

ZERO VARIETIES FOR THE NEVANLINNA CLASS ON ALL CONVEX DOMAINS OF FINITE TYPE

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Abstract. It is shown, that the so-called Blaschke condition characterizes in any bounded smooth convex domain of finite type exactly the divisors which are zero sets of functions of the Nevanlinna class on the domain. The main tool is a non-isotropic L^1 estimate for solutions of the Cauchy-Riemann equations on such domains, which are obtained by estimating suitable kernels of Berndtsson-Andersson type.

§1. Introduction and results

In their article [2] J. Bruna, Ph. Charpentier and Y. Dupain did a large step towards a complete characterization of the zero sets of the Nevanlinna class on all convex domains of finite type in \mathbb{C}^n . The purpose of this article is to complete their result.

For the convenience of the reader we recall at first briefly the following definitions and facts:

If $D \subset\subset \mathbb{C}^n$ is a C^∞ -smoothly bounded domain and ρ a smooth defining function for D , then the Nevanlinna class $N(D)$ is

$$N(D) = \left\{ f \in \mathcal{O}(D) : \sup_{\varepsilon} \int_{\partial D_\varepsilon} |\log |f(z)|| d\sigma_\varepsilon(z) < +\infty \right\}.$$

Here $\partial D_\varepsilon = \{z : \rho(z) = -\varepsilon\}$ for $\varepsilon > 0$ small enough and $d\sigma_\varepsilon(z)$ denotes the Euclidean surface measure on ∂D_ε . Furthermore, if $X \subset D$ is a complex analytic hypersurface with irreducible decomposition

$$X = \bigcup_k X_k$$

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and $n_k \in \mathbb{N}$ are positive integers for each k , the divisor $\hat{X} := \{X_k, n_k\}$ is said to satisfy the Blaschke condition if

$$(1.1) \quad \sum_k n_k \int_{X_k} |\rho(z)| d\mu_{X_k}(z) < \infty.$$

It is a well-known fact, that for any function $f \in N(D)$ the zero divisor X_f of f satisfies the Blaschke condition (1.1). And it is conjectured, that the inverse also is always true.

For a survey of the partial positive answers to this conjecture known so far see [2]. The basic work for $n > 1$ was done independently by G. Henkin in [7] and H. Skoda in [9] for the case of strictly pseudoconvex domains in \mathbb{C}^n .

In [2] the authors consider the class of bounded convex smooth domains $D \subset\subset \mathbb{C}^n$ of finite type with the goal of showing, that condition (1.1) characterizes precisely the zero divisors of the functions in $N(D)$. A large part of their work goes, in fact, through for all such domains. However, in order to accomplish the final analytically most critical step of their method of proof (which, actually, is the complete analogue of the method introduced by H. Skoda), they need the following additional condition on the domains D :

DEFINITION 1.1. A smooth convex domain $D \subset\subset \mathbb{C}^n$ is called of strict type if there is a constant $c = c(D)$ such that for all $\zeta \in \partial D$ and all $v \in T_\zeta^{10} \partial D$ with $|v| = 1$ and for all small t

$$c^{-1} \rho(\zeta + tv) \leq \rho(\zeta + tiv) \leq c \rho(\zeta + tv).$$

If D is in addition of finite type $\leq m$, it is called of finite strict type $\leq m$.

Altogether, J. Bruna, Ph. Charpentier and Y. Dupain showed in [2], that any divisor \hat{X} on a bounded smooth convex domain D of finite strict type in \mathbb{C}^n satisfying the Blaschke condition comes from a function $f \in N(D)$. The crucial analytic step, where the strict type condition is needed in [2] is the construction of $\bar{\partial}$ solutions with L^1 estimates on the boundary (see Theorem 1.3 below).

The analysis on domains D of finite strict type is much easier than on other convex domains of finite type, since for such D the family $T_\zeta^{10} \partial D =$

$\{w \in \mathbb{C}^n : \langle \partial\rho(\zeta), w - \zeta \rangle = 0\}$ for $\zeta \in \partial D$ is a perfect family of holomorphic support functions which can be used in the construction of $\bar{\partial}$ -solving integral kernels. The difference between smooth convex domains which are of finite strict type and others which are just of finite type becomes clear, if one compares among the standard model domains the so-called complex pseudoellipsoids

$$D := \left\{ z : \sum_{k=1}^n |z_k|^{2m_k} < 1 \right\}$$

with positive integers m_k one of which being > 1 , with the real ellipsoids

$$\hat{D} := \left\{ z : \sum_{k=1}^n (x_k^{2m_k} + y_k^{2q_k}) < 1 \right\}$$

where, of course, $z_k = x_k + iy_k$, m_k, q_k are again positive integers, at least one $m_k > 1$ and the corresponding $q_k > m_k$. The domains D are of finite strict type, whereas \hat{D} are only of finite type, but not of strict type. Concerning the question, how to do analysis on convex domains of finite type which are not of strict type, $\bar{\partial}$ -solving integral kernels for real pseudoellipsoids were constructed and estimated in [6]. In continuation of this work, smooth families of holomorphic support functions with best possible non-isotropic estimates were constructed for all convex domains of finite type in [5]. In [4] they were used for constructing $\bar{\partial}$ -solving integral kernels with best possible Hölder estimates. Here we will now use them in order to show essentially by Skoda’s method the following

THEOREM 1.2. *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type. Then there is a constant $C > 0$, such that there is for any smooth $(0, 1)$ -form ω on \bar{D} with $\bar{\partial}\omega = 0$ a smooth function u on \bar{D} such that $\bar{\partial}u = \omega$ and*

$$(1.2) \quad \int_{\partial D} |u(z)| d\sigma(z) \leq C \int_D \|\omega(z)\|_k dV(z).$$

Here the anisotropic norm $\|\cdot\|_k$ for $(0, 1)$ -forms is defined by

$$\|\omega(z)\|_k = \sup \left\{ \frac{|\omega(z)(v)|}{k(z, v)} : v \in \mathbb{C}^n \setminus \{0\} \right\}$$

where $k(z, v)$ is the usual weighted boundary distance of z in D in the direction v (for details see for instance [2]).

It was already shown in [2], that the following theorem is a consequence of this:

THEOREM 1.3. *Let D be as in Theorem 1.2. Then there is for every divisors $\hat{X} = \{X_k, n_k\}$ in D satisfying the Blaschke condition (1.1) a function $f \in N(D)$, such that \hat{X} is the zero divisor of f .*

Remark 1.4. a) The article [3] of Anne Cumenge also contains this theorem. She uses a different approach.

b) This article appeared as a preprint on November 26, 1998, and was distributed as such among many colleagues.

Large parts of the following sections containing the proof of Theorem 1.2 are based on the work done in [2]. However, our situation is more complicated, such that at some stages a different approach has to be used. We will almost always use the same notations as in [2].

§2. Integral kernels revisited

For proving Theorem 1.2 we want to use an integral kernel of the type H. Skoda. In order to construct it we use the methods of [1]. For them a good \mathcal{C}^∞ -family $S(z, \zeta)$ of support functions on D , holomorphic in $z \in \bar{D}$ and \mathcal{C}^∞ in ζ chosen in a suitable neighborhood of ∂D together with the corresponding Leray section $s(z, \zeta)$ is needed. We obtain it from the family $S(z, \zeta)$ constructed in [5]. The necessary estimates for S and s are contained in [4].

More precisely, Proposition 4.1 from [4] says:

PROPOSITION 2.1. *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type m and ρ a convex defining function of D in a neighborhood U of ∂D . Then the function $S(z, \zeta) \in \mathcal{C}^\infty(\bar{D} \times U)$, holomorphic in z , constructed in [5], has the following property:*

Let, for $\zeta \in U$, n_ζ denote the outer unit normal to the level set $\{\rho = \rho(\zeta)\}$ and let v be any unit vector complex tangential to this level set at ζ . Define

$$a_{\alpha\beta}(\zeta, v) := \frac{\partial^{\alpha+\beta}}{\partial \lambda^\alpha \partial \bar{\lambda}^\beta} \rho(\zeta + \lambda v)|_{\lambda=0}.$$

Then there are constants $K, c, d > 0$, such that one has for all points z written as $z = \zeta + \mu n_\zeta + \lambda v$ with $\mu, \lambda \in \mathbb{C}$ the estimate

$$(2.1) \quad \operatorname{Re} S(z, \zeta) \leq - \left| \frac{\operatorname{Re} \mu}{2} \right| - \frac{K}{2} (\operatorname{Im} \mu)^2 - c \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)| |\lambda|^j + d \sup \{0, \rho(z) - \rho(\zeta)\}.$$

Remark 2.2. In order to avoid that the constant d appears again and again in the estimates of this article, we replace from now on the original $S(z, \zeta)$ by $\frac{1}{2d} S(z, \zeta)$ calling this new function again $S(z, \zeta)$.

In (5) and (6) of [4] \mathcal{C}^∞ functions $Q_j(z, \zeta)$, $j = 1, \dots, n$, holomorphic in z , were defined, such that

$$(2.2) \quad S(z, \zeta) = \sum_{j=1}^n Q_j(z, \zeta)(z_j - \zeta_j)$$

for $(z, \zeta) \in \overline{D} \times U$.

Let us now choose a number $\eta_0 > 0$ such that $U_0 := \{w : |\rho(w)| < \eta_0\} \subset U$ and a \mathcal{C}^∞ cut-off function χ with $\chi(w) = 0$ for $\rho(w) \geq -\frac{\eta_0}{2}$ and $\chi(w) = 1$ for $\rho(w) \leq -\eta_0$. We put for $(\zeta, z) \in \overline{D} \times \overline{D}$

$$(2.3) \quad s(\zeta, z) := -\rho(z) \sum_{i=1}^n (\overline{\zeta}_i - \overline{z}_i) d\zeta_i + (1 - \chi(z)) \overline{S(\zeta, z)} \sum_{i=1}^n Q_i(\zeta, z) d\zeta_i$$

and

$$(2.4) \quad \mathcal{Q}(\zeta, z) := \frac{1}{\rho(\zeta)} \left[(1 - \chi(\zeta)) \sum_{i=1}^n Q_i(z, \zeta) d\zeta_i + \chi(\zeta) \sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i}(\zeta) d\zeta_i \right].$$

For any compact subset $A \subset D$ we obviously can find a constant $K > 0$ such that

$$(2.5) \quad |\langle s(\zeta, z), \zeta - z \rangle| \geq K \|\zeta - z\|^2$$

for all $\zeta \in \overline{D}$ and $z \in A$. Furthermore, $s(\zeta, z)$ is holomorphic in z for $\zeta \in D$.

Following the method of Berndtsson-Andersson [1], we define the kernel

$$(2.6) \quad K(\zeta, z) := c_n \sum_{k=0}^{n-1} c_k \frac{s \wedge (\bar{\partial}_\zeta s)^{n-1-k} \wedge (\bar{\partial}_\zeta \mathcal{Q})^k}{\langle s(\zeta, z), \zeta - z \rangle^{n-k}} \times \frac{(\rho(\zeta))^{k+1}}{[\rho(\zeta) + (1 - \chi(\zeta))S(z, \zeta) + \chi(\zeta) \sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i}(\zeta)(z_i - \zeta_i)]^{k+1}}.$$

Observe, that because of Proposition 2.1 and the classical estimate for the last sum in the denominator, the expression $[\dots]$ in the denominator does not have zeros for $\zeta \in \bar{D}$ and $z \in D$. Therefore, and because of (2.5), the theorem of Berndtsson-Andersson says

PROPOSITION 2.3. *Let ω be a $\bar{\partial}$ -closed $(0, 1)$ -form smooth on \bar{D} , and put*

$$(2.7) \quad u(z) := \int_D \omega(\zeta) K(\zeta, z)$$

for $z \in D$. Then $\bar{\partial}u = \omega$ on D .

§3. Estimation of the solution to the $\bar{\partial}$ -equation

We would like to know at first, that for ω as in Proposition 2.3 the function u from (2.7) is continuous up to ∂D and its boundary values are given by the formula

$$(3.1) \quad u(z) = c_n \int_D \omega(\zeta) K(\zeta, z).$$

According to the theorem of Skoda it suffices to prove for this that $K(\zeta, z)$ is uniformly integrable with respect to ζ for $z \in \partial D$.

However, for these z we have

$$(3.2) \quad s(\zeta, z) = \overline{S(\zeta, z)} \sum_{i=1}^n Q_i(\zeta, z) d\zeta_i$$

and $Q_i(\zeta, z)$ is holomorphic in ζ . Hence

$$s \wedge \bar{\partial}_\zeta s = 0$$

such that it suffices to use $k = n - 1$ in the definition of $K(\zeta, z)$ and we obtain for $z \in \partial D$

$$(3.3) \quad K(\zeta, z) = c_n \frac{\sum_{i=1}^n Q_i(\zeta, z) d\zeta_i \wedge (\bar{\partial}_\zeta Q)^{n-1} [\rho(\zeta)]^n}{S(\zeta, z) \left[\rho(\zeta) + (1 - \chi(\zeta))S(z, \zeta) + \chi(\zeta) \sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) \right]^n}.$$

Observe, that for $z \in \partial D$ according to Proposition 2.3 the denominator becomes 0 only for $\zeta = z$ (if we would have used the classical support function as used for instance in [2], then there could be whole pieces of real lines in ∂D as zeros of the denominator in the case of convex domains D of finite type which are not of strict type). Therefore, the corresponding arguments of Section 4 of Bruna-Charpentier-Dupain apply. Hence, for showing the desired uniform integrability of $K(\zeta, z)$, it suffices according to (24) of [2] to prove

LEMMA 3.1. *If m denotes the type of D , then one for $\varepsilon > 0$ sufficiently small*

$$\int_{D \cap P_\varepsilon(z)} |K(\zeta, z)| dV(\zeta) = O\left(\varepsilon^{1/m}\right)$$

for all $z \in \partial D$. Here $P_\varepsilon(z)$ denotes the pseudoball at z with radius ε .

We will reduce the proof of this lemma to two other lemmas. We introduce the notation

$$Q(\zeta, z) := \sum_{i=1}^n Q_i(\zeta, z) d\zeta_i$$

$$Q(z, \zeta) := \sum_{i=1}^n Q_i(z, \zeta) d\zeta_i.$$

Notice, that for $\varepsilon > 0$ small enough and $z \in \partial D$ one has $P_\varepsilon(z) \subset U_0$. Hence, $\chi(\zeta) = 0$ for $\zeta \in P_\varepsilon(z)$ and the kernel K can be simplified in the following way:

$$K(\zeta, z) = K_1(\zeta, z) + K_2(\zeta, z)$$

with

$$(3.4) \quad K_1(\zeta, z) = \frac{Q(\zeta, z) \wedge \bar{\partial} \rho(\zeta) \wedge Q(z, \zeta) \wedge (\bar{\partial}_\zeta Q(z, \zeta))^{n-2}}{S(\zeta, z)[\rho(\zeta) + S(z, \zeta)]^n}$$

$$=: \frac{W_1(\zeta, z)}{S(\zeta, z)[\rho(\zeta) + S(z, \zeta)]^n}$$

and

$$(3.5) \quad K_2(\zeta, z) = \frac{Q(\zeta, z) \wedge (\bar{\partial}_\zeta Q(z, \zeta))^{n-1} \rho(\zeta)}{S(\zeta, z)[\rho(\zeta) + S(z, \zeta)]^n} =: \frac{W_2(\zeta, z)}{S(\zeta, z)[\rho(\zeta) + S(z, \zeta)]^n}.$$

We want to show at first the following estimate

LEMMA 3.2. *We have uniformly in $z \in \partial D$ and in $\varepsilon > 0$ (sufficiently small)*

$$\int_{D \cap P_\varepsilon(z)} |W_i(\zeta, z)| = O\left(\varepsilon^{1/m+n+1}\right)$$

for $i = 1, 2$.

Proof. We treat only the case $i = 1$, since the other case is similar. We have

$$W_1(\zeta, z) = Q(\zeta, z) \wedge Q(z, \zeta) \wedge \bar{\partial} \rho(\zeta) \wedge [\bar{\partial}_\zeta Q(z, \zeta)]^{n-2}.$$

We choose ε -extremal coordinates $w = (w_1, \dots, w_n)$ at z and transform the forms $Q(z, \zeta)$ and $Q(z, \zeta)$ by pulling back the ζ -variable, obtaining

$$Q^*(w, z) := \sum_{i=1}^n Q_i(\zeta(w), z) d\zeta_i(w), \quad Q^*(z, w) := \sum_{i=1}^n Q_i(z, \zeta(w)) d\zeta_i(w).$$

We write

$$Q^*(w, z) = \sum_{i=1}^n Q_i^*(w, z) dw_i, \quad Q^*(z, w) = \sum_{i=1}^n Q_i^*(z, w) dw_i.$$

We then can estimate $|W_1(\zeta, z)|$ from above by the sum over all terms of the form

$$(3.6) \quad |Q_{i_1}^*(w, z)| |Q_{i_2}^*(z, w)| \left| \frac{\partial \rho}{\partial \bar{w}_{j_2}} \prod_{l=3}^n \left| \frac{\partial Q_{i_l}^*(z, w)}{\partial \bar{w}_{j_l}} \right| \right|$$

where (i_1, \dots, i_n) is a multiindex with values from $\{1, \dots, n\}$ and $i_l \neq i_k$ for $k \neq l$ and (j_2, \dots, j_n) is a multiindex with values in $\{1, \dots, n\}$ with $j_k \neq j_l$ for $k \neq l$.

Each single term in (3.6) can be estimated as in [4] using Proposition 3.1 from [4], which consists of material mostly collected from results of McNeal from [8]. Namely, according to Proposition 3.1, iv) and vii) of [4] we have

$$(3.7) \quad \left| \frac{\partial \rho}{\partial \bar{w}_{j_2}}(w) \right| \lesssim \frac{\varepsilon}{\tau_{j_2}(z, \varepsilon)} \quad \forall \zeta(w) \in P_\varepsilon(z)$$

and Lemma 5.1 of [4] gives

$$(3.8) \quad \left| \frac{\partial Q_{i_l}^*}{\partial w_{j_l}}(z, w) \right| \lesssim \frac{\varepsilon}{\tau_{i_l}(z, \varepsilon)\tau_{j_l}(z, \varepsilon)} \quad \forall \zeta(w) \in P_\varepsilon(z).$$

Furthermore, the same lemma also contains

$$(3.9) \quad |Q_{i_l}^*(z, w)| \lesssim \frac{\varepsilon}{\tau_{i_l}(z, \varepsilon)} \quad \forall \zeta(w) \in P_\varepsilon(z).$$

It remains to estimate the first factor in (3.6). This is a simplified version of Lemma 5.4 of [4]. We have

$$(3.10) \quad |Q_k^*(w, z)| \lesssim \frac{\varepsilon}{\tau_k(z, \varepsilon)}$$

where the τ_k are always the weighted directional radii of the pseudoballs also used in [2].

Altogether we get for each expression (3.6) the upper estimate by

$$(3.11) \quad \frac{\varepsilon}{\tau_{i_1}(z, \varepsilon)} \frac{\varepsilon}{\tau_{i_2}(z, \varepsilon)} \frac{\varepsilon}{\tau_{j_2}(z, \varepsilon)} \prod_{l=3}^n \frac{\varepsilon}{\tau_{i_l}(z, \varepsilon)\tau_{j_l}(z, \varepsilon)}.$$

If we put this together into $W_1(\zeta, z)$ we get

$$(3.12) \quad |W_1(\zeta, z)| \lesssim \varepsilon^{n+1} \tau_n(z, \varepsilon) \prod_{l=1}^n \frac{1}{\tau_l^2(z, \varepsilon)}.$$

The estimate of the lemma for $i = 1$ follows directly from this. □

In order to formulate the next lemma, we choose a constant $C_1 > 0$ such that $C_1 P_{\varepsilon/2}(z) \supset \frac{1}{2} P_\varepsilon(z)$ for all z, ε (see Proposition 3.1,ii) of [4]) and put for positive integers i

$$P_\varepsilon^i(z) := C_1 P_{2^{-i}\varepsilon}(z) \setminus P_{2^{-i}\varepsilon}(z).$$

Then the family $\{P_\varepsilon^i(z)\}$ is a covering of $P_\varepsilon(z)$, see (11) of [4].

Next we want to show

LEMMA 3.3. *One has*

- i) $|S(\zeta, z)| \gtrsim 2^{-i}\varepsilon$
- ii) $|\rho(\zeta) + S(z, \zeta)| \gtrsim 2^{-i}\varepsilon$

uniformly in $z \in \partial D, \zeta \in P_\varepsilon^i(z) \cap D$ and all i .

Proof. For showing i) we write $\zeta = \mu n_z + \lambda v$ as in Proposition 2.1. According to Proposition 3.1 iii) there is a uniform constant k_1 such that in either of the following cases

- (1) $|\mu| \leq k_1 \tau(z, n_z, 2^{-i} \varepsilon),$
- (2) $|\lambda| \leq k_1 \tau(z, v, 2^{-i} \varepsilon)$

necessarily $\zeta \in \frac{1}{2} P_{2^{-i} \varepsilon}(z)$. In other words, for $\zeta \notin \frac{1}{2} P_{2^{-i} \varepsilon}(z)$, one has

$$|\lambda| \geq k_1 \tau(z, v, 2^{-i} \varepsilon) \text{ or } |\mu| \geq k_1 \tau(z, n_z, 2^{-i} \varepsilon).$$

We now show at first that for any $k_0 < k_1$ the inequality $|\lambda| \geq k_0 \tau(z, v, 2^{-i} \varepsilon)$ implies i). Namely, we immediately get from Proposition 2.1 together with Proposition 3.1 vi) of [4] the estimate

$$|S(\zeta, z)| \geq ck_0^m \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(z, v)| \tau^j(z, v, 2^{-i} \varepsilon) \gtrsim 2^{-i} \varepsilon.$$

If, on the other hand, $|\lambda| \leq k_0 \tau(z, v, 2^{-i} \varepsilon)$, we must have $|\mu| \geq k_1 \tau(z, n_z, 2^{-i} \varepsilon)$. Therefore, Proposition 2.1 and Lemma 4.2 of [4] give in this case

$$\begin{aligned} |S(\zeta, z)| &\gtrsim -\operatorname{Re} S(\zeta, z) + |\operatorname{Im} S(\zeta, z)| \\ &\gtrsim -\operatorname{Re} \mu + 3|\operatorname{Im} \mu| - |2K \operatorname{Re} \mu \operatorname{Im} \mu| - c \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(z, v)| |\lambda|^j \\ &\gtrsim |\mu| - ck_0^2 \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(z, v)| \tau^j(z, v, 2^{-i} \varepsilon) \\ &\gtrsim k_1 \tau(z, n_z, 2^{-i} \varepsilon) - ck_0^2 2^{-i} \varepsilon \cong 2^{-i} \varepsilon. \end{aligned}$$

This finishes the proof of i). In order to show ii), we use that we get from Proposition 3.1 of [4] the following implications: there are uniform constants such that

$$\begin{aligned} \zeta \notin \frac{1}{2} P_{2^{-i} \varepsilon}(z) &\Rightarrow \zeta \notin P_{2^{-i} c\varepsilon}(z) \\ &\Rightarrow d(\zeta, z) \geq Kc2^{-i} \varepsilon \\ (3.13) \quad &\Rightarrow d(z, \zeta) \geq KK'c2^{-i} \varepsilon \\ &\Rightarrow z \notin P_t(\zeta) \forall t \leq K''KK'c2^{-i} \varepsilon \\ &\Rightarrow z \notin P_{K_0 2^{-i} \varepsilon}(\zeta) \\ &\Rightarrow z \notin K_1 P_{2^{-i} \varepsilon}(\zeta). \end{aligned}$$

On the other hand, there exists a constant $k_1 > 0$, such that for any point of the form $z' = \mu'n_\zeta + \lambda'v'$, the following two inequalities together

$$|\mu'| \leq k_1\tau(\zeta, n_\zeta, 2^{-i}\varepsilon) \text{ and } |\lambda'| \leq k_1\tau(\zeta, v', 2^{-i}\varepsilon)$$

imply

$$z' \in K_1P_{2^{-i}\varepsilon}(\zeta).$$

Hence, we have for a pair (z, ζ) , $z = \mu n_\zeta + \lambda v$, chosen as in ii) of the lemma necessarily

$$|\mu| \geq k_1\tau(\zeta, n_\zeta, 2^{-i}\varepsilon) \text{ or } |\lambda| \geq k_1\tau(\zeta, v, 2^{-i}\varepsilon).$$

Let us again choose any $k_0 < k_1$ and suppose that $|\lambda| \geq k_0\tau(\zeta, v, 2^{-i}\varepsilon)$. Then we get from Proposition 2.1

$$\begin{aligned} & |\rho(\zeta) + S(z, \zeta)| \\ (3.14) \quad & \gtrsim -\rho(\zeta) - \operatorname{Re} S(z, \zeta) \\ & \gtrsim \frac{|\operatorname{Re} \mu|}{2} + \frac{K}{2}(\operatorname{Im} \mu)^2 + c \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)| |\lambda|^j. \end{aligned}$$

From (3.14) together with Proposition 3.1 of [4] we get

$$(3.15) \quad |\rho(\zeta) + S(z, \zeta)| \gtrsim 2^{-i}\varepsilon.$$

We still have to consider the case $|\lambda| \leq k_0\tau(\zeta, v, 2^{-i}\varepsilon)$. Hence, necessarily, $|\mu| \geq k_1\tau(\zeta, n_\zeta, 2^{-i}\varepsilon)$. Therefore, we can estimate

$$\begin{aligned} |\rho(\zeta) + S(z, \zeta)| & \gtrsim -\rho(\zeta) - \operatorname{Re} S(z, \zeta) + |\operatorname{Im} S(z, \zeta)| \\ & \gtrsim \frac{|\operatorname{Re} \mu|}{2} + 3|\operatorname{Im} \mu| - |2K \operatorname{Re} \mu \operatorname{Im} \mu| \\ & \quad - c \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)| |\lambda|^j \\ & \gtrsim |\operatorname{Re} \mu| + |\operatorname{Im} \mu| - ck_0^2 2^{-i}\varepsilon \\ & \gtrsim k_1 2^{-i}\varepsilon - ck_0^2 2^{-i}\varepsilon \\ & \gtrsim 2^{-i}\varepsilon. \end{aligned}$$

This finishes the proof of the lemma. □

With the aid off the last two lemmas we now can easily prove Lemma 3.1:

Proof of Lemma 3.1. Since the family $\{P_\varepsilon^i(z)\}_{i=0}^\infty$ is a covering of $P_\varepsilon(z)$ we get

$$\begin{aligned} & \int_{D \cap P_\varepsilon(z)} |K(\zeta, z)| dV(\zeta) \\ & \leq \sum_{i=0}^\infty \left(\int_{D \cap P_\varepsilon^i(z)} \frac{|W_1(\zeta, z)|}{|S(\zeta, z)| |\rho(\zeta) + S(z, \zeta)|^n} dV(\zeta) \right. \\ & \qquad \left. + \int_{D \cap P_\varepsilon^i(z)} \frac{|W_2(\zeta, z)|}{|S(\zeta, z)| |\rho(\zeta) + S(z, \zeta)|^n} dV(\zeta) \right). \end{aligned}$$

Together with Lemma 3.3, parts i) and ii) this gives

$$\begin{aligned} & \int_{D \cap P_\varepsilon(z)} |K(\zeta, z)| dV(\zeta) \\ & \leq \sum_{i=0}^\infty \left(\frac{1}{(2^{-i}\varepsilon)^{n+1}} \int_{D \cap C_1 P_{2^{-i}\varepsilon}(z)} |W_1(\zeta, z)| dV(\zeta) \right. \\ & \qquad \left. + \frac{1}{(2^{-i}\varepsilon)^{n+1}} \int_{D \cap C_1 P_{2^{-i}\varepsilon}(z)} |W_2(\zeta, z)| dV(\zeta) \right). \end{aligned}$$

According to Lemma 3.2 applied to $\zeta \in C_1 P_{2^{-i}\varepsilon}(z) \subset P_{c2^{-i}\varepsilon}(z)$ we can continue this estimate as follows

$$\begin{aligned} \int_{D \cap P_\varepsilon(z)} |K(\zeta, z)| dV(\zeta) & \lesssim \sum_{i=0}^\infty \frac{1}{(2^{-i}\varepsilon)^{n+1}} (e2^{-i}\varepsilon)^{1/m+n+1} \\ & \lesssim \sum_{i=0}^\infty \varepsilon^{1/m} 2^{-i/m} \cong \varepsilon^{1/m}. \end{aligned}$$

This proves Lemma 3.1. □

§4. Proof of Theorem 1.2

Comparing what we have done so far with [2], we can observe, that our Lemma 3.1 says, that the equation (24) from [2] still holds for all smooth convex domains of finite type. Furthermore, our Lemma 3.2 is the generalization of Lemma 4.3 of [2]. However, the rest of the proof of Theorem 1.2 is for us now very easy. The reader just has to convince himself, that the complete subsection 4.4 of [2] is only based on there equation (24) and there Lemma 4.3. Together with our Lemmas 3.1 and 3.2 it carries over literally to our more general situation. We do not repeat it here. □

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