AN EXTENSION OF POINCARÉ FORMULA IN INTEGRAL GEOMETRY

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1. A curve c_2 of finite length L_2 moves on a euclidean plane. Let the number of points of intersection of c_2 with the fixed curve c_1 of length L_1 be n, and the element of kinematic measure of the position of c_2 be dK. Then, owing to Poincaré, we have

$$\int n\, dK = 4L_1L_2,$$

where the integration extends over all the positions of the moving curve c_2. An analogous formula was obtained by Santaló [1] in the case of a curve and a surface in the euclidean 3-space, and by Blaschke [2] in the case of two surfaces. Here I extend these to the case of general Klein spaces by the method of moving frames of E. Cartan [3]. The method used is analogous to that of the paper of S. S. Chern [4], but I have worked out independently. Moreover I show examples which may be of some interest.

2. In Kein spaces, whose fundamental group is a Lie group G, we call the left cosets aH of G by a Lie subgroup H points, and let F_1 and F_2 be manifolds which consist of points x, the former being space fixed and the latter moving. Hereafter we assume the differentiability to the order we need. We attach to every point of F_1 and F_2 Frenet's frames, whose motion along F_1 and F_2 is denoted by S_1 and S_2. Then we take one of the intersection points and call it O. Let the motion, which removes the Frenet's frame of F_1 at O to the one of F_2 at O, be T and let the fixed frame of F_1 be R_0 and the frame that is relatively fixed to F_2 be R. Then R can be represented as

$$R = S_1TS_2^{-1}R_0,$$

which can be understood by the fact that the relative position between R and S_1TR_0 is represented by S_2^{-1}. So when we put

$$S = S_1TS_2^{-1}, \tag{1}$$

the position of moving manifold F_2 can be determined by S. We denote the

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parameters of the fundamental Lie group G by a symbolically and those of S_1, S_2 and T by a_1, a_2 and t respectively. Hereafter we use the notation δ for the infinitesimal relative motion; for example $\delta S = S(a)^{-1}S(a+da)$. Hence we get by (1)

$$\delta S = S(a)^{-1}S(a+da) = S_2(a_2)\,T(t)^{-1}S_1(a_1)^{-1}S_1(a_1+da_1)\,T(t+dt)\,S_2^{-1}(a_2+da_2).$$

So we have

$$S_2^{-1}\cdot\delta S\cdot S_2 = (T^{-1}\cdot\delta S\cdot T)\,\delta T\,(\delta S_2)^{-1}. \tag{2}$$

Let the dimension of Lie group G be r and the relative components of δS be $\omega_i\,(i=1, 2, \dots, r)$ among which the principal components of points are $\omega_i\,(i=1, 2, \dots, n)$. It is easily seen that the relative components of the product of several infinitesimal motions are the sum of the corresponding relative components of each infinitesimal motion. So, when we denote the relative components of

$$T^{-1}\cdot\delta S_1\cdot T,\quad \delta T,\quad (\delta S_2)^{-1}$$

by

$$\sum_{j=1}^{r} t_{ij}\omega_j^{(1)},\quad \omega_i^{(0)},\quad -\omega_i^{(2)}\quad (i=1, 2, \dots, r),$$

those of $S_2^{-1}\cdot\delta S\cdot S_2$ are

$$\sum_{j=1}^{r} t_{ij}\omega_j^{(1)} + \omega_i^{(0)} - \omega_i^{(2)}\quad (i=1, 2, \dots, r). \tag{3}$$

Here $\omega_j^{(1)}$ $(j=1, 2, \dots, r)$ are the relative components of δS_1 and the transformation matrix (t_{ij}) means the matrix of the linear adjoint group that corresponds to T.

Now we assume that G has measure which is independent of the choice of the frame attached to the moving manifold. Then the measure element dK of δS is equal to that of $S_2^{-1}\cdot\delta S\cdot S_2$. So by virtue of (2) and (3) we have

$$dK = \prod_{i=1}^{r}\Big(\sum_{j=1}^{r} t_{ij}\omega_j^{(1)} - \omega_i^{(2)} + \omega_i^{(0)}\Big),$$

where the product of pfaffian forms means alternating product. For the relative motion δT of T, which is the rotation about the point O, the principal relative components $\omega_i^{(0)}$ $(i=1, 2, \dots, n)$ are all zero. So we have

$$dK = \prod_{i=1}^{n}\Big(\sum_{j=1}^{r} t_{ij}\omega_j^{(1)} - \omega_i^{(2)}\Big)\prod_{i=n+1}^{r}\Big(\sum_{j=1}^{r} t_{ij}\omega_j^{(1)} - \omega_i^{(2)} + \omega_i^{(0)}\Big). \tag{4}$$

Now we treat the case in which the relation $n = p + q$ holds, where p and q are dimensions of the manifold F_1 and F_2 respectively. As the matrix (t_{ij}) which operates on $\omega_1, \dots, \omega_r$ keeps the relation $\omega_1 = 0, \dots, \omega_n = 0$ invariant, we have the relation

$$t_{ij} = 0 \quad (i = 1, \ldots, n;\ j = n+1, \ldots, r).$$

On the other hand let $\pi_i^{(1)}$ $(i = 1, \ldots, p)$ and $\pi_j^{(2)}$ $(j = 1, \ldots, q)$ be the principal relative components of the Frenet's frames of F_1 and F_2 that are linearly independent with respect to the differential. Then all the principal relative components of F_1 and F_2 are represented by

$$(5) \qquad \omega_i^{(1)} = \sum_{j=1}^{p} \lambda_{ij}^{(1)} \pi_j^{(1)}, \quad \omega_i^{(2)} = \sum_{j=1}^{q} \lambda_{ij}^{(2)} \pi_j^{(2)} \quad (i = 1, \ldots, n).$$

After these considerations we get from (4)

$$(6) \qquad dK = \varDelta \prod_{i=1}^{p} \pi_i^{(1)} \cdot \prod_{j=1}^{q} \pi_j^{(2)} \cdot \prod_{k=n+1}^{r} \omega_k^{(0)}.$$

Here

$$(7) \qquad \varDelta = \sum \pm T\begin{pmatrix} j_1' & j_2' & \ldots & j_p' \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} \varLambda^{(1)}(i_1 i_2 \ldots i_p)\, \varLambda^{(2)}(j_1 j_2 \ldots j_q),$$

and $\varLambda^{(1)}(i_1 \ldots i_p)$ is the determinant constructed from i_1-th, i_2-th, ..., and i_p-th row of the matrix $\left(\lambda_{ij}^{(1)}\right)$, and the same for $\varLambda^{(2)}(j_1 j_2 \ldots j_q)$, while $T\begin{pmatrix} j_1' & j_2' & \ldots & j_p' \\ i_1 & i_2 & \ldots & i_p \end{pmatrix}$ is the minor of the matrix (t_{ij}) and $j_1', j_2', \ldots, j_p', j_1, j_2, \ldots j_q$ is any permutation of $1, 2, \ldots, n$, and the summation in (7) extends over all the permutations of $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$.

3. We now make two assumptions and integrate (6). One of them is that

(A) $\varDelta$ depends only on (t_{ij}),

while the other is that

(B) $c = \int \left| \varDelta \prod_{k=n+1}^{r} \omega_k^{(0)} \right|$ is finite, where the integration extends over all the rotations about O.

Then, when the number of intersection points of F_1 and F_2 is ν, we get

$$(8) \qquad \int \nu\, dK = cM_1 M_2,$$

where

$$\int \left| \prod_{i=1}^{p} \pi_i^{(1)} \right| = M_1 \quad \text{and} \quad \int \left| \prod_{j=1}^{q} \pi_j^{(2)} \right| = M_2$$

the integration ranging over the whole mamifold F_1 and F_2. (8) is a generalization of Poincaré's theorem.

The assumption (A) is satisfied if all $\lambda_{ij}^{(1)}$ and $\lambda_{ij}^{(2)}$ are constant. This means that F_1 has only one p-dimensional area and F_2 only one q-dimensional area.

4. Before we show some examples, we make one remark. When the fundamental Lie group is a linear group, we can take PI^0 as the frame which is obtained by the transformation $P=(p_{ij})$ from the fundamental frame $I^0=(I_1^0, I_2^0, \ldots, I_n^0)$, I_0 being a set of independent vectors. Then the point x, which has $(x_1, x_2, \ldots, x_n)$ as coordinates with respect to I^0, is transformed into the point

$$x'=\sum_{i=1}^{n} x_i I_i=\sum_{i,j=1,\ldots,n} x_i p_{ij} I_j^0$$

namely into $x'=(x_1', x_2', \ldots, x_n')$ that is determined by $x'=P'x$, P' being the transposed matrix of P. Thus if we denote the point transformation by $x'=P'x$, the frame transformation is $I=PI^0$. So the order of the products of the matrices of the point transformation and the frame transformation are inverse. For example from the frame transformation $I=PI^0$, we get for the infinitesimal relative motion $dP.P^{-1}$, whose elements are relative components.

5. In the euclidean space of dimension n the infinitesimal relative motion can be represented by

$$dA=\sum_{i=1}^{n}\omega_i I_i, \quad dI_i=\sum_{j=1}^{n}\omega_{ij} I_j \quad (i=1, \ldots, n),$$

where A is the vertex of the frame and $I_1, \ldots, I_n$ are unit orthogonal system. Let F_1 and F_2 be respectively p- and q-dimensional surfaces and their independent principal relative components be $\omega_i^{(1)}$ $(i=1, 2, \ldots, p)$ and $\omega_j^{(2)}$ $(j=1, 2, \ldots, q)$, the other principal components being zero. Moreover we denote by $T=(t_{ij})$ the rotation about O that removes the Frenet's frame of F_1 at O to that of F_2 at O. When the relation $n=p+q$ holds, we get by calculation

$$dK=(-1)^q \Delta \prod_{j=1}^{q}\omega_j^{(2)} \prod_{i=1}^{p}\omega_i^{(1)} \cdot dK_0,$$

where

$$\Delta=\begin{vmatrix} t_{q+1,1} & \ldots & t_{q+1,p} \\ & \ldots\ldots & \\ t_{n,1} & \ldots & t_{n,p} \end{vmatrix}$$

and dK_0 is the measure element of rotation about O. Thus the assumptions (A) and (B) are both satisfied and we get (8).

The case $n=2$, $p=1$, $q=1$ is the Poincaré's and the case $n=2$, $p=1$, $q=2$ is the Santalo's. Moreover the above consideration is available for the case of spherical space. When $n<p+q$, we get the extension of Blaschke's

theorem. Let the area element of the intersection manifold F of F_1 and F_2 be $d\sigma$, and the rotation which removes the Frenet's frame at a certain point O of F_1 to the Frenet's frame of F_2 at O be $T = (t_{ij})$. When we denote by dK_0 the measure element of the rotation about the point O, we get

$$d\sigma \circ dK = (-1)^q \Delta \prod_{j=1}^{q} \omega_j^{(2)} \circ \prod_{i=1}^{p} \omega_i^{(1)} \circ dK_0,$$

where the meaning of Δ is as follows. The rotation T is decomposed into three parts, namely T_1, T_2 and U, where T_i $(i = 1, 2)$ is the one which removes the Frenet's frame of F_i at O to the frame R_i $(i = 1, 2)$ whose first $p + q - n$ axes touch F, and U is the rotation which removes R_1 to R_2. If we put

$$U = \begin{pmatrix} * & * \\ A & * \end{pmatrix}$$

where A has $n - q$ rows and p columns, then $\pm\Delta$ is the determinant which is constructed from the $r + 1$, $r + 2$, . . . and p-th columns of the matrix A. We omit the proof.

6. Next we treat the case when the assumptions (A) and (B) do not hold. Let c_1 and c_2 be the one-parametric sets of straight lines on a euclidean plane and let the angles, which any straight line of c_i $(i = 1, 2)$ makes with the fixed line of c_i $(i = 1, 2)$ be θ_i $(i = 1, 2)$. We call the intersection of c_1 and c_2 the line that is common to c_1 and c_2, and let the distance of two points, at which the intersection line touches the enveloping curves of c_1 and c_2, be λ. Then by calculation the formula (6) reduces to

$$dK = \lambda d\lambda d\theta_1 d\theta_2.$$

So, although the assumption (A) is satisfied, the assumption (B) is not.

7. The example where (A) does not hold can be obtained if we consider two line congruences of the euclidean 3-space. In a euclidean 3-space we take a point A and three unit vectors I_1, I_2, I_3 that are orthogonal to each other and call (A, I_1, I_2, I_3) a frame. Then the infinitesimal motion can be represented by

$$dA = \sum_{i=1}^{3} \omega_i I_i, \quad dI_i = \sum_{j=1}^{3} \omega_{ij} I_j \quad (\omega_{ij} + \omega_{ji} = 0).$$

Now we take the straight line which passes through A and has the direction I_3. We introduce the new frame determined by

$$\overline{A} = A + \lambda I_3, \quad \overline{I}_1 = I_1 \cos\theta + I_2 \sin\theta, \quad \overline{I}_2 = -I_1 \sin\theta + I_2 \cos\theta, \quad \overline{I}_3 = I_3. \tag{9}$$

Then the infinitesimal motion can be represented as

$$d\overline{A} = \sum_{i=1}^{3} \overline{\omega}_i \overline{I}, \quad d\overline{I}_i = \sum_{j=1}^{3} \overline{\omega}_{ij} \overline{I}_j,$$

where

$$\begin{aligned}
\overline{\omega}_1 &= \omega_1 \cos\theta + \omega_2 \sin\theta + \lambda(\omega_{31}\cos\theta + \omega_{32}\sin\theta) \\
\overline{\omega}_2 &= -\omega_1 \sin\theta + \omega_2 \cos\theta + \lambda(-\omega_{31}\sin\theta + \omega_{32}\cos\theta) \\
\overline{\omega}_{31} &= \omega_{31}\cos\theta + \omega_{32}\sin\theta \\
\overline{\omega}_{32} &= -\omega_{31}\sin\theta + \omega_{32}\cos\theta.
\end{aligned}$$

Quadratic differential forms

$$\omega_{31}^2 + \omega_{32}^2, \qquad \begin{vmatrix} \omega_1 & \omega_2 \\ \omega_{31} & \omega_{32} \end{vmatrix} = \omega_1\omega_{32} - \omega_2\omega_{31}$$

are invariant for the transformation (9) and consequently $\omega_{31}^2 + \omega_{32}^2 + c(\omega_1\omega_{32} - \omega_2\omega_{31})$ is invariant for any constant c. Hence we get invariant forms

$$\begin{aligned}
&A = [\omega_{31}\omega_{32}], \qquad B = [\omega_{31}\omega_1] + [\omega_{32}\omega_2], \\
&C = [\omega_1\omega_{32}]^2 + [\omega_2\omega_{31}]^2 - 2[\omega_2\omega_{32}][\omega_1\omega_{31}] - 2[\omega_1\omega_2][\omega_{31}\omega_{32}],
\end{aligned} \tag{10}$$

where the brackt [ ] means alternating product. Here and hereafter we omit the details of calculation.

For a line congruence which has the property $A = [\omega_{31}\omega_{32}] \neq 0$ and satisfies a certain reality condition we can select λ and θ in (9) suitably and make one of ω_1 and ω_2 identically zero. So we assume that by this choice of frame we have $\omega_2 = 0$. This frame is Frenet's frame of the line congruence. If we put $\omega_1 = a\omega_{31} + b\omega_{32}$, then we have by (10)

$$B = bA, \quad C = a^2A^2.$$

So a and b are invariants of line congruences which correspond to λ_{ij} in (5).

New we take line congruences F_1 and F_2, the former being space fixed and the latter moving. Let the common line of F_1 and F_2 be l and the motion which removes the Frenet's frame of F_1 at l to the one of F_2 at l be represented by (9). Denote the area elèment $[\omega_{31}^{(i)}, \omega_{32}^{(i)}]$ $(i = 1, 2)$ of the spherical representation of the line congruences by $d\sigma_i$ $(i = 1, 2)$ and the invariants a and b of F_i $(i = 1, 2)$ by a_i and b_i. Then the measure element dK of the position of the moving congruence F_2 which has a common line with the space-fixed F_1 can be written as

$$dK = \Delta\, d\lambda\, d\theta\, d\sigma_1 d\sigma_2,$$

where

$$\Delta = -\lambda^2 + (a_1 - a_2)\lambda - (a_1b_2 - a_2b_1)\cos\theta\sin\theta + (a_1a_2 + b_1b_2)\sin^2\theta.$$

This does not satisfy the assumption (A), except when $a_1 - a_2$, $a_1 b_2 - a_2 b_1$ and $a_1 a_2 + b_1 b_2$ are all constant.

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