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REAL CANONICAL CYCLE AND ASYMPTOTICS OF OSCILLATING INTEGRALS

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Abstract. Let $X_{\mathbb{R}} \subset \mathbb{R}^N$ a real analytic set such that its complexification $X_{\mathbb{C}} \subset \mathbb{C}^N$ is normal with an isolated singularity at 0. Let $f_{\mathbb{R}} : X_{\mathbb{R}} \to \mathbb{R}$ a real analytic function such that its complexification $f_{\mathbb{C}} : X_{\mathbb{C}} \to \mathbb{C}$ has an isolated singularity at 0 in $X_{\mathbb{C}}$. Assuming an orientation given on $X^*_{\mathbb{R}}$, to a connected component A of $X^*_{\mathbb{R}}$ we associate a compact cycle $\Gamma(A)$ in the Milnor fiber of $f_{\mathbb{C}}$ which determines completely the poles of the meromorphic extension of $\int_A f^\lambda \square$ or equivalently the asymptotics when $\tau \to \pm\infty$ of the oscillating integrals $\int_A e^{i\tau f}\square$. A topological construction of $\Gamma(A)$ is given. This completes the results of [BM] paragraph 6.

§0. Introduction

Let $X_{\mathbb{C}}$ be a normal complex space of dimension $n+1$ $(n \geq 1)$ having an isolated singularity at 0, and let $f : X_{\mathbb{C}} \to \mathbb{C}$ be an holomorphic function on $X_{\mathbb{C}}$ with an isolated singularity at 0. We shall assume that $(X_{\mathbb{C}}, f)$ is the complexification of a real analytic function $(X_{\mathbb{R}}, f_{\mathbb{R}})$ on a real analytic space $X_{\mathbb{R}}$. In such a situation, we shall consider A, non zero, in $H^0(X^*_{\mathbb{R}}, \mathbb{C})$. Assuming that an orientation is given on the smooth real manifolds $X^*_{\mathbb{R}}$, we have defined in [BM] a compactly supported cohomology class $\gamma(A) \in H^n_c(F, \mathbb{C})_1$ associated to A, where F denotes the complex Milnor's fiber of f on $X_{\mathbb{C}}$ and $H^n_c(F, \mathbb{C})_1$ the spectral part for the eigenvalue 1 of the monodromy acting on $H^n_c(F, \mathbb{C})$. The definition is the following:

For any $e \in H^n(F, \mathbb{C})_1$ represented by semi-meromorphic forms on $X_{\mathbb{C}}$, with poles along $f = 0, w_0, \dots, w_k$, so satisfying the conditions

$$(A) \begin{cases} 1) \ \ dw_j = \dfrac{df}{f} \wedge w_{j-1} & \forall j \in [1, k],\ w_0 = 0 \\ 2) \ \ [w_k/_F] = e \in H^n(F, \mathbb{C})_1 \end{cases}$$

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we have

$$I\big(e, \overline{\gamma(A)}\big) = (2i\pi)^{-n} \operatorname{Res}\Big(\lambda = 0, \int_A f^\lambda \rho w \wedge \frac{df}{f}\Big).$$

Here $I : H^n_c(F, \mathbb{C}) \times H^n(F, \mathbb{C}) \to \mathbb{C}$ denotes the hermitian Poincaré duality defined by $I(a, b) = \frac{1}{(2i\pi)^n} \int_F a \wedge \bar{b}$ and ρ is in $C^\infty_c(X_\mathbb{R})$ with $\rho \equiv 1$ near 0. We use here the notation $i\pi \int_A f^\lambda \frac{df}{f} \wedge \square$ for the $\mathbb{R}^*$-Mellin transform of the function defined on $\mathbb{R}^*$ by $\varphi(s) = \int_{f(s)\cap A} \square$ where $\square$ is a semi-meromorphic n-form on $X_\mathbb{R}$ with compact support$^{(*)}$ and poles in $\{f_\mathbb{R} = 0\}$.

Recall that the $\mathbb{R}^*$-Mellin transform of φ is given (see [B99]) by definition by

$$i\pi M\varphi(\lambda) := \int_0^{+\infty} x^{\lambda-1}\varphi(x)\, dx - e^{-i\pi\lambda} \int_0^{+\infty} x^{\lambda-1}\varphi(-x)\, dx.$$

The purpose of this article is to give a topological construction of a compact n-cycle whose class in $H^n_c(F, \mathbb{C})_1$ is $\gamma(A)$. This complete the results of the paragraph 6 in [BM]. In fact it appears from our proof and [BM] results that the class of our cycle $\Gamma(A)$ in $H^n_c(F, \mathbb{C})$ controls completely the poles of $\int_A f^\lambda \square$ for the given A. So the same holds for the asymptotics when $\tau \to \pm\infty$ of the oscillating integrals $\int_A e^{i\tau f} \square$ where $\square$ denotes a C^∞-compactly supported $(n+1)$-form on $X_\mathbb{R}$.

So to prove existence of a pole in our context it is enough (but also necessary) to prove that the class of $\Gamma(A)$ in $H_n(F, \mathbb{C}) \simeq H^n_c(F, \mathbb{C})$ is not zero. This gives some new light on Jeddi's proof of Palamodov's conjecture (see [J]).

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§1. Some more notations

We continue with the hypothesis and notations introduced in paragraph 0 (still assuming that an orientation is given on $X^*_\mathbb{R}$).

We shall fix a Milnor representative of f, denoted by $f : X_\mathbb{C} \to D_\delta$ by choosing a real embedding $X_\mathbb{R} \hookrightarrow \mathbb{R}^N$ (so $X_\mathbb{C} \hookrightarrow \mathbb{C}^N$ and $X_\mathbb{C} \cap \mathbb{R}^N = X_\mathbb{R}$) and choosing $0 < \varepsilon \ll 1$ and $0 < \delta \ll \varepsilon$ such that $X_\mathbb{C} := B(0, \varepsilon) \cap f^{-1}(D_\delta)$; we fix a base point $s_0 \in D_\delta \cap \mathbb{R}^{+*}$ and define the Milnor fiber of f to be $f^{-1}(s_0) = F$.

$^{(*)}$ Remark that f-proper support is enough to define the polar parts of $\int_A f^\lambda \frac{df}{f} \wedge \square$.

Recall now that, given A in $H^0(X_{\mathbb{R}} - \{f_{\mathbb{R}} = 0\}, \mathbb{C})$ (we shall use here the obvious restriction map from $H^0(X^*_{\mathbb{R}}, \mathbb{C})$ to $H^0(X_{\mathbb{R}} - f_{\mathbb{R}}^{-1}(0), \mathbb{C})$), we have defined in [BM] the closed n cycles $\delta(A)^+ := A \cap f_{\mathbb{R}}^{-1}(s_0)$, oriented as the boundary of the (oriented) open set $A^+ \cap \{f_{\mathbb{R}} < s_0\}$ and $\delta(A)^- := A \cap f_{\mathbb{R}}^{-1}(-s_0)$ oriented as the boundary of the open set $A^- \cap \{-s_0 < f_{\mathbb{R}}\}$, where $A = A^+ + A^-$ is the decomposition of the sum $A = \sum_\alpha a_\alpha A_\alpha$ according to the sign of $f_{\mathbb{R}}$ on each connected component A_α of $X_{\mathbb{R}} - f_{\mathbb{R}}^{-1}(0)$.

Now, using a C^∞ trivialization of Milnor's fibration along the half-circle $\{s_0 e^{i\theta},\ \theta \in [-\pi, 0]\}$ we define $M^{1/2}\delta(A)^-$ as the closed oriented n-cycle in F obtained from $\delta(A)^-$ by direct image along the projection on F given by this trivialisation.

Then we set $\delta(A) := \delta(A)^+ - M^{1/2}\delta(A)^-$ and denote by $[\delta(A)]$ the classe defined by $\delta(A)$ in $H^n(F, \mathbb{C})$. To be precise, the class $[\delta(A)]$ is defined via the hermitian duality I by the formula

$$I\big([a], \overline{\delta(A)}\big) := \tfrac{1}{(2i\pi)^n} \int_{\delta(A)} a$$

where a is a compactly supported closed C^∞ n-form on F defining the class $[a]$ in $H^n_c(F, \mathbb{C})$.

§2. Construction of $\Gamma(A)$ in $H_n(F, \mathbb{C}) \simeq H^n_c(F, \mathbb{C})$

We fix $\varepsilon' < \varepsilon'' < \varepsilon$ with $\varepsilon - \varepsilon' \ll \varepsilon$ and define

$$\begin{aligned} X &:= B(0, \varepsilon) \cap f^{-1}(D_\delta) \\ X' &:= B(0, \varepsilon') \cap f^{-1}(D_\delta) \\ X'' &:= B(0, \varepsilon'') \cap f^{-1}(D_\delta) \end{aligned}$$

and then $\partial A := A \cap \partial X''$ for our given non zero A in $H^n(X^*_{\mathbb{R}}, \mathbb{C})$. The orientation of this compact n-cycle in $X^*_{\mathbb{R}}$ is given as the boundary of the open set $A \cap X''$. As a compact n-chain ∂A has three pieces:

$$\partial A = \big(\delta(A^-) \cap \overline{X}''\big) \cup \big(\delta(A)^+ \cap \overline{X}''\big) \cup \Delta$$

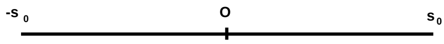
where the two "vertical" pieces $\delta(A)^\pm \cap \overline{X}''$ are obtained by cutting $\delta(A)^\pm$ by $\overline{B(0, \varepsilon'')}$, and where the compact n-chain Δ lies in $X - \overline{X}'$ is fibered by f over $[-s_0, s_0]$ as a family of compact $(n-1)$-cycles which gives an homology in $X - \overline{X}'$ between $\delta(A)^- \cap \partial B(0, \varepsilon'')$ and $\delta(A)^+ \cap \partial B(0, \varepsilon'')$.

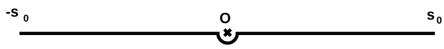
The proof of our theorem will follow precisely a move from this compact n-cycle ∂A to a compact n-cycle $\Gamma(A)$ contained in F. To move ∂A to $\Gamma(A)$, first fix a C^∞ trivialisation of Milnor's fibration over the punctured half disc $\overline{D}^-_{s_0} - \{0\} = \{s \in \mathbb{C} / \operatorname{Im} s \leq 0,\ |s| \leq s_0,\ s \neq 0\}$ which induces the previously fixed trivialisation on the half-circle $\{s_0 e^{i\theta},\ \theta \in [-\pi, 0]\}$ used to define $M^{1/2}\delta(A)^-$. We shall also fix a C^∞ trivialisation of $f | X - \overline{X \cap B(0, \varepsilon')} \to D_\delta$ which corresponds to a commutative diagram

$$\begin{array}{ccc} X - \overline{X \cap B(0,\varepsilon')} & \longrightarrow & D_\delta \times F' \\ \Big\downarrow f & & \Big\downarrow P_1 \\ D_\delta & = & D_\delta \end{array}$$

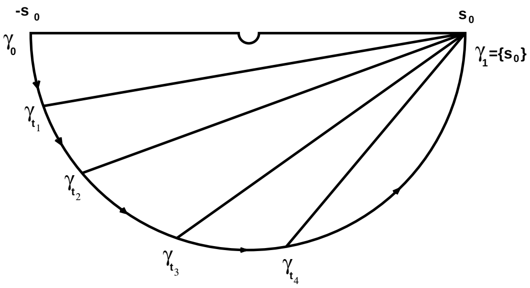
where $F' := F \cap \big(X - \overline{X \cap B(0, \varepsilon')}\big)$, also compatible with the previous trivialisation.

First we begin by moving Δ to Δ_0 using the above trivialisation (recall that $\Delta \subset X - \overline{X \cap B(0, \varepsilon')}$) without moving its boundary part, so that we get

Δ : 

Δ_0: 

Then we move the compact n-cycle $\partial A_0 := \partial A - \Delta + \Delta_0 \subset f^{-1}\big(\overline{D}^-_{s_0} - \{0\}\big)$ using the above trivialisation of f over this set, so that the vertical part $\delta(A)^- \cap \overline{X}''$ will follow the half-circle $\{s_0 e^{i\theta},\ \theta \in [-\pi, 0]\}$, the vertical part $\delta(A)^+ \cap \overline{X}''$ will be fixed, and the Δ_0 part moves, using the trivialisation of f on $X - \overline{X \cap B(0, \varepsilon')}$ from the path γ_0 to the constant path γ_1 equal to $\{s_0\}$ as follows



Let us call $(\Delta_t)_{t\in[0,1]}$ this deformation. We shall denote by $(\partial A_t)_{t\in[0,1]}$ the family of compact n-cycles in $f^{-1}\big(\overline{D}^-_{s_0} - \{0\}\big)$ defined for $t \in [0,1]$ by

$$(\partial A)_t := -\widetilde{\delta}(A)^-_{s_0 e^{-i\pi(1-t)}} + \widetilde{\delta}(A)^+ + \Delta_t$$

where $\widetilde{\delta}(A)^\pm$ is $\delta(A)^\pm \cap \overline{X}''$ and where $\widetilde{\delta}(A)^-_{s_0 e^{i\theta}}$ is obtained by following the compact $\widetilde{\delta}(A)^-$ in the above trivialisation along the half-circle.

So we define now the compact oriented n-cycle

$$\Gamma(A) := (\partial A)_1 \subset F.$$

By definition we have

$$\Gamma(A) = \widetilde{\delta}(A)^+ - M^{1/2}\widetilde{\delta}(A)^- + \Delta_1$$

where Δ_1 is a compact n-chain in F' so that $\partial\Gamma(A) = \emptyset$.

Remark that this already shows that we have

$$\operatorname{can}[\Gamma(A)] = [\delta(A)] \quad \text{in } H^n(F,\mathbb{C})$$

because our initial chain Δ was the boundary in $X - X \cap \overline{B(0,\varepsilon')}$ of the closed $(n+1)$-chain

$$\big(A - A \cap B(0,\varepsilon'')\big) \cap f^{-1}(\overline{D}_{s_0})$$

(and Δ_0 similarily).

As we know that $\partial\big(\widetilde{\delta}(A)^+\big)$ and $\partial\big(\widetilde{\delta}(A)^-\big)$ are homologuous in $X - \overline{X \cap B(0,\varepsilon')}$ as $(n-1)$-compact cycles, for any choice of a compact n-chain Δ_2 in F' such that $\widetilde{\delta}(A)^+ - M^{1/2}\widetilde{\delta}(A)^- + \Delta_2$ is a compact n-cycle in F ($M^{1/2}$ preserves the homology between boundaries), we obtain a compact n-cycle in F whose image by

$$\operatorname{can} : H_n(F,\mathbb{C}) \simeq H^n_c(F,\mathbb{C}) \longrightarrow H^n(F,\mathbb{C})$$

is the class $[\delta(A)]$. But the choice of Δ_2 is defined up to a compact n-cycle of F'. As $H_n(F') \simeq H_n(\partial F) \simeq H^{n-1}(\partial F)$ is exactly the kernel of can (via the exact sequence $0 \to \operatorname{Ker}\operatorname{can} \simeq H^{n-1}(\partial F) \to H^n_c(F) \xrightarrow{\text{can}} H^n(F)$) to make this construction is just to lift $[\delta(A)]$ to $H^n_c(A)$, and this is possible by [BM].

What we have done in the construction of $\Gamma(A)$ is to make a precise choice of $\Delta_2 \subset F'$ by using the component A again.

The following theorem shows that our choice is the good one.

THEOREM. *The cycle $\Gamma(A)$ constructed above satifies the following property*:

For any $e \in H^n(F,\mathbb{C})_1$ we have

$$I\big(e, \overline{\Gamma(A)}\big) = (2i\pi)^{-n} \operatorname{Res}\Big(\lambda = 0, \int_A f^\lambda w_k \wedge \frac{df}{f}\Big)$$

where $w_1, \ldots, w_k$ are semi-meromorphic n-forms representing e (i.e. satisfying the condition (A) of paragraph 0).

Moreover we have $\operatorname{can}([\Gamma(A)]) = [\delta(A)]$ so, using [BM], we deduce that $[\Gamma(A)]$ satisfies also:

For any $e \in H^n(F,\mathbb{C})_1$ represented by $w_1, \ldots, w_k$

$$h\big(e,\ \operatorname{can}(\overline{\Gamma(A)}_1)\big) = (2i\pi)^n P_2\Big(\lambda = 0, \int_A f^\lambda w_k \frac{df}{f}\Big)$$

where $P_2(\lambda = 0, F(\lambda))$ is the coefficient of $1/\lambda^2$ of the Laurent expansion of the meromorphic function f at 0, and where

$$h : H^n(F,\mathbb{C})_1 \times H^n(F,\mathbb{C})_1 \longrightarrow \mathbb{C}$$

is the canonical hermitian form defined in [BM] in our context.

For any $e \in H^n(F,\mathbb{C})_{e^{-2i\pi r}}$, $0 < r < 1$, represented by $w_1, \ldots, w_k$(†) we have

$$I\big(e, \overline{\Gamma(A)}\big) = \frac{s_0^r}{(2i\pi)^n} \operatorname{Res}\Big(\lambda = -r, \int_A f^\lambda w_k \wedge \frac{df}{f}\Big).$$

As an easy consequence we obtain the following corollary, which completes results of [BM] paragraph 6.

COROLLARY.

1) *we have* $[\Gamma(A)]_1 = \gamma(A)$

2) $\int_A f^\lambda \square$ *has no poles iff* $[\Gamma(A)] = 0$ *in* $H_n(F,\mathbb{C})$.

Proof of the theorem. In order to follow easily the moving cycle $(\partial A)_t$ and integral on it, it is convenient to introduce a d-closed n-form W associated to $e \in H^n(F,\mathbb{C})_1$. Let us fix the logarithm function on $D_\delta - D_\delta \cap i\mathbb{R}^+$ such that the argument is in $]-3\pi/2,\ \pi/2[$. Now define

$$W := \sum_{j=0}^{k-1} (-1)^j w_j \frac{(\operatorname{Log} f)^j}{j!}$$

(†) in this case we have replaced (A) of paragraph 0 by $dw_j = r\frac{df}{f} \wedge w_j + \frac{df}{f} \wedge w_{j-1}$ $\forall j \in [1,k]$ $(w_0 = 0)$ and $[w_k/_F] = e$

on the open set $f^{-1}(D_\delta - D_\delta \cap i\mathbb{R}^+)$ of X, where $w_1, \dots, w_k$ are semi-meromorphic n-forms on X satisfying (A) of paragraph 0.

Then we have $dW = 0$ and also

$$[W|_F] = \sum_{k-1}^{j=0} (-1)^j \frac{(\operatorname{Log} s_0)^j}{j!} e_{k-j}$$

where $e_{k-j} := [w_j|_F]$ for $j \in [0, k-1]$ in $H^n(F, \mathbb{C})$. So $e_0 = e$.

So we have

$$\int_{(\partial A)_0} W = \int_{\Gamma(A)} W = \sum_{j=0}^{k-1} (-1)^j \frac{(\operatorname{Log} s_0)^j}{j!} (2i\pi)^n I\big(e_{k-j}, \overline{\Gamma(A)}\big).$$

Now we have to go from $(\partial A)_0$ to ∂A.

We define $\int_{\partial A} W$ as follows:

$$\int_{\partial A} W := Pf\Big(\lambda = 0, \int_{\partial A} f^\lambda W\Big).$$

And we shall precise later on why $\int_{\partial A} W = \int_{(\partial A)_0} W$. But thanks to Stokes formula (for $\operatorname{Re}\lambda \gg 0$) and analytic continuation we get

$$\int_{\partial A} W = \operatorname{Res}\Big(\lambda = 0, \int_{A \cap \overline{X}''} f^\lambda \frac{df}{f} \wedge W\Big).$$

So, modulo our Lemma 2 which will allow us to replace the integration on $A \cap \overline{X}''$ by a smooth cut off function $\rho \in C_c^\infty(X_\mathbb{R})$, $\rho \equiv 1$ near $\overline{X}''$, without changing the polar parts, we obtain, by the definition of $\gamma(A) \in H_c^n(F, \mathbb{C})_1$ recalled in paragraph 0

$$\sum_{j=0}^{k-1} (-1)^j \frac{(\operatorname{Log} s_0)^j}{j!} I\big(e_{k-j}, \overline{\gamma(A)}\big) = \sum_{j=0}^{k-1} (-1)^j \frac{(\operatorname{Log} s_0)^j}{j!} I\big(e_{k-j}, \overline{\Gamma(A)}\big).$$

Now we can conclude easily because we already know from [BM] that $\operatorname{can}[\gamma(A)] = \operatorname{can}[\Gamma(A)]_1$. So for $e_1, \dots, e_{k-1}$ which are in $\operatorname{Im}\operatorname{can}$ (because $\operatorname{Im}(T-1) \subset \operatorname{Im}\operatorname{can}$) we know that $I\big(e_{k-j}, \overline{\gamma(A)}\big) = I\big(e_{k-j}, \overline{\Gamma(A)}\big)$ for $j \in [1, k-1]$. So we conclude that

$$I\big(e_k, \overline{\gamma(A)}\big) = I\big(e_k, \overline{\Gamma(A)}\big)$$

and we obtain $[\gamma(A)] = [\Gamma(A)]_1$ in $H_n(F, \mathbb{C})_1$.

To pass from Δ to Δ_0 we first remark that now we are considering a compact n-chain (with fixed boundary) in $X - \overline{X \cap B(0,\varepsilon')}$ where no singularity occurs for $X_{\mathbb{R}}$ or $f_{\mathbb{R}}$. So locally we can assume that $A = \mathbb{R}^{n+1}$ and $f = x_0$ is the first coordinnate. Now let us define a push down of W on $\mathbb{C}$: via our fixed C^∞ trivialisation of $X - \overline{X \cap B(0,\varepsilon')} \to D_\delta$ we can consider Δ near $f_{\mathbb{R}}^{-1}(0)$ as a family $(\delta_t)_{t\in[-\eta,\eta]}$ of compact $(n-1)$-cycles in F' which are smooth. Let us then consider the submanifold ∇ defined near δ_0 as the union of all $(t + i\tau,\ \delta_t)$ for $\tau \in [-\xi,\xi]$ and t near 0. So, in fact, we just translate Δ near 0 along the imaginary axis in our trivialisation compatible with f.

Now ∇ is a piece of smooth $(n+1)$-submanifold containing Δ and with a proper smooth fibration $f|_\nabla : \nabla \to \mathbb{C}$ near 0 in $\mathbb{C}$.

Define $\alpha := (f|\nabla)_*(W|\nabla)$. Then α is a semi-meromorphic n-form near 0 in $\mathbb{C}$ which is d-closed because W is semi-meromorphic and d-closed. Now the following lemma with allow us to pass from $\int_{\partial A} W$ to $\int_{(\partial A)_0} W$:

Lemma 1. *Let $\eta > 0$ and denote by α a d-closed semi-meromorphic* 1-*form* (*with pole at $s = 0$*) *in a neighbourhood of $\overline{D(0,\eta)}$ in $\mathbb{C}$.*

Then we have

$$Pf\Big(\int_0^\eta s^\lambda i_+^*(\alpha) - e^{-i\pi\lambda}\int_0^\eta s^\lambda i_-^*(\alpha)\Big) = \int_{-\pi}^0 j^*(\alpha)$$

where $i_+ : [0,\eta] \to \overline{D(0,\eta)}$ and $i_- : [-\eta,0] \to \overline{D(0,\eta)}$ are the obvious inclusion and where j is given by $j : [-\pi,0] \to \overline{D(0,\eta)}$, $j(\theta) = \eta e^{i\theta}$.

Proof. After reduction to the case $\alpha = ds/s^k$ this is an elementary exercice. ∎

To finish the proof of our theorem, it is enough to prove our second lemma:

Lemma 2. *Let $\rho \in C_c^\infty(X_{\mathbb{R}})$ with $\rho \equiv 1$ near $\overline{X}''$ and let w be a semi-meromorphic n-form on $X - \overline{X \cap B(0,\varepsilon')}$ with poles in $f = 0$.*

Then for any $k \in \mathbb{N}$ the meromorphic function

$$\lambda \longrightarrow \int_{(X-\overline{X}'')\cap A} f^\lambda (\operatorname{Log} f)^k \rho \frac{df}{f} \wedge w$$

has no pole.

Proof. Of course we have here our previous choice of logarithm. Our assertion is local on $(X-\overline{X}'')\cap\operatorname{Supp}\rho$ so we can assume again that $A=\mathbb{R}^{n+1}$ and that $f_{\mathbb{R}}=x_0$ is the first coordinnate.

Far from $\{f_{\mathbb{R}}=0\}$ there is nothing to prove (and this is the case along the vertical boundary parts of $\overline{X}''$ for instance, where $f_{\mathbb{R}}=\pm s_0$).

Far from $\partial B(0,\varepsilon')$ (i.e. far from Δ) we are reduced to the case of

$$(*)\quad \int_0^{+\infty} x_0^{\lambda-j}(\operatorname{Log} x_0)^k\sigma(x_0)\frac{dx_0}{x_0} - e^{-i\pi\lambda}\int_0^{+\infty}(-1)^j x^{\lambda-j}(\operatorname{Log} x_0 - i\pi)^k\sigma(-x_0)\frac{dx_0}{x_0}$$

where $\sigma\in C_c^\infty(\mathbb{R})$ is obtained by integrating first in $x_1,\dots,x_n$ (x_0^{-j} comes from the poles of the semi-meromorphic form ρw).

Near $\partial B(0,\varepsilon'')$ we are reduced to the same situation but $\sigma\in\mathbb{C}_c^\infty(\mathbb{R})$ is now obtained by integration of w along $x_1\geq 0$, $x_2,\dots,x_n$ where we assume the coordinnates chosen in such a way that X'' is locally defined by $x_1<0$.

To treat $(*)$, use a Taylor expansion of σ at $x_0=0$ to reduce to the case of

$$(**)\qquad \int_0^{\eta} x^{\lambda-j}(\operatorname{Log} x)^k\frac{dx}{x} - e^{-i\pi\lambda}\int_0^{\eta}(-1)^j x^{\lambda-j}(\operatorname{Log} x - i\pi)^k\frac{dx}{x}$$

which is given, thanks to Cauchy's theorem, by the integral over the half circle $\{z=\eta e^{i\theta},\ \theta\in[-\pi,0]\}$

$$\int z^{\lambda-j}(\operatorname{Log} z)^k\frac{dz}{z}$$

But this is clearly an entire function of λ. □

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