CONSTRUCTION AND CHARACTERIZATION OF GALOIS ALGEBRAS WITH GIVEN GALOIS GROUP

TADASI NAKAYAMA

Recently H. Hasse[1] has given an interesting theory of Galois algebras, which generalizes the well known theory of Kummer fields; an algebra $\mathfrak{A}$ over a field Ω is called a Galois algebra with Galois group G when $\mathfrak{A}$ possesses G as a group of automorphisms and $\mathfrak{A}$ is (G, Ω)-operator-isomorphic to the group ring $G(\Omega)$ of G over Ω.[2] On assuming that the characteristic of Ω does not divide the order of G and that absolutely irreducible representations of G lie in Ω, Hasse constructs certain Ω-basis of $\mathfrak{A}$, called factor basis, in accord with Wedderburn decomposition of the group ring and shows that a characterization of $\mathfrak{A}$ is given by a certain matrix factor system which defines the multiplication between different parts of the factor basis belonging to different characters of G. Now the present work is to free the theory from the restriction on the characteristic. We can indeed embrace the case of non-semisimple modular group ring $G(\Omega)$.

1. Decomposition of group ring.[3] Let G be a finite group whose absolutely irreducible representations lie in a field Ω. Let $\mathfrak{G} = G(\Omega)$ be its group ring over Ω. Let

$$1 = \sum_{\kappa=1}^{k} \sum_{i=1}^{f(\kappa)} e_i^{(\kappa)} \tag{1}$$

be a decomposition of 1 into a sum of mutually orthogonal primitive idempotent elements in $\mathfrak{G}$, where the left-(or, right-)ideals generated by $e_1^{(\kappa)}, \ldots, e_{f(\kappa)}^{(\kappa)}$ are isomorphic while those generated by $e_i^{(\kappa)}$, $e_j^{(\lambda)}$ with $\kappa \neq \lambda$ are not. Let $c_{ij}^{(\kappa)}$ be, for each κ, a corresponding system of matric units. For simplicity's sake we denote $e_1^{(\kappa)}$ by $e^{(\kappa)}$. Let

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1) [2].

2) Hasse demands further that $\mathfrak{A}$ be associative, commutative and, moreover, semisimple.

3) Cf. e.g. [3].

$$\mathfrak{v}^{(\kappa)} = \begin{pmatrix} e^{(\kappa)} \\ t_2^{(\kappa)} \\ \vdots \\ t_{v(\kappa)}^{(\kappa)} \end{pmatrix} \tag{2}$$

be an (independent) Ω-basis of the right ideal $e^{(\kappa)}\mathfrak{G}$ taken in accord with a composition series. We have, for $z \in \mathfrak{G}$,

$$\mathfrak{v}^{(\kappa)} z = V^{(\kappa)}(z)\mathfrak{v}^{(\kappa)} \tag{3}$$

with a representation $V^{(\kappa)}$ of $\mathfrak{G}$ in Ω. We assume that $e^{(1)}$ corresponds to the 1-representation of G. We can, and shall, take $\sum_{z \in G} z$ for $t_{v(1)}^{(1)}$.

As for the left-ideal $\mathfrak{G}e^{(\kappa)}$ we take its basis

$$\mathfrak{u}^{(\kappa)} = (e^{(\kappa)}, s_2^{(\kappa)}, \ldots, s_{v(\kappa)}^{(\kappa)}) \tag{4}$$

in the following more specified manner. Let namely the q-th residue-module in a composition series of $\mathfrak{G}e^{(\kappa)}$ correspond to $e^{(\kappa_q(\kappa))}$ (i.e. be isomorphic to $\mathfrak{G}e^{(\kappa_q(\kappa))}/\mathfrak{N}e^{(\kappa_q(\kappa))}$, where $\mathfrak{N}$ denotes the radical of $\mathfrak{G}$), and take a generator $r_q^{(\kappa)} (\in \mathfrak{G})$ of the residue-module; $r_q^{(\kappa)}$ may be taken from $e^{(\kappa_q(\kappa))}\mathfrak{G}e^{(\kappa)}$, and we really employ $e^{(\kappa)}$ as $r_1^{(\kappa)}$. Then

$$((e^{(\kappa)}, c_{21}^{(\kappa)}, \ldots), \quad (e^{(\kappa_2(\kappa))}, c_{21}^{(\kappa_2(\kappa))}, \ldots) r_2^{(\kappa)}, \ldots\ldots) \tag{5}$$

forms a basis of $\mathfrak{G}e^{(\kappa)}$, which we take for $\mathfrak{u}^{(\kappa)}$ in (4).

Now we introduce a matrix

$$\mathfrak{T}^{(\kappa)} = (\mathfrak{v}^{(\kappa)}, s_2^{(\kappa)}\mathfrak{v}^{(\kappa)}, \ldots, s_{v(\kappa)}^{(\kappa)}\mathfrak{v}^{(\kappa)}) \tag{6}$$

in $\mathfrak{G}$; it is the transpose of the Kronecker product, so to speak, of the transposes of $\mathfrak{u}^{(\kappa)}$, $\mathfrak{v}^{(\kappa)}$. Denote the matrix consisting of the first $f(\kappa)$ columns of $\mathfrak{T}^{(\kappa)}$ by $\mathfrak{V}^{(\kappa)}$, i.e.

$$\mathfrak{V}^{(\kappa)} = (\mathfrak{v}^{(\kappa)}, c_{21}^{(\kappa)}\mathfrak{v}^{(\kappa)}, \ldots, c_{f(\kappa)1}^{(\kappa)}\mathfrak{v}^{(\kappa)}) . \tag{7}$$

We have

$$\mathfrak{T}^{(\kappa)} = S^{(\kappa)} \begin{pmatrix} \mathfrak{V}^{(\kappa)} & & \\ & \mathfrak{V}^{(\kappa_2(\kappa))} & \\ & & \ddots \end{pmatrix} \tag{8}$$

with a matrix $S^{(\kappa)}$ in Ω. Here

$$\begin{pmatrix} \mathfrak{V}^{(\kappa)} & & \\ & \mathfrak{V}^{(\kappa_2(\kappa))} & \\ & & \ddots \end{pmatrix} = K^{(\kappa)*} \begin{pmatrix} \mathfrak{V}^{(1)} & & \\ & \ddots & \\ & & \mathfrak{V}^{(k)} \end{pmatrix} K^{(\kappa)} \tag{9}$$

with matrices $K^{(\kappa)}$, $K^{(\kappa)*}$ possessing one 1 in each column or row respectively. Thus

$$\mathfrak{T}^{(\kappa)} = S^{(\kappa)} K^{(\kappa)*} \begin{pmatrix} \mathfrak{V}^{(1)} & & \\ & \ddots & \\ & & \mathfrak{V}^{(k)} \end{pmatrix} K^{(\kappa)}. \tag{10}$$

If $\mathfrak{x}$ is any column of elements of $\mathfrak{G}$ satisfying $\mathfrak{x}z = V^{(\kappa)}(z)\mathfrak{x}$, then there exists an element x in $\mathfrak{G}$ such that $\mathfrak{x} = x\mathfrak{v}^{(\kappa)}$. Hence

$$\mathfrak{x} = \mathfrak{T}^{(\kappa)} X$$

with a column X in Ω; in fact X is the first column of the matrix corresponding to x in the representation of $\mathfrak{G}$ defined by $\mathfrak{G}e^{(\kappa)}$ with respect to our basis $\mathfrak{u}^{(\kappa)}$.

Now, Kronecker products of $V^{(\kappa)}$ are decomposed, directly, into certain numbers of $V^{(\kappa)}$.[4] Thus

$$V^{(\kappa)} \times V^{(\lambda)} = P^{-1}_{\kappa,\lambda} \begin{pmatrix} V^{(\omega_1(\kappa,\lambda))} & & \\ & V^{(\omega_2(\kappa,\lambda))} & \\ & & \ddots \end{pmatrix} P_{\kappa,\lambda} \tag{11}$$

with a non-singular matrix $P_{\kappa,\lambda}$ in Ω. There is a matrix $G_{\kappa,\lambda}$ possessing one 1 in each column such that

$$\begin{gathered} \begin{pmatrix} V^{(\omega_1(\kappa,\lambda))} & & \\ & V^{(\omega_2(\kappa,\lambda))} & \\ & & \ddots \end{pmatrix} = G'_{\kappa,\lambda} \begin{pmatrix} V^{(1)} & & \\ & \ddots & \\ & & V^{(k)} \end{pmatrix} G_{\kappa,\lambda}, \\ \begin{pmatrix} \mathfrak{T}^{(\omega_1(\kappa,\lambda))} & & \\ & \mathfrak{T}^{(\omega_2(\kappa,\lambda))} & \\ & & \ddots \end{pmatrix} = G'_{\kappa,\lambda} \begin{pmatrix} \mathfrak{T}^{(1)} & & \\ & \ddots & \\ & & \mathfrak{T}^{(k)} \end{pmatrix} G_{\kappa,\lambda}. \end{gathered} \tag{12}$$

We have next

$$V^{(\lambda)} \times V^{(\kappa)} = J^{-1}_{\kappa,\lambda}(V^{(\kappa)} \times V^{(\lambda)}) J_{\kappa,\lambda} \tag{13}$$

with permutation matrix $J_{\kappa,\lambda}$. Further we may assume

$$\omega_i(\lambda, \kappa) = \omega_i(\kappa, \lambda), \quad G_{\lambda,\kappa} = G_{\kappa,\lambda} \text{ and } P_{\lambda,\kappa} = P_{\kappa,\lambda} J_{\kappa,\lambda}. \tag{14}$$

Finally we quote the following particular case of the Nesbitt-Brauer-Nakayama orthogonality relation[5]

$$\sum_{z \in G} V^{(1)}(z) = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \quad (\gamma \neq 0), \quad \sum_{z \in G} V^{(\kappa)}(z) = 0 \quad (\kappa \neq 1). \tag{15}$$

2. Galois algebra. Let $\mathfrak{A}$ be an algebra, not necessarily associative, over Ω

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4) See [5].

5) See [1], [4].

which has G as a group of automorphisms. We call $\mathfrak{A}$ a Galois algebra, with Galois group G, when the right $\mathfrak{G}(=G(\Omega))$-module $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$ itself (i.e. when $\mathfrak{A}$ possesses a normal basis). Let, with such a Galois algebra $\mathfrak{A}$, $\sim$ denote an isomorphism of $\mathfrak{G}$ and $\mathfrak{A}$. We have

$$\tilde{\mathfrak{V}}^{(\kappa)}z = V^{(\kappa)}(z)\tilde{\mathfrak{V}}^{(\kappa)} \tag{16}$$

for $z \in \mathfrak{G}$. So $(\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)})z = (V^{(\kappa)} \times V^{(\lambda)})(z)(\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)})$, or

$$P_{\kappa,\lambda}(\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)})z = \begin{pmatrix} V^{(\omega_1(\kappa,\lambda))}(z) & & \\ & V^{(\omega_2(\kappa,\lambda))}(z) & \\ & & \ddots \end{pmatrix} P_{\kappa,\lambda}(\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)}).$$

Hence, from an observation in 1 (and the isomorphism property of $\sim$),

$$P_{\kappa,\lambda}(\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)}) = \begin{pmatrix} \tilde{\mathfrak{X}}^{(\omega_1(\kappa,\lambda))} & & \\ & \tilde{\mathfrak{X}}^{(\omega_2(\kappa,\lambda))} & \\ & & \ddots \end{pmatrix} P_{\kappa,\lambda}A_{\kappa,\lambda} = G'_{\kappa,\lambda}\begin{pmatrix} \tilde{\mathfrak{X}}^{(1)} & & \\ & \ddots & \\ & & \tilde{\mathfrak{X}}^{(k)} \end{pmatrix} G_{\kappa,\lambda}P_{\kappa,\lambda}A_{\kappa,\lambda}$$

i.e.

$$\tilde{\mathfrak{V}}^{(\kappa)} \times \tilde{\mathfrak{V}}^{(\lambda)} = P_{\kappa,\lambda}G'_{\kappa,\lambda}\begin{pmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(k)} \end{pmatrix}\begin{pmatrix} K^{(1)*} & & \\ & \ddots & \\ & & K^{(k)*} \end{pmatrix}\left(I_k \times \begin{pmatrix} \tilde{\mathfrak{V}}^{(1)} & & \\ & \ddots & \\ & & \tilde{\mathfrak{V}}^{(k)} \end{pmatrix}\right) \tag{17}$$
$$\begin{pmatrix} K^{(1)} & & \\ & \ddots & \\ & & K^{(k)} \end{pmatrix} G_{\kappa,\lambda}P_{\kappa,\lambda}A_{\kappa,\lambda}$$

with uniquely determined matrix $A_{\kappa,\lambda}$ of type $(v(\kappa)v(\lambda), f(\kappa)f(\lambda))$ in Ω, I_k being unit matrix of degree k. Taking $A_{\kappa,\lambda}$ for each pair (κ, λ) we obtain a system $\{A_{\kappa,\lambda};\ \kappa, \lambda = 1, 2, \ldots, k\}$ of matrices in Ω.

Conversely any system $\{A_{\kappa,\lambda}\}$, with each $A_{\kappa,\lambda}$ possessing type $(v(\kappa)v(\lambda), f(\kappa)f(\lambda))$, defines a Galois algebra with Galois group G. Namely, if we introduce $g = \sum f(\kappa)v(\kappa)$ elements, arrange them into k matrices $\tilde{\mathfrak{V}}^{(\kappa)}$ of respective type $(v(\kappa), f(\lambda))$, define by virtue of (17) an Ω-linear multiplication in the Ω-module $\mathfrak{A}$ spanned by the elements, considered as being independent, and set (16), then we see that $\mathfrak{A}$ becomes a Galois algebra with Galois group G corresponding to the given system $\{A_{\kappa,\lambda}\}$.

Now, similar consideration can be made for $\tilde{\mathfrak{X}}^{(\kappa)} \times \tilde{\mathfrak{X}}^{(\lambda)}$ too, to give

$$\tilde{\mathfrak{X}}^{(\kappa)} \times \tilde{\mathfrak{X}}^{(\lambda)} = P_{\kappa,\lambda}G'_{\kappa,\lambda}\begin{pmatrix} \tilde{\mathfrak{X}}^{(1)} & & \\ & \ddots & \\ & & \tilde{\mathfrak{X}}^{(k)} \end{pmatrix} G_{\kappa,\lambda}P_{\kappa,\lambda}B_{\kappa,\lambda} \tag{18}$$

with again uniquely determined matrix $B_{\kappa,\lambda}$, of degree $v(\kappa)v(\lambda)$; $A_{\kappa,\lambda}$ is composed of certain $f(\kappa)f(\lambda)$ columns of $B_{\kappa,\lambda}$. Also the system $\{B_{\kappa,\lambda}\}$ characterizes $\mathfrak{A}$, but it must be observed that it can not, in general, be taken arbitrarily, contrary to $\{A_{\kappa,\lambda}\}$. Indeed, elements of $B_{\kappa,\lambda}$ can be expressed linearly by those of A's, the expression depending on G (and Ω) only (but not on $\mathfrak{A}$), which we write in

$$B_{\kappa,\lambda} = B_{\kappa,\lambda}(\{A_{\kappa,\lambda}\}). \tag{19}$$

Let, with our same $\mathfrak{A}$, a second ($\mathfrak{G}$-) isomorphism of $\mathfrak{G}$ and $\mathfrak{A}$ be denoted by $\overline{}$. There exists a regular element a in $\mathfrak{G}$ such that $\bar{x} = \widetilde{ax}$ $(x \in \mathfrak{G})$. We see that

$$P_{\kappa,\lambda}^{-1}G'_{\kappa,\lambda}\begin{pmatrix} U^{(1)}(a)^{-1} & & \\ & \ddots & \\ & & U^{(k)}(a)^{-1}\end{pmatrix} G_{\kappa,\lambda}P_{\kappa,\lambda}B_{\kappa,\lambda}(U^{(\kappa)}(a) \times U^{(\lambda)}(a)) \tag{20}$$

plays for $\overline{\mathfrak{T}}^{(\kappa)}$ the roll of $B_{\kappa,\lambda}$ for $\tilde{\mathfrak{T}}^{(\kappa)}$, where $U^{(\kappa)}$ denotes the representation defined by the basis $u^{(\kappa)}$ of $\mathfrak{G}e^{(\kappa)}$. Thus:

(Under the assumption that all absolutely irreducible representations of G lie in Ω) *to each Galois algebra* $\mathfrak{A}$ *over* Ω *with Galois group* G *is associated a class of systems* $\{A_{\kappa,\lambda}\}$ *mutually related by transformation which takes* $A_{\kappa,\lambda}$ *into the matrix consisting of the* 1*st*, ..., $f(\lambda)$-*th*, $v(\lambda)+1$*st*, ..., $v(\lambda)+f(\lambda)$-*th*, ... *columns of* (20) *with* $B_{\kappa,\lambda} = B_{\kappa,\lambda}(\{A_{\kappa,\lambda}\})$, *a being regular element in* $\mathfrak{G}$. *Multiplication in* $\mathfrak{A}$ *is given in terms of* $\{A_{\kappa,\lambda}\}$ *by* (17), *and operation of* G *on* $\mathfrak{A}$ *by* (16). *Conversely, any system* $\{A_{\kappa,\lambda}\}$ *of matrices in* Ω, *with respective type* $(v(\kappa)v(\lambda), f(\kappa)f(\lambda))$, *gives rise to a Galois algebra over* Ω *with Galois group* G.

3. **Associativity, commutativity and semisimplicity.** If $\mathfrak{A}$ is commutative, then the permutation matrix $J_{\kappa,\lambda}$ in (13) gives also $J_{\kappa,\lambda}^{-1}(\tilde{\mathfrak{T}}^{(\kappa)} \times \tilde{\mathfrak{T}}^{(\lambda)})J_{\kappa,\lambda} = \tilde{\mathfrak{T}}^{(\lambda)} \times \tilde{\mathfrak{T}}^{(\kappa)}$. We have then

$$B_{\lambda,\kappa} = J_{\kappa,\lambda}^{-1}B_{\kappa,\lambda}J_{\kappa,\lambda}, \tag{21}$$

which gives in fact necessary and sufficient condition for the commutativity of $\mathfrak{A}$; if we take (19) (and perhaps its trivial inverse) into account, the condition can be regarded as being in terms of $\{A_{\kappa,\lambda}\}$.

On considering $V^{(\kappa)} \times V^{(\lambda)} \times V^{(\mu)}$, let next $H_{\kappa,\lambda,\mu}$ and $L_{\kappa,\lambda,\mu}$ be permutation matrices satisfying

$$H_{\kappa,\lambda,\mu}^{-1}\begin{pmatrix} V^{(\kappa)} \times V^{(\omega_1(\lambda,\mu))} & & \\ & \ddots & \\ & & \cdot \end{pmatrix} H_{\kappa,\lambda,\mu} = V^{(\kappa)} \times \begin{pmatrix} V^{(\omega_1(\lambda,\mu))} & & \\ & \ddots & \\ & & \cdot \end{pmatrix}, \tag{22}$$

$$L_{\kappa,\lambda,\mu}^{-1}\begin{pmatrix} V^{(\omega_1(\omega_1(\kappa,\lambda),\mu))} & & \\ & \ddots & \\ & & \cdot \end{pmatrix} L_{\kappa,\lambda,\mu} = \begin{pmatrix} V^{(\omega_1(\kappa,\omega_1(\lambda,\mu)))} & & \\ & \ddots & \\ & & \cdot \end{pmatrix}. \tag{23}$$

Then

$$(24)\quad L_{\kappa,\lambda,\mu}\begin{pmatrix} P_{\kappa,\omega_1(\lambda,\mu)} & & \\ & \cdot & \\ & & \cdot \end{pmatrix} H_{\kappa,\lambda,\mu}(I_{v(\kappa)} \times P_{\lambda,\mu}) = Q_{\kappa,\lambda,\mu}\begin{pmatrix} P_{\omega_1(\kappa,\lambda),\mu} & & \\ & \cdot & \\ & & \cdot \end{pmatrix}(P_{\kappa,\lambda} \times I_{v(\mu)})$$

with a non-singular matrix $Q_{\kappa,\lambda,\mu}$ commutative with the representation $\begin{pmatrix} V^{(\omega_1(\omega_1(\kappa,\lambda),\mu))} & & \\ & \cdot & \\ & & \cdot \end{pmatrix}$. There is matrix $R_{\kappa,\lambda,\mu}$ such as

$$(25)\qquad Q^{-1}_{\kappa,\lambda,\mu}\begin{pmatrix} \mathfrak{T}^{(\omega_1(\omega_1(\kappa,\lambda)\mu)} & & \\ & \cdot & \\ & & \cdot \end{pmatrix} = \begin{pmatrix} \mathfrak{T}^{(\omega_1(\omega_1(\kappa,\lambda)\mu)} & & \\ & \cdot & \\ & & \cdot \end{pmatrix} R_{\kappa,\lambda,\mu}.$$

These matrices $H_{\kappa,\lambda,\mu}$, $L_{\kappa,\lambda,\mu}$, $Q_{\kappa,\lambda,\mu}$ and $R_{\kappa,\lambda,\mu}$ are all determined by G (and Ω) only. Calculating $(\tilde{\mathfrak{T}}^{(\kappa)} \times \tilde{\mathfrak{T}}^{(\lambda)}) \times \tilde{\mathfrak{T}}^{(\mu)}$ and $\tilde{\mathfrak{T}}^{(\kappa)} \times (\tilde{\mathfrak{T}}^{(\lambda)} \times \tilde{\mathfrak{T}}^{(\mu)})$, we find that

$$(26)\qquad \begin{pmatrix} P^{-1}_{\omega_1(\kappa,\lambda),\mu} & & \\ & \cdot & \\ & & \cdot \end{pmatrix}\begin{pmatrix} B_{\omega_1(\kappa,\lambda),\mu} & & \\ & \cdot & \\ & & \cdot \end{pmatrix}(P_{\kappa,\lambda} \times I_{v(\mu)})(B_{\kappa,\lambda} \times I_{v(\mu)})$$

$$= R_{\kappa,\lambda,\mu}L_{\kappa,\lambda,\mu}\begin{pmatrix} P_{\kappa,\omega_1(\lambda,\mu)} & & \\ & \cdot & \\ & & \cdot \end{pmatrix}\begin{pmatrix} B_{\kappa,\omega_1(\lambda,\mu)} & & \\ & \cdot & \\ & & \cdot \end{pmatrix} H_{\kappa,\lambda,\mu}(I_{v(\kappa)} \times P_{\lambda,\mu})(I_{v(\kappa)} \times B_{\lambda,\mu})$$

is necessary and sufficient for the associativity of $\mathfrak{A}$; the condition may be seen again as being in terms of $\{A_{\kappa,\lambda}\}$.

Finally, since the system $\{A_{\kappa,\lambda}\}$ gives, by (17), the multiplication table of $\mathfrak{A}$, the regular discriminant of $\mathfrak{A}$ can be expressed by means of $\{A_{\kappa,\lambda}\}$. Provided $\mathfrak{A}$ is associative (that is, (26) holds) its non-vanishing is necessary and sufficient in order that $\mathfrak{A}$ be absolutely semisimple and its capacities be all indivisible by the characteristic of Ω. However, on assuming both the commutativity and associativity we can obtain a second expression for the discriminat (whose non-vanishing is now necessary and sufficient for the absolute semisimplicity of $\mathfrak{A}$) as follows. Namely, the trace of an element of $\mathfrak{A}$ may then be given as the sum of its transforms by G. So the matrix composed of the traces of elements of $\tilde{\mathfrak{B}}^{(\kappa)} \times \tilde{\mathfrak{B}}^{(\lambda)}$ is given by

$$(27)\qquad \sum_{z\in G}(\tilde{\mathfrak{B}}^{(\kappa)} \times \tilde{\mathfrak{B}}^{(\lambda)})z = \sum_{z}(P^{-1}_{\kappa,\lambda}\begin{pmatrix} \tilde{\mathfrak{T}}^{(\omega_1(\kappa,\lambda))} & & \\ & \cdot & \\ & & \cdot \end{pmatrix} P_{\kappa,\lambda}A_{\kappa,\lambda})z.$$

Making use of the orthogonality relation (15) we find that this is equal to

$$(28)\quad P^{-1}_{\kappa,\lambda}G'_{\kappa,\lambda}\begin{pmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(k)} \end{pmatrix}\begin{pmatrix} K^{(1)*} & & \\ & \ddots & \\ & & K^{(k)*} \end{pmatrix}(I_k \times \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix})\begin{pmatrix} K^{(1)} & & \\ & \ddots & \\ & & K^{(k)} \end{pmatrix} G_{\kappa,\lambda}P_{\kappa,\lambda}A_{\kappa,\lambda}$$

multiplied by γ and the trace of $\tilde{1}$ (which is not 0). Now the part of our discriminant matrix corresponding to the products of the basis elements from $\mathfrak{B}^{(\kappa)}$ and $\mathfrak{B}^{(\lambda)}$ can be obtained from (28) by virtue of a certain, easily describable rearrangement of elements (except a non-zero scalar factor independent of κ, λ).

These together offer criterion for the *associativity, commutativity and absolute semisimplicity* of the Galois algebra $\mathfrak{A}$ given by $\{A_{\kappa,\lambda}\}$.

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Nagoya University