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ON THE COHOMOLOGICAL COMPLETENESS OF q-COMPLETE DOMAINS WITH CORNERS

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Abstract. We prove the vanishing and non-vanishing theorems for an intersection of a finite number of q-complete domains in a complex manifold of dimension n. When q does not divide n, it is stronger than the result naturally obtained by combining the approximation theorem of Diederich-Fornaess for q-convex functions with corners and the vanishing theorem of Andreotti-Grauert for q-complete domains. We also give an example which implies our result is best possible.

Introduction

Let D be a complex manifold of dimension n and let q be an integer with $1 \leq q \leq n$. A continuous function from D to $\mathbb{R}$ is called q-convex with corners if it is locally a maximum of a finite number of q-convex functions. In [D-F] Diederich-Fornaess proved that every q-convex function with corners defined on D can be approximated by $\widetilde{q}$-convex functions whole on D, where $\widetilde{q} := n - [n/q] + 1$ and $[x]$ denotes the integral part of x. They moreover showed that the number $\widetilde{q}$ is best possible for any (n, q), i.e., there exist an open subset D in $\mathbb{C}^n$ and a finite number of q-convex functions $\varphi_1, \dots, \varphi_s$ defined on D such that the function $\varphi := \max\{\varphi_1, \dots, \varphi_s\}$ cannot be approximated by $(\widetilde{q} - 1)$-convex functions.

A complex manifold D is called q-complete (resp. q-complete with corners) if D has an exhaustion function which is q-convex (resp. q-convex with corners) on D. Combining the above theorem of Diederich-Fornaess with the theorem of Andreotti-Grauert ([A-G]) it follows at once that if D is q-complete with corners then D is cohomologically $\widetilde{q}$-complete.

Now the following problem arises naturally.

Problem. *Is there a complex manifold which is q-complete with corners but not cohomologically $(\widetilde{q} - 1)$-complete?*

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It is easy to find such examples if q divides n (cf. [S-V], [E-S], [M-1] and [M-2]). However, it seems that such an example is still unknown if q does not divide n.

The purpose of this article is to prove the following.

THEOREM. *Let M be a complex manifold of dimension n and let $D_1, \dots, D_t$ be q-complete open subsets in M. Let $\mathcal{F}$ be a coherent analytic sheaf on M such that $H^n(M, \mathcal{F}) = 0$. Then*

$$H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) = 0 \qquad \text{if} \quad j \geq \widehat{q}_t.$$

Here

$$\widehat{q}_t := \min\{\widehat{q}, t(q-1)+1\}$$

and

$$\widehat{q} := n - \left[\frac{n-1}{q}\right] = \begin{cases} \widetilde{q} & \text{if} \quad q \mid n \\ \widetilde{q} - 1 & \text{if} \quad q \nmid n. \end{cases}$$

Moreover, the number $\widehat{q}_t$ in the above theorem is best possible for any (n, q, t). In particular, for any (n, q) there exist a finite number of q-complete open subsets $D_1, \dots, D_s$ in $\mathbb{C}^n$ such that $H^{\widehat{q}-1}(D_1 \cap \cdots \cap D_s, \mathcal{O}) \neq 0$, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on $\mathbb{C}^n$ (see §3).

The author, in general, does not know whether the cohomologically $\widehat{q}$-complete set $D_1 \cap \cdots \cap D_t$ in the above theorem is $\widehat{q}$-complete, i.e., it has a $\widehat{q}$-convex exhaustion function, even in the case $M = \mathbb{C}^n$.

§1. The key proposition

First we show the following proposition which is a key step to prove Theorem.

PROPOSITION 1. *Let M be a topological space, let $\{D_1, \dots, D_t\}$ be a family of open subsets in M and let $\mathcal{S}$ be a sheaf of Abelian groups on M. Let $p \in \mathbb{N}$ be fixed and suppose that for any k with $1 \leq k \leq t-1$ the family $\{D_1, \dots, D_t\}$ satisfies the condition*

$$C(k,p) \qquad H^j(D_{i_1} \cap \cdots \cap D_{i_k}, \mathcal{S}) = 0 \quad \text{for all } j \geq p \text{ and all } i_1, \dots, i_k \in \{1, 2, \dots, t\}.$$

Then

$$(1) \quad H^j(D_1 \cap \cdots \cap D_t, \mathcal{S}) \cong H^{j+t-1}(D_1 \cup \cdots \cup D_t, \mathcal{S}) \qquad \text{if} \quad j \geq p;$$

$$(2) \quad H^{p-1}(D_1 \cap \cdots \cap D_t, \mathcal{S}) \twoheadrightarrow H^{p+t-2}(D_1 \cup \cdots \cup D_t, \mathcal{S})$$

Remark. The family $\{D_1, \ldots, D_t\}$ satisfies the condition $C(k, p)$ for all k with $1 \leq k \leq t-1$ if it satisfies only $C(t-1, p)$.

Proof of Proposition 1. We shall prove the proposition by induction on $t \in \mathbb{N}$.

Step 1. When $t = 1$, (1) and (2) are trivial.

Step 2. When $t \geq 2$, it follows by Mayer-Vietories that the sequence

$$\begin{aligned} H^j(D_1 \cap \cdots \cap D_{t-1}, \mathcal{S}) &\oplus H^j(D_t, \mathcal{S}) \\ &\longrightarrow H^j((D_1 \cap \cdots \cap D_{t-1}) \cap D_t, \mathcal{S}) \\ &\longrightarrow H^{j+1}((D_1 \cap \cdots \cap D_{t-1}) \cup D_t, \mathcal{S}) \\ &\longrightarrow H^{j+1}(D_1 \cap \cdots \cap D_{t-1}, \mathcal{S}) \oplus H^{j+1}(D_t, \mathcal{S}) \end{aligned}$$

is exact for each j. Since $\{D_1, \ldots, D_t\}$ satisfies $C(t-1, p)$ and $C(1, p)$ by assumption, we have

$$H^j(D_1 \cap \cdots \cap D_{t-1}, \mathcal{S}) = H^j(D_t, \mathcal{S}) = 0 \qquad \text{if} \quad j \geq p.$$

Therefore, if we put $E_i := D_i \cup D_t$ for $i = 1, 2, \ldots, t-1$, then

(3) $\quad H^j(D_1 \cap \cdots \cap D_t, \mathcal{S}) \cong H^{j+1}(E_1 \cap \cdots \cap E_{t-1}, \mathcal{S}) \qquad \text{if} \quad j \geq p;$

(4) $\quad H^{p-1}(D_1 \cap \cdots \cap D_t, \mathcal{S}) \twoheadrightarrow H^p(E_1 \cap \cdots \cap E_{t-1}, \mathcal{S}).$

In particular, this means that the proposition holds in the case $t = 2$.

Step 3. When $t \geq 3$, we assume that the proposition has been proved for $1, 2, \ldots, t-1$. We first show the following.

LEMMA 1. *Under the above situation, the family* $\{E_1, \ldots, E_{t-1}\}$ *satisfies the condition* $C(t-2, p+1)$.

Proof. We shall prove by induction that for any l with $1 \leq l \leq t-2$ the family $\{E_1, \ldots, E_{t-1}\}$ satisfies the condition

$C(l, p+1)$
$$H^{j+1}(E_{i_1} \cap \cdots \cap E_{i_l}, \mathcal{S}) = 0 \quad \text{for all } j \geq p \text{ and all } i_1, \ldots, i_l \in \{1, 2, \ldots, t-1\}.$$

By the assumption of the proposition $\{D_1, \dots, D_t\}$ satisfies $C(t-1,p)$ and particularly $C(1,p)$ and $C(2,p)$. Since the proposition holds in the case $t=2$ we have

$$\begin{aligned} H^{j+1}(E_{i_1}, \mathcal{S}) &= H^{j+1}(D_{i_1} \cup D_t, \mathcal{S}) \\ &\cong H^j(D_{i_1} \cap D_t, \mathcal{S}) = 0 \qquad \text{if} \quad j \geq p. \end{aligned}$$

Therefore, $\{E_1, \dots, E_{t-1}\}$ satisfies $C(1, p+1)$.

Next let $2 \leq l \leq t-2$ and assume that the lemma has been proved for all m with $1 \leq m \leq l-1$. Then the family $\{E_{i_1}, \dots, E_{i_l}\}$, where $i_1, \dots, i_l \in \{1, 2, \dots, t-1\}$, also satisfies the condition $C(m, p+1)$ for all m with $1 \leq m \leq l-1$. Moreover, since $\{E_1, \dots, E_{t-1}\}$ satisfies $C(l-1, p+1)$ by the inductive hypothesis and since the proposition holds for l,

$$\begin{aligned} &H^{j+1}(E_{i_1} \cap \cdots \cap E_{i_l}, \mathcal{S}) \\ &\qquad \cong H^{(j+1)+l-1}(E_{i_1} \cup \cdots \cup E_{i_l}, \mathcal{S}) \\ &\qquad = H^{j+(l+1)-1}(D_{i_1} \cup \cdots \cup D_{i_l} \cup D_t, \mathcal{S}) \qquad \text{if} \quad j \geq p. \end{aligned}$$

On the other hand, $\{D_1, \dots, D_t\}$ satisfies $C(l,p)$ and $C(l+1,p)$ because $l+1 \leq t-1$. Since the proposition holds for $l+1$,

$$\begin{aligned} &H^{j+(l+1)-1}(D_{i_1} \cup \cdots \cup D_{i_l} \cup D_t, \mathcal{S}) \\ &\qquad \cong H^j(D_{i_1} \cap \cdots \cap D_{i_l} \cap D_t, \mathcal{S}) = 0 \qquad \text{if} \quad j \geq p. \end{aligned}$$

Hence we obtain

$$H^{j+1}(E_{i_1} \cap \cdots \cap E_{i_l}, \mathcal{S}) = 0 \qquad \text{if} \quad j \geq p,$$

which proves that $\{E_1, \dots, E_{t-1}\}$ satisfies $C(l, p+1)$ for all l with $1 \leq l \leq t-2$. ∎

End of Proof of Proposition 1. If $t \geq 3$ and if $\{D_1, \dots, D_t\}$ satisfies $C(t-1,p)$ then $\{E_1, \dots, E_{t-1}\}$ satisfies $C(t-2, p+1)$, where $E_i := D_i \cup D_t$ for $i = 1, 2, \dots, t-1$. Therefore, by the inductive hypothesis, we have

$$H^{j+1}(E_1 \cap \cdots \cap E_{t-1}, \mathcal{S}) \cong H^{j+t-1}(E_1 \cup \cdots \cup E_{t-1}, \mathcal{S}) \quad \text{if} \quad j \geq p; \tag{5}$$

$$H^p(E_1 \cap \cdots \cap E_{t-1}, \mathcal{S}) \twoheadrightarrow H^{p+t-2}(E_1 \cup \cdots \cup E_{t-1}, \mathcal{S}). \tag{6}$$

Notice here that $E_1 \cup \cdots \cup E_{t-1} = D_1 \cup \cdots \cup D_t$. Then we can obtain (1) and (2) by (3), (4), (5) and (6).

This completes the proof of the proposition. ∎

§2. Proof of Theorem

Let M be a complex manifold of dimension n, let $D_1, \dots, D_t$ be q-complete open subsets in M and let $\mathcal{F}$ be a coherent analytic sheaf on M such that $H^n(M, \mathcal{F}) = 0$.

Since the intersection $D_1 \cap \cdots \cap D_t$ is q-complete with corners it follows from the theorem of Diederich-Fornaess and the theorem of Andreotti-Grauert that

$$H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) = 0 \qquad \text{if} \quad j \geq \widetilde{q}_t.$$

Here $\widetilde{q}_t := \min\{\widetilde{q}, t(q-1)+1\}$ and $\widetilde{q} := n - [n/q] + 1$.

We put

$$\widehat{q} := n - \left[\frac{n-1}{q}\right] = \begin{cases} \widetilde{q} & \text{if} \quad q \mid n \\ \widetilde{q} - 1 & \text{if} \quad q \nmid n. \end{cases}$$

For the proof of Theorem it is enough to prove the following.

LEMMA 2. *Under the above situation,*

$$H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) = 0 \qquad \textit{if} \quad j \geq \widehat{q}.$$

Proof. We put $m := [n/q]$ and $r := n - mq$. Then $n = mq + r$ and $0 \leq r \leq q - 1$. We shall prove the lemma by induction on $t \in \mathbb{N}$.

First if $t \leq m$,

$$t(q-1) + 1 \leq m(q-1) + 1 = n - m + 1 - r = \widetilde{q} - r.$$

If $q \mid n$ or $r = 0$ then $\widetilde{q} - r = \widetilde{q} = \widehat{q}$; and if $q \nmid n$ or $r \geq 1$ then $\widetilde{q} - r \leq \widetilde{q} - 1 = \widehat{q}$. Hence if $t \leq m$ we have $t(q-1) + 1 \leq \widehat{q} \leq \widetilde{q}$ and

$$\widetilde{q}_t := \min\{\widetilde{q}, t(q-1)+1\} = t(q-1) + 1 \leq \widehat{q}.$$

Therefore, by the theorem of Diederich-Fornaess, the lemma holds if $t \leq m$.

Next if $t \geq m + 1$ and if the lemma holds for $1, 2, \dots, t-1$, then for any k with $1 \leq k \leq t - 1$ the family $\{D_1, \dots, D_t\}$ satisfies the condition

$$C(k, \widehat{q}) \qquad H^j(D_{i_1} \cap \cdots \cap D_{i_k}, \mathcal{F}) = 0 \quad \text{for all } j \geq \widehat{q} \text{ and all } i_1, \dots, i_k \in \{1, 2, \dots, t\}.$$

Hence by Proposition 1

$$H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) \cong H^{j+t-1}(D_1 \cup \cdots \cup D_t, \mathcal{F}) \qquad \text{if} \quad j \geq \widehat{q}.$$

Notice here that if $t \geq m+1$ and $j \geq \widehat{q}$ then $j+t-1 \geq \widehat{q}+m \geq \widetilde{q}-1+m = n$.

Since the set $D_1 \cup \cdots \cup D_t$ is open in M and since $H^n(M, \mathcal{F}) = 0$ by assumption we have $H^n(D_1 \cup \cdots \cup D_t, \mathcal{F}) = 0$ (see Remark below). Therefore we obtain

$$H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) = 0 \qquad \text{if} \quad j \geq \widehat{q},$$

which proves the lemma. □

Theorem is the direct result of the above lemma and the theorem of Diederich-Fornaess (cf. [D-F], §5).

Remark. By the theorem of Greene-Wu ([G-W]), a connected complex manifold of dimension n is n-complete if and only if it is noncompact. Therefore, if D is noncompact complex manifold of dimension n then by the theorem of Andreotti-Grauert $H^n(D, \mathcal{F}) = 0$ for any coherent analytic sheaf $\mathcal{F}$ on D. It is obvious that if $H^n(M, \mathcal{F}) = 0$ then $H^n(D, \mathcal{F}) = 0$ for any connected (and not necessarily noncompact) component D of M.

§3. Example

As in Section 2 we put $n = mq + r$. In $\mathbb{C}^n$, consider the complex linear subspaces defined by

$$L_i := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{(i-1)q+1} = \ldots = z_{iq} = 0\}$$

and put $D_i := \mathbb{C}^n \setminus L_i$ for $i = 1, 2, \ldots, m$. Then each D_i is q-complete but not $(q-1)$-complete (cf. [W]). If $q \nmid n$ or $r \geq 1$, we moreover put

$$L_{m+1} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{mq+1} = \ldots = z_n = 0\}$$

and $D_{m+1} := \mathbb{C}^n \setminus L_{m+1}$. Then D_{m+1} is r-complete and particularly q-complete because $r < q$.

The number $\widehat{q}_t$ in Theorem is best possible for any (n, q, t), where

$$\widehat{q}_t := \min\{\widehat{q}, t(q-1)+1\} = \begin{cases} t(q-1)+1 & \text{if} \quad t \leq m \\ \widehat{q} & \text{if} \quad t > m \end{cases}$$

and

$$\widehat{q} := n - \left[\frac{n-1}{q}\right] = \begin{cases} n-m+1 & \text{if} \quad q \mid n \\ n-m & \text{if} \quad q \nmid n. \end{cases}$$

In fact, we have the following.

EXAMPLE. *Under the above notations,* $H^{t(q-1)}(D_1 \cap \cdots \cap D_t, \mathcal{O}) \neq 0$ *for* $t = 1, 2, \ldots, m$. *Moreover,* $H^{n-m-1}(D_1 \cap \cdots \cap D_{m+1}, \mathcal{O}) \neq 0$ *if* $q \nmid n$.

In the example above, $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. The example is a part of the following.

PROPOSITION 2. *Let* $\alpha_0, \alpha_1, \ldots, \alpha_t$ *and* n_0 *be integers such that* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_t = n_0 \leq n$. *In* $\mathbb{C}^n$, *consider the complex linear subspaces defined by*

$$L_i := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{\alpha_{i-1}+1} = z_{\alpha_{i-1}+2} = \cdots = z_{\alpha_i} = 0\}$$

and put $D_i := \mathbb{C}^n \setminus L_i$ *for* $i = 1, 2, \ldots, t$. *Then*

$$\begin{cases} H^{n_0-t}(D_1 \cap \cdots \cap D_t, \mathcal{O}) \neq 0 \\ H^j(D_1 \cap \cdots \cap D_t, \mathcal{O}) = 0 \qquad \text{if} \quad j \geq n_0 - t + 1. \end{cases}$$

Proof. Since $\operatorname{codim} L_i \leq n_0 - (t-1)$ each D_i is at least $(n_0 - t + 1)$-complete. Hence if we put $p := n_0 - t + 1$ then $H^j(D_i, \mathcal{O}) = 0$ for all $j \geq p$ and all i with $1 \leq i \leq t$.

We shall now prove by induction that for any k with $1 \leq k \leq t-1$ the family $\{D_1, \ldots, D_t\}$ satisfies the condition

$$C(k,p) \qquad H^j(D_{i_1} \cap \cdots \cap D_{i_k}, \mathcal{O}) = 0 \quad \text{for all } j \geq p \text{ and all } i_1, \ldots, i_k \in \{1, 2, \ldots, t\}.$$

First $\{D_1, \ldots, D_t\}$ satisfies $C(1,p)$. Next if it satisfies $C(k-1,p)$ where $k \geq 2$, it follows from Proposition 1 that

$$H^j(D_{i_1} \cap \cdots \cap D_{i_k}, \mathcal{O}) \cong H^{j+k-1}(D_{i_1} \cup \cdots \cup D_{i_k}, \mathcal{O}) \qquad \text{if} \quad j \geq p.$$

Since $D_{i_1} \cup \cdots \cup D_{i_k} = \mathbb{C}^n \setminus (L_{i_1} \cap \cdots \cap L_{i_k})$ and since $\operatorname{codim}(L_{i_1} \cap \cdots \cap L_{i_k}) \leq n_0 - (t-k) = p+k-1$, the set $D_{i_1} \cup \cdots \cup D_{i_k}$ is at least $(p+k-1)$-complete. Hence for any k with $1 \leq k \leq t-1$ we have

$$H^j(D_{i_1} \cap \cdots \cap D_{i_k}, \mathcal{O}) = 0 \qquad \text{if} \quad j \geq p,$$

which implies that $\{D_1, \ldots, D_t\}$ satisfies $C(t-1,p)$.

Therefore, by Proposition 1 we obtain

$$(7) \quad H^j(D_1 \cap \cdots \cap D_t, \mathcal{O}) \cong H^{j+t-1}(D_1 \cup \cdots \cup D_t, \mathcal{O}) \qquad \text{if} \quad j \geq p;$$

$$(8) \quad H^{p-1}(D_1 \cap \cdots \cap D_t, \mathcal{O}) \twoheadrightarrow H^{p+t-2}(D_1 \cup \cdots \cup D_t, \mathcal{O}).$$

On the other hand,

$$\begin{cases} H^{n_0-1}(D_1 \cup \cdots \cup D_t, \mathcal{O}) \neq 0 \\ H^j(D_1 \cup \cdots \cup D_t, \mathcal{O}) = 0 \qquad \text{if} \quad j \geq n_0 \end{cases}$$

because $D_1 \cup \cdots \cup D_t = \mathbb{C}^n \setminus (L_1 \cap \cdots \cap L_t)$ and $\operatorname{codim}(L_1 \cap \cdots \cap L_t) = n_0$. Since $p := n_0 - t + 1$ we thus obtain

$$\begin{cases} H^{n_0-t}(D_1 \cap \cdots \cap D_t, \mathcal{O}) \neq 0 \\ H^j(D_1 \cap \cdots \cap D_t, \mathcal{O}) = 0 \qquad \text{if} \quad j \geq n_0 - t + 1. \end{cases}$$

This completes the proof of the proposition. □

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