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HYPERCONVEXITY AND BERGMAN COMPLETENESS

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Abstract. We show that any bounded hyperconvex domain is Bergman complete.

Let $D \subset \mathbb{C}^n$ be a bounded domain. By b_D we denote the Bergman distance on D which is defined as the integrated form of the Bergman metric

$$\beta_D(z;X) := \sqrt{\sum_{i,j=1}^{n} \frac{\partial^2 \log K_D(z,z)}{\partial z_i \partial \bar{z}_j} X_i \bar{X}_j},$$

i.e. $b_D(z',z'') := (\int \beta_D)(z',z'')$, $z',z'' \in D$, where $K_D(\cdot,\cdot)$ is the Bergman kernel of D (for more details see [9, Chapter IV]).

It is an old problem asked by Kobayashi (cf. [11], see also [12]) which bounded pseudoconvex domain $D \subset \mathbb{C}^n$ is Bergman complete. Observe that pseudoconvexity is necessary. There is a long list of papers treating this question (cf. [5], [7], [10], [13], [14], [15], [16]). The state of affair is that the Bergman kernel K_D tends to infinity near the boundary if D is hyperconvex (cf. [14]). Recall that a bounded domain D is called to be *hyperconvex* if there is a continuous negative plurisubharmonic exhaustion function. Observe that D is already hyperconvex if a negative (not necessarily continuous) plurisubharmonic exhaustion function of D exists (cf. [17], [3]). Using a result of P. Pflug (cf. [16]) density of $H^\infty(D)$ in $L^2_h(D)$ would imply that D is Bergman complete. Following this line Chen (cf. [5]) proved recently that any bounded pseudoconvex domain with Lipschitz boundary is b-complete. Observe that such domain is automatically hyperconvex (cf. [6]). In his paper, Chen asks the question whether any bounded hyperconvex domain is Bergman complete. In fact, his paper itself contains the key to solve that question in the affirmative. Namely, the following lemma is there.

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LEMMA. (cf. [5]) *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Then there is a positive constant C such that if $f \in L^2_h(D)$ and $a \in D$ then there exists $F \in L^2_h(D)$ such that*

$$F(a) = 0 \text{ and } \|F - f\|_{L^2_h(D)} \leq C\|f\|_{L^2_h(D_a)},$$

where $D_a := \{z \in D: g_D(a, z) < -1\}$. Here $g_D(a, \cdot)$ denotes the pluricomplex Green function of D with pole at a.

Using an old result by Pflug (cf. [16]) we get (see also the paper of Chen):

PROPOSITION A. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Assume for any boundary sequence $(a_\nu)_\nu \subset D$, $\lim a_\nu =: a \in \partial D$, that $\operatorname{vol}(D_{a_\nu}) \to 0$. Then D is b_D-complete.*

Proof. Assume that D is not Bergman complete. Then, according to [16] we find a boundary sequence $(a_\nu) - \nu \subset D$ and real numbers $(\Theta_\nu)_\nu$ such that

$$\left(\frac{K_D(\cdot, a_\nu)}{\sqrt{K_D(a_\nu, a_\nu)}} e^{i\Theta_\nu}\right)_\nu$$

is a Cauchy sequence in the Hilbert space $L^2_h(D)$ that converges to a function $f \in L^2_h(D)$ with $\|f\|_{L^2_h(D)} = 1$.

Hence we get taking scalar product that

$$\frac{|f(a_\nu)|}{\sqrt{K_D(a_\nu, a_\nu)}} \longrightarrow \|f\|^2_{L^2_h(D)} = 1.$$

On the other side using the above lemma we see that for suitable functions $F_\nu \in L^2_h(D)$ we have

$$\frac{|f(a_\nu)|}{\sqrt{K_D(a_\nu, a_\nu)}} = \left|\left(f - F_\nu, \frac{K_D(\cdot, a_\nu)}{\sqrt{K_D(a_\nu, a_\nu)}} e^{i\Theta_\nu}\right)_{L^2_h(D)}\right|$$
$$\leq \|F_\nu - f\|_{L^2_h(D)} \leq C\|f\|_{L^2_h(D_{a_\nu})},$$

which contradicts the assumption that $\operatorname{vol}(D_{a_\nu}) \to 0$. □

So the main point to prove is the following statement.

PROPOSITION B. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $(a_\nu)_\nu \subset D$ be a boundary sequence in D. Then*

$$\operatorname{vol}(\{z \in D : g_D(a_\nu, z) < -1\}) \to 0,$$

if $\nu \to \infty$.

Proof. According to a result from [2] we find a function $u \in PSH(D) \cap C(\bar{D})$ with $(dd^c u)^n = d\Lambda$ and $u|_{\partial D} = 0$. Using an estimate proven in [1] we obtain

$$\begin{aligned}\int_D (-g_D(a_\nu, \cdot))^n d\Lambda &= \lim_{j\to\infty} \int_D (-\max\{g_D(a_\nu, \cdot),\, -j\})^n (dd^c u)^n \\ &\leq n!\|u\|^{n-1}_{L^\infty(D)} \lim_{j\to\infty} \int_D (-u)(dd^c \max\{g_D(a_\nu, \cdot),\, -j\})^n \\ &= n!(2\pi)^n \|u\|^{n-1}_{L^\infty(D)} |u(a_\nu)|,\end{aligned}$$

where the last equality is due to Demailly [6]. Therefore,

$$\lim_{\nu\to\infty} \int_D (-g_D(a_\nu, \cdot))^n d\Lambda = 0,$$

from which the assertion immediately follows. □

Combining Propositions A and B we reach the following result

THEOREM. *Any bounded hyperconvex domain $D \subset \mathbb{C}^n$ is Bergman complete.*

In particular, we get (cf. [10] and [5] (see also [6])) the following corollary.

COROLLARY. a) *Any bounded complete circular domain of holomorphy with continuous Minkowski functional is Bergman complete.*

b) *Any bounded pseudoconvex domain with Lipschitz boundary is Bergman complete.*

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