$K_\omega = \frac{1}{4}({H_\omega}^2 + 4\Delta_\omega D_\omega + 2H_\omega).$$

PROPOSITION 2.

(B.13) $$K_\omega u^j = 0 \qquad (j \geq 1)$$

Proof. By caluculation we have

$$\begin{aligned} K_\omega u^{(1)} &= \frac{1}{4}({H_\omega}^2 u^{(1)} + 4\Delta_\omega D_\omega u^{(1)} + 2H_\omega) u^{(1)} \\ &= \frac{1}{4}((-2)^2 u^{(1)} + 4\Delta_\omega \omega + 2(-2) u^{(1)}) = 0, \end{aligned}$$

$$\begin{aligned} K_\omega u^{(j)} &= \frac{1}{4}((-2j)^2 u^{(j)} + 4\Delta_\omega (-j)(j-1) u^{(j-1)} + 2(-2j) u^{(j)}) \\ &= \frac{1}{4}(4j^2 u^{(j)} - 4j(j-1) u^{(j)} - 4j u^{(j)}) = 0. \end{aligned}$$

PROPOSITION 3.

$$K_\omega \xi^{(l)} = \frac{1}{4}\omega(\omega+2)\xi^{(l)} \qquad (l \geq 0) \tag{B.14}$$

Proof. By calculatation, we have

$$\begin{aligned} K\xi^{(l)} &= \frac{1}{4}(H_\omega{}^2\xi^{(l)} + 4\Delta_\omega D_\omega \xi^{(l)} + 2H_\omega \xi^{(l)}) \\ &= \frac{1}{4}((\omega-2l)^2 + 4l(\omega-l+1) + 2(\omega-2l))\xi^{(l)} = \frac{1}{4}\omega(\omega+2)\xi^{(l)}. \end{aligned}$$

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