NOTE ON SUBDIRECT SUMS OF RINGS

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In my previous paper "On the theory of semi-local rings,"[1] we saw that if a semi-local ring R with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ is a subdirect sum of local rings $R_{[\mathfrak{p}_i]}$,[2] then R is the direct sum of $R_{[\mathfrak{p}_i]}$ (proposition 15, (slr)[1]) and that a complete semi-local ring is a direct sum of complete local rings (Remark to proposition 5, (slr)).

The main purpose of the present note is to prove two kinds of generalization (also for non-commutative case) of the first assertion mentioned above (Theorems 2 and 3). We first introduce in §1 the concept of n-rings and then we define the concepts of semi-local rings, local rings and so on; it is proved here that a commutative (semi-) local ring is a (semi-) local ring in the sense of (slr). It is also remarked that the assumption in Proposition 15, (slr), is a necessary and sufficient condition in order that a commutative semi-local ring is a direct sum of local rings. In §2, we prove our main theorems. In §3, we prove a generalization of the second assertion mentioned above for non-commutative case; in §4 we study rings which are subdirect sums of (a finite number of) n-rings.

1. Definitions and remarks to commutative case

DEFINITION 1. A ring[3] R is called an n-ring if $R^2 = R$ and if for any proper ideal[4] $\mathfrak{a}$ in R there exists a maximal ideal[5] containing $\mathfrak{a}$.

DEFINITION 2. A quasi-semi-local ring is a non-zero n-ring which contains only a finite number of maximal ideals. A quasi-local ring is a non-zero n-ring which contains only one maximal ideal.

DEFINITION 3. A quasi-semi-local ring R with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ is called a semi-local ring if $\bigcap_{i,n} \mathfrak{p}_i^n = (0)$. In this case we introduce a topology in R by taking $\{\bigcap_{i=1}^{h} \mathfrak{p}_i^n;\ n = 1, \ldots, k, \ldots\}$ as a system of neighbour-

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2) This notation is same as in (slr); this denotes the topological quotients ring of $\mathfrak{p}_i$ with respect to R: See Chapter I, (slr).

3) A ring means an associative ring.

4) An ideal means a two-sided ideal.

5) Since $R^2 = R$, any maximal ideal is prime (we say an ideal $\mathfrak{p}$ in a ring R is maximal if $R \neq \mathfrak{p}$ and if there exists no ideal $\mathfrak{a}$ such as $R \supset \mathfrak{a} \supset \mathfrak{p}$).

hoods of zero; thus a semi-local ring is a topological ring. A local ring is a semi-local, quasi-local ring.

LEMMA 1. Let R be a ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be proper prime ideals in R. Then $\bigcup_{i=1}^{h} \mathfrak{p}_i \neq R$.[6)]

Proof. For $h = 1$, our assertion is trivial. So, we assume that $\bigcup_{i=1}^{h-1} \mathfrak{p}_i \neq R$. Let a be an element of R which is not contained in $\bigcup_{i=1}^{h-1} \mathfrak{p}_i$. If $a \notin \mathfrak{p}_h$, our assertion is true; if not, we take an element b of R such as $b \in \bigcap_{i=1}^{h-1} \mathfrak{p}_i$, $b \notin \mathfrak{p}_h$,[*)] then $a + b \notin \mathfrak{p}_i$ for any i $(1 \leqq i \leqq h)$. This proves our assertion.

COROLLARY. Let R be an n-ring. Then any union of a finite number of proper ideals does not coincide with R.

PROPOSITION 1. A commutative quasi-semi-local ring contains the identity.

Proof. This follows from our Lemma 1 (or Corollary to it) and the fact that a commutative ring $R \neq (0)$ contains the identity if (and only if) there exists an element a of R such that $aR = R$.

COROLLARY. A commutative semi-local ring is a semi-local ring in the sense of (slr).

We mention, by the way,

PROPOSITION 2. Let a commutative ring R which contains the identity be a direct sum of rings R_i $(i = 1, \ldots, n)$ $(R_i \neq (0))$. Let $\{\mathfrak{p}_{i\lambda};\ \lambda \in \Lambda_i\}$ (for each $i = 1, \ldots, n$) be the totality of maximal ideals whose images in R_i are different from R_i. Then R_i is the ring of quotients of S_i with respect to R, where S_i is the complementary set of $\bigcup_{\lambda \in \Lambda_i} \mathfrak{p}_{i\lambda}$ with respect to R. If R is a semi-local ring (or more generally, generalized semi-local ring in the sense of (slr)) then R_i coincides also with the topological quotients ring of S_i with respect to R.

Proof. Easy.

2. Main theorems

LEMMA 2. Let a ring R be a subdirect sum of rings R_i $(i = 1, \ldots, n)$. If $\mathfrak{p}$ is a proper prime ideal in R, then for at least one i the image of $\mathfrak{p}$ in R_i does not coincide with R_i.

Proof. Let $\mathfrak{n}_i$ be the kernel of natural homomorphism of R onto R_i, for each i. Then $\bigcap_{i=1}^{n} \mathfrak{n}_i = (0)$. Therefore $\mathfrak{n}_i \subseteqq \mathfrak{p}$ for at least one i.

COROLLARY. Let an n-ring R be a subdirect sum of rings (necessarily n-rings) R_i $(i = 1, \ldots, n)$. If $\mathfrak{a}$ is a proper ideal in R, then for at least one i the

6) Set theoretical union.

*) We may assume without loss of generality that $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ $(i \neq j)$.

image of $\mathfrak{a}$ in R_i is different from R_i.

THEOREM 1. *Let a ring R be a subdirect sum of n-rings R_i $(i = 1, \ldots, n)$ $(n > 1)$. Then R contains R_i if (and only if) the following condition is satisfied: If $\bar{\mathfrak{p}}_1$ and $\bar{\mathfrak{p}}_2$ are two maximal ideals in the direct sum $\bar{R}$ of R_i $(i = 1, \ldots, n)$ such that $\bar{\mathfrak{p}}_1 \supseteqq R_1$, $\bar{\mathfrak{p}}_2 \not\supseteqq R_1$, then $\bar{\mathfrak{p}}_1 \cap R \neq \bar{\mathfrak{p}}_2 \cap R$.*

Proof. We set $R_1 \cap R = \mathfrak{a}$. We assume that $\mathfrak{a} \neq R_1$. Let $\mathfrak{p}_1$ be a maximal ideal in R_1 cntaining $\mathfrak{a}$. Then $\mathfrak{p} = R \cap (\mathfrak{p}_1 + R_2 + \ldots + R_n)$ is a maximal prime ideal in R. On the other hand, $R/\mathfrak{a}$ is a subdirect sum of rings R_i $(i = 2, \ldots, n)$. Therefore, for a suitable k $(k > 1)$, the image of $\mathfrak{p}$ in R_k is different from R_k: Let $\mathfrak{p}_k$ be a maximal ideal in R_k containing the image of $\mathfrak{p}$ in R_k. Then $\mathfrak{p}$ is contained in $R_1 + \ldots + R_{k-1} + \mathfrak{p}_k + R_{k+1} + \ldots + R_n$. This shows that $R \cap (\mathfrak{p}_1 + R_2 + \ldots + R_n) = R \cap (R_1 + \ldots + R_{k-1} + \mathfrak{p}_k + R_{k+1} + \ldots + R_n)$, contrary to our assumption.

THEOREM 2. *Let a ring R be a subdirect sum of n-rings $R_1, \ldots, R_n$. Then R is the direct sum of R_i $(i = 1, \ldots, n)$ if (and only if) the following condition is satisfied: If $\bar{\mathfrak{p}}_1$ and $\bar{\mathfrak{p}}_2$ are distinct maximal ideals in the direct sum $\bar{R}$ of R_i $(i = 1, \ldots, n)$, then $\bar{\mathfrak{p}}_1 \cap R \neq \bar{\mathfrak{p}}_2 \cap R$.*

Proof. This is an immediate consequence of Theorem 1.

COROLLARY 1. If a ring R is a subdirect sum of (quasi-) semi-local rings $R_1, \ldots, R_n$ and if the number of maximal prime ideals[7] of R is the sum of those of R_i, then R is the direct sum of R_i $(i = 1, \ldots, n)$.

COROLLARY 2. A semi-local ring R with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ is a direct sum of local rings if and only if each $\mathfrak{p}_i$ is the unique maximal ideal containing $\bigcap_{n=1}^{\infty} \mathfrak{p}_i^n$.

COROLLARY 3. Let a ring R be a subdirect sum of n-rings $R_1, \ldots, R_n$. If $R_i/\mathfrak{p}_i$ and $R_j/\mathfrak{p}_j$ are non-isomorphic to each other for any maximal ideals $\mathfrak{p}_i$ in R_i and $\mathfrak{p}_j$ in R_j $(i \neq j)$, then R is the direct sum of $R_1, \ldots, R_n$.

THEOREM 3. *If an n-ring is a subdirect sum of (quasi-) local rings R_i $(i = 1, \ldots, n)$, then R is a direct sum of suitable m $(\leqq n)$ (quasi-) local rings. (If moreover R contains n distinct maximal ideals, R is the direct sum of R_i.)*

Proof. Our assertion is trivial for the case $n = 1$. Now, assuming that our assertion is true for the case $n < h$, we prove the case $n = h$. Let $\bar{R}$ be the direct sum of rings $R_1, \ldots, R_h$. We set $\mathfrak{a}_i = R \cap R_i$. Then $R/\mathfrak{a}_i$ is a subdirect sum of $R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_h$. Hence $R/\mathfrak{a}_i$ is a direct sum of m_i $(< h)$

7) Evidently this number is finite.

(quasi-) local rings. If $m_i < h-1$ for some i, our assertion is true because R is a subdirect sum of m_i+1 (quasi-) local rings. Therefore we assume that $R/\mathfrak{a}_i \cong R_1 + \ldots + R_{i-1} + R_{i+1} + \ldots + R_h$ for any i. Whence, if $\mathfrak{a}_i = R_i$ for some i, our assertion is true, i.e., in this case, $R = \overline{R}$. Now, we assume that $\mathfrak{a}_i \neq R_i$ (for at least one, therefore any, i). Let $\overline{\mathfrak{p}}_1, \ldots, \overline{\mathfrak{p}}_h$ be the maximal ideals in $\overline{R}$, where $\overline{\mathfrak{p}}_i \cap R_i \neq R_i$. Set $\overline{\mathfrak{p}}_i \cap R = \mathfrak{p}_i$. Since $R/\mathfrak{a}_i$ contains only $h-1$ maximal ideals, one $\mathfrak{p}_j$, say $\mathfrak{p}_h$, coincides with some $\mathfrak{p}_k$, say with $\mathfrak{p}_{h-1}$. Therefore, if $h=2$, R is itself a (quasi-) local ring. If $h>2$, R contains elements $(b_1, 0, \ldots, 0, a_h)$ and $(b_2, 0, \ldots, 0, a_{h-1}, 0)$ with suitable $b_1, b_2 \in R_1$ and $a_h \in R_h$, $a_{h-1} \in R_{h-1}$, *such that each* a_i *is not contained in the maximal ideal in* R_i. This is a contradiction to our assumption that $\mathfrak{p}_{h-1} = \mathfrak{p}_h$.

Remark. If a semi-local ring R is a direct sum of semi-local rings R_i $(i = 1, \ldots, n)$, R is a product space of R_i.

3. Complete[8] semi-local rings

LEMMA 3. Let R be a ring such that $R^2 = R$. If $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are ideals in R such that $\mathfrak{a} + \mathfrak{b} = R$ and $\mathfrak{a} + \mathfrak{c} = R$, then $\mathfrak{a}^m + \mathfrak{b}^n = R$ for any integers m and n, and $\mathfrak{a} + \mathfrak{b}\mathfrak{c} = R$ (therefore $\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = R$).

Proof. Since $\mathfrak{a} + \mathfrak{b}^2 \supseteqq R^2 = R$, we have $\mathfrak{a} + \mathfrak{b}^2 = R$. This proves our first assertion. The second one follows from $R = R^2 \subseteqq \mathfrak{a} + \mathfrak{b}\mathfrak{c}$.

THEOREM 4. *A complete semi-local ring is a direct sum of complete local rings.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be the totality of maximal ideals in a complete semi-local ring R. We set $\mathfrak{a}_i^{(n)} = \bigcap_{j \neq i} \mathfrak{p}_j^n$. By Lemma 3, $\mathfrak{p}_i^n + \mathfrak{a}_i^{(n)} = R$. Let a be an element of R. Then we can find an element $a_{i,n}$ of $\mathfrak{a}_i^{(n)}$ such that $a_{i,n} \equiv a$ (mod. $\mathfrak{p}_i^n$). Then the sequence $(a_{i,n})$ $(n = 1, 2, \ldots)$ is convergent (for each i). Let $f_i(a)$ be its limit. Then $f_i(a) \equiv a\,(\text{mod.} \bigcap_{n=1}^{\infty} \mathfrak{p}_i^n)$, $f_i(a) \in \bigcap_{n=1}^{\infty} \mathfrak{a}_i^{(n)}$.[9] This *proves that each* $\mathfrak{p}_i$ *is the unique maximal ideal containing* $\bigcap_{n=1}^{\infty} \mathfrak{p}_i^n$, i.e., that R is the direct sum of local rings $R_i = R/(\bigcap_{n=1}^{\infty} \mathfrak{p}_i^n)$ $(i = 1, \ldots, h)$. Completeness of each R_i is evident.

4. Subdirect sums of n-rings

THEOREM 5. *Let a ring* R *be a subdirect sum of* n*-rings* $R_1, \ldots, R_n$. *Then*

(i) R *is an* n*-ring if (and only if)* $R^2 = R$, *and*

(ii) R^n *is an* n*-ring.*

Proof. Let $\mathfrak{n}_i$ be the kernel of natural homomorphism of R onto R_i (for

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8) This means topological completeness.

9) This shows that $\sum_{i=1}^{h} f_i(a) = a$ and that R is the direct sum of ideals $\bigcap_{n=1}^{\infty} \mathfrak{a}_i^{(n)}$ $(i = 1, \ldots, h)$.

each i).

(1) *Proof of* (i).

Let $\mathfrak{a}$ be an ideal in R such that there exists no maximal ideal containing $\mathfrak{a}$. Then $\mathfrak{a} + \mathfrak{n}_i = R$ for each i. Therefore $\mathfrak{a} + (\bigcap_{i=1}^{n} \mathfrak{n}_i) = R$, by Lemma 3, i.e., $\mathfrak{a} = R$.

(2) *Proof of* (ii).

It is clear that R^n is a subdirect sum of $R_1, \ldots, R_h$. Hence, it is sufficient to prove that $R^{n+1} = R^n$, by virtue of (i). Evidently $R^2 + \mathfrak{n}_i = R$ for each i. Therefore it is easy to see that $R^{n+1} + \mathfrak{n}_1 \mathfrak{n}_2 \ldots \mathfrak{n}_n \supseteqq R^n$, i.e., $R^{n+1} = R^n$.

Example. Let R be a ring such that $R^2 = (0)$ $(R \neq (0))$. Using the notation $(1, R)$[10] as in my paper "On the theory of radicals in a ring"[11] we construct a ring $S = R + (1, R)$ (direct sum). Let $\mathfrak{n}_1 = R$, $\mathfrak{n}_2 = \{a + (0, a);\ a \in R\}$. Then S is a subdirect sum of n-rings $S/\mathfrak{n}_1$ and $S/\mathfrak{n}_2$. On the other hand, S is not an n-ring because $S^2 = (1, R)$.

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[10] $(1, R)$ is a typical over-ring of a ring R which contains the identity and in which R is an ideal.

[11] To appear in J. Math. Soc. Jap.