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$L^2(\mathbb{R}^n)$ BOUNDEDNESS FOR THE COMMUTATORS OF CONVOLUTION OPERATORS

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Abstract. The commutators of convolution operators are considered. By localization and Fourier transform estimates, a sufficient condition such that these commutators are bounded on $L^2(\mathbb{R}^n)$ is given. As applications, some new results about the $L^2(\mathbb{R}^n)$ boundedness for the commutators of homogeneous singular integral operators are established.

§1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. For T a linear operator from $C_0^\infty(\mathbb{R}^n)$ to $\mathcal{M}(\mathbb{R}^n)$, the set of measurable functions on $\mathbb{R}^n$, define the k-th order commutator of T and b by

$$T_{b,k}f(x) = T((b(x)-b(\cdot))^k f)(x),\ f \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

A celebrated result of Coifman and Meyer [3] states that if T is a standard Calderón-Zygmund singular integral operator, then for $1 < p < \infty$, the $L^p(\mathbb{R}^n)$ boundedness for $T_{b,1}$ could be obtained from the weighted L^p estimate with A_p weights for the operator T, where A_p denotes the weight function class of Muckenhoupt (see [9, Chapter 5] for definition and properties of A_p). Alvarez, Bagby, Kurtz and Pérez [1] developed the idea of Coifman and Meyer, and established a generalized boundedness criterion for the commutators of linear operators. Let E be a Banach space with norm $\|\cdot\|_E$, denote by $\mathcal{M}(E)$ the set of E-valued measurable functions on $\mathbb{R}^n$. Let u be a weight function on $\mathbb{R}^n$, that is, u is real-valued, non-negative and locally integrable. For $1 \leq p < \infty$, define the Banach space $L^p_u(E)$ by

$$L^p_u(E) = \Big\{f:\ f \in \mathcal{M}(E),\ \|f\|_{L^p_u(E)} = \Big(\int_{\mathbb{R}^n} \|f(x)\|^p_E u(x)dx\Big)^{1/p} < \infty\Big\}.$$

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The result of Alvarez, Bagby, Kurtz and Pérez (see [1, Theorem 2.13]) can be stated as follows.

THEOREM ABKP. *Let E be a Banach space, $1 < p, q < \infty$. Suppose that the linear operator $T\colon C_0^\infty(\mathbb{R}^n) \to \mathcal{M}(E)$ satisfies the weighted estimate*

$$\|Tf\|_{L^p_w(E)} \le \bar{C}\|f\|_{p,\,w}$$

for all $w \in A_q$, and $\bar{C}$ depends only on n, p and $\widetilde{C}_q(w)$ (the A_q constant of w), but not on the weight w. Then for any positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$ and any weight function $u \in A_q$, the operator $T_{b,\,k}$, the k-th order commutator of T defined by (1), is bounded from $L^p_u(\mathbb{R}^n)$ to $L^p_u(E)$ with bound $C(n, p, k, \widetilde{C}_q(u))\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

This result is of great interest and is suitable for many classical operators in harmonic analysis, such as the Calderón-Zygmund singular integral operators, the Bochner-Riesz operators at critical index, the oscillatory singular integral operator of Ricci and Stein, etc.. But for many important operators, Theorem ABKP does not applies. A typical example is the following homogeneous singular integral operator.

Let Ω be homogeneous of degree zero, have mean value zero on the unit sphere S^{n-1} $(n \ge 2)$. Define the homogeneous singular integral operator $\widetilde{T}$ by

$$\widetilde{T}f(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy.$$

For positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, define the k-th order commutator of $\widetilde{T}$ by

$$\widetilde{T}_{b,\,k}f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))^k \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{2}$$

The well-known result of Coifman, Rochberg and Weiss [2] tells us that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$, then the commutator $\widetilde{T}_{b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. By the result of Duoandikoetxea [4], we see that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then for $p > q'$ $(q' = q/(q-1))$ and $w \in A_{p/q'}$, the operator $\widetilde{T}$ is bounded on $L^p_w(\mathbb{R}^n)$ with bound depending only on n, p and the $A_{p/q'}$ constant of w. This together with Theorem ABKP shows that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then for positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator $\widetilde{T}_{b,\,k}$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $q' < p < \infty$, and then by the standard duality and interpolation argument, is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. But if $\Omega \notin \bigcup_{q>1} L^q(S^{n-1})$, we do not know $\widetilde{T}$ satisfies weighted $L^p(\mathbb{R}^n)$ estimate with general A_q weights for any fixed $1 < p, q < \infty$. In this case, the $L^p(\mathbb{R}^n)$ boundedness for the corresponding commutator has not been known.

The purpose of this paper is to give a sufficient condition such that the commutators of convolution operators are bounded on $L^2(\mathbb{R}^n)$, and this sufficient condition is based on Fourier transform estimate of the kernel of the convolution operator. As applications, we will establish the $L^2(\mathbb{R}^n)$ boundedness for the commutator $\widetilde{T}_{b,k}$ when $\Omega \notin \bigcup_{q>1} L^q(S^{n-1})$. We remark that in this paper, we are very much motivated by the work of Pérez [8], some ideas are from the paper of Hu, Lu and Ma [7]. For function f on $\mathbb{R}^n$, denote by $\widehat{f}$ the Fourier transform of f. Our first result in this paper is

THEOREM 1. *Let $K(x)$ be a function on $\mathbb{R}^n\backslash\{0\}$ and $K(x) = \sum_{j\in\mathbb{Z}} K_j(x)$. Let k be a positive integer. Suppose that there are some constants $C > 0$, $0 < A \le 1/2$ and $\alpha > k+1$ such that for each $j \in \mathbb{Z}$*

$$\|K_j\|_1 \le C, \ \ \|\nabla \widehat{K_j}\|_\infty \le C2^j; \tag{3}$$

$$|\widehat{K_j}(\xi)| \le C \min\{A|2^j\xi|, \ \log^{-\alpha}(2+|2^j\xi|)\}. \tag{4}$$

Then for $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \nu < 1$ such that $\alpha\nu > k+1$, the commutator

$$T_{b,k}f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))^k K(x-y) f(y)dy, \ \ f \in C_0^\infty(\mathbb{R}^n)$$

is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n, k, \alpha, \nu)\log^{-\alpha\nu+k+1}(\frac{1}{A})\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

Remark 1. In our applications, we only use the case $A = 1/2$. Theorem 1 for the case $A < 1/2$ seems useful in the study of the $L^p(\mathbb{R}^n)$ boundedness for the commutators of convolution operators.

As applications of Theorem 1, we will have

THEOREM 2. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, Ω be homogeneous of degree zero and have mean value zero. Suppose that for some $\alpha > k+1$,*

$$\sup_{\zeta\in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \Big(\log\frac{1}{|\theta\cdot\zeta|}\Big)^\alpha d\theta < \infty. \tag{5}$$

Then the commutator $\widetilde{T}_{b,k}$ defined by (2) is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n, k, \alpha)\ \|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

Remark 2. The size condition (5) for $\alpha \geq 1$ was introduced by Grafakos and Stefanov [6] in order to study the $L^p(\mathbb{R}^n)$ boundedness for the operator $\widetilde{T}$. It has been proved in [6] that there exist integrable functions on S^{n-1} which are not in $H^1(S^{n-1})$ (the Hardy space on S^{n-1}), but satisfy (5) for all $\alpha > 0$.

THEOREM 3. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, Ω be homogeneous of degree zero and have mean value zero, $h(r)$ be a function on $(0, \infty)$ which satisfies*

$$\sup_{R>0} \int_R^{2R} |h(r)|^s \frac{dr}{r} < \infty \text{ for some } s > 1.$$

Suppose that for some $\alpha > k+1$, Ω satisfies the size condition

$$\int_{S^{n-1}} |\Omega(x')| \log^{\alpha}(2 + |\Omega(x')|) dx' < \infty.$$

Then the commutator defined by

$$\bar{T}_{b,\,k} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$$

is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n, k, \alpha)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. For a locally integrable function f, a positive integer m and a cube I, define

$$\|f\|_{L(\log L)^m,\, I} = \inf \Big\{ \lambda > 0 : \frac{1}{|I|} \int_I \frac{|f(y)|}{\lambda} \log^m \Big(2 + \frac{|f(y)|}{\lambda} \Big) dy \leq 1 \Big\}$$

and

$$\|f\|_{\exp(L)^{1/m},\, I} = \inf \Big\{ \lambda > 0 : \frac{1}{|I|} \int_I \exp\Big(\frac{|f(y)|}{\lambda} \Big)^{1/m} dy \leq 2 \Big\}.$$

Since that $\Phi(t) = t \log^m(2+t)$ is a Young function on $[0, \infty)$ and its complementary Young function is $\Psi(t) \approx \exp t^{1/m}$, the generalized Hölder inequality

$$\frac{1}{|I|} \int_I |f(y)h(y)| dy \leq C \|f\|_{L(\log L)^m,\, I} \|h\|_{\exp(L)^{1/m},\, I}$$

holds for locally integrable functions f and h, see [8, page 168] for details.

§2. Proof of Theorems

We begin with some lemmas.

LEMMA 1. *Let* $\phi \in C_0^\infty(\mathbb{R}^n)$ *be a radial function such that* $\operatorname{supp}\phi \subset \{\xi : 1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1,\ |\xi| \ne 0.$$

Denote by S_l *the multiplier operator* $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$*. For any positive integer* k *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$*, denote by* $S_{l;\,b,\,k}$ *the* k*-th order commutator of* S_l *defined as in* (1)*. Then for* $1 < p < \infty$*, the inequality*

$$\Big\|\Big(\sum_{l\in\mathbb{Z}} |S_{l;\,b,\,k}f|^2\Big)^{1/2}\Big\|_p \le C(n,\,k,\,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p$$

holds.

By the weighted Littlewood-Paley theory, it is easy to see that for $1 < p < \infty$ and $w \in A_p$,

$$\Big\|\Big(\sum_{l\in\mathbb{Z}} |S_l f|^2\Big)^{1/2}\Big\|_{p,\,w} \le C(n,\,p,\,\widetilde{C}_p(w))\|f\|_{p,\,w}.$$

Thus Lemma 1 follows from Theorem ABKP directly. See also [7, page 361].

LEMMA 2. *Let* $m_\delta \in C^1(\mathbb{R}^n)$ $(0 < \delta < \infty)$ *be a family of multipliers such that* $\operatorname{supp} m_\delta \subset \{\xi : |\xi| \le \delta\}$ *and for some constants* C*,* $0 < A \le 1/2$ *and* $\alpha > 1$*,*

$$\|m_\delta\|_\infty \le C\min\{A\delta,\, \log^{-\alpha}(2+\delta)\},\ \|\nabla m_\delta\|_\infty \le C.$$

Let T_δ *be the multiplier operator defined by* $\widehat{T_\delta f}(\xi) = m_\delta(\xi)\widehat{f}(\xi)$*. For positive intger* k *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$*, denote by* $T_{\delta;\,b,\,k}$ *the* k*-th order commutator of* T_δ*. Then for any* $0 < \varepsilon < 1$*, there exists a positive constant* $C = C(n,\,k,\,\varepsilon)$ *such that*

$$\|T_{\delta;\,b,\,k}f\|_2 \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}(A\delta)^{1-\varepsilon}\log^k\big(\frac{1}{A}\big)\|f\|_2,\ \text{if}\ \delta < 10/\sqrt{A};$$
$$\|T_{\delta;\,b,\,k}f\|_2 \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\log^{-\alpha(1-\varepsilon)+k}(2+\delta)\|f\|_2,\ \text{if}\ \delta > 1/\sqrt{A}.$$

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. Let ψ be a radial function such that $\operatorname{supp}\psi\subset\{x:\ 1/4\le|x|\le 4\}$, and

$$\sum_{l\in\mathbb{Z}}\psi(2^{-l}x)=1,\ \ |x|>0.$$

Set $\psi_0(x)=\sum_{l=-\infty}^{0}\psi(2^{-l}x)$ and $\psi_l(x)=\psi(2^{-l}x)$ for positive integer l. Let $K_\delta(x)=m_\delta^\vee(x)$, the inverse Fourier transform of m_δ. Splite K_δ as

$$K_\delta(x)=K_\delta(x)\psi_0(x)+\sum_{l=1}^{\infty}K_\delta(x)\psi_l(x)=\sum_{l=0}^{\infty}K_{\delta,l}(x).$$

Let $T_{\delta,\,l}$ be the convolution operator whose kernel is $K_{\delta,\,l}$. Recall that $m_\delta\subset\{\xi:\ |\xi|\le\delta\}$. Trivial computation shows that $\|K_{\delta,\,l}\|_\infty\le\|K_\delta\|_\infty\le\|m_\delta\|_1\le C\delta^n$. This via the Young inequality says that

$$\|T_{\delta,\,l}f\|_\infty\le C\delta^n\|f\|_1. \tag{6}$$

Note that $\int\widehat{\psi}(\eta)d\eta=0$. Thus

$$\begin{aligned}\|\widehat{K_{\delta,\,l}}\|_\infty&=\Big\|\int_{\mathbb{R}^n}(m_\delta(\xi-2^{-l}\eta)-m_\delta(\xi))\widehat{\psi}(\eta)d\eta\Big\|_\infty\\&\le C2^{-l}\|\nabla m_\delta\|_\infty\int_{\mathbb{R}^n}|\eta||\widehat{\psi}(\eta)|d\eta\le C2^{-l}.\end{aligned}$$

On the other hand, by the Young inequality,

$$\|\widehat{K_{\delta,\,l}}\|_\infty\le\|\widehat{K_\delta}\|_\infty\|\widehat{\psi_l}\|_1\le C\min\{A\delta,\ \log^{-\alpha}(2+\delta)\}.$$

For each fixed $0<\varepsilon<1$, let $t_0=\varepsilon/3$ $(0<t_0<1/3)$, we then have

$$\|\widehat{K_{\delta,\,l}}\|_\infty\le C2^{-t_0l}\Big(\min\{A\delta,\ \log^{-\alpha}(2+\delta)\Big)^{1-t_0},$$

which together with the Plancherel theorem tells us that

$$\|T_{\delta,\,l}f\|_2\le C2^{-t_0l}\Big(\min\{A\delta,\ \log^{-\alpha}(2+\delta)\}\Big)^{1-t_0}\|f\|_2. \tag{7}$$

Let $T_{\delta,\,l;\,b,\,k}$ be the k-th order commutator of $T_{\delta,\,l}$. We want to show the following refined estimates

$$\|T_{\delta,\,l;\,b,\,k}f\|_2\le C(A\delta)^{1-3t_0}2^{-t_0l/4}\log^k\big(\frac{1}{A}\big)\|f\|_2,\ \text{if}\ \delta<10/\sqrt{A} \tag{8}$$

and

$$\|T_{\delta,l;b,k}f\|_2 \le C2^{-t_0l/4}\log^{-\alpha(1-3t_0)+k}(2+\delta)\|f\|_2, \text{ if } \delta > 1/\sqrt{A}. \tag{9}$$

If we can do this, summing over these inequalities respectively for all nonnegative integer l completes the proof of Lemma 2.

Let $T^*_{\delta,l}$ be the dual operator of $T_{\delta,l}$, that is,

$$T^*_{\delta,l}f(x) = \int_{\mathbb{R}^n} K_{\delta,l}(y-x)f(y)dy.$$

To prove the inequality (8) and (9), we will use some basic estimates for $T^*_{\delta,l}$. Let I be a cube with side length 2^l. We claim that if $\operatorname{supp} f \subset I$, then for nonnegative integer m,

$$\begin{aligned} &\|(T^*_{\delta,l}f)^2\|_{L(\log L)^{2m},I} \\ &\quad \le C(A\delta)^{2(1-2t_0)}2^{-nl}2^{-t_0l}\log^{2m}(\frac{1}{A})\|f\|_2^2, \quad \text{if } \delta < 10/\sqrt{A} \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\|(T^*_{\delta,l}f)^2\|_{L(\log L)^{2m},I} \\ &\quad \le C2^{-nl}2^{-t_0l}\log^{-2\alpha(1-2t_0)+2m}(2+\delta)\|f\|_2^2, \quad \text{if } \delta \ge 1/\sqrt{A}, \end{aligned} \tag{11}$$

(for $m=0$, $\|(T^*_{\delta,l}f)^2\|_{L(\log L)^{2m},I} = |I|^{-1}\|T^*_{\delta,l}f\|_2^2$). In fact, without loss of generality, we may assume that $\|f\|_2 = 1$. By the Schwarz inequality and the fact that $\|K_{\delta,l}\|_\infty \le C\delta^n$, it follows that

$$\|T^*_{\delta,l}f\|_\infty \le C\delta^n\|f\|_1 \le C\delta^n 2^{nl/2}. \tag{12}$$

We consider the following two cases.

Case I. $\delta < 10/\sqrt{A}$. Take

$$\lambda_1 = (A\delta)^{2(1-2t_0)}2^{-nl}2^{-t_0l}\log^{2m}(\frac{1}{A}).$$

By the estimate (7) and (12), we have

$$\begin{aligned} &\int_I |T^*_{\delta,l}f(x)|^2\log^{2m}\Big(2+\frac{|T^*_{\delta,l}f(x)|^2}{\lambda_1}\Big)dx \\ &\le C2^{-2t_0l}(A\delta)^{2(1-t_0)}\log^{2m}\Big(\frac{2^{(2n+1)l}}{A^n(A\delta)^{2(1-2t_0)}}\Big) \\ &\le C2^{-t_0l}(A\delta)^{2(1-2t_0)}\log^{2m}(\frac{1}{A}). \end{aligned}$$

Therefore,

$$\|(T^*_{\delta,l}f)^2\|_{L(\log L)^{2m},I} \le C\lambda_1.$$

Case II. $\delta > 1/\sqrt{A}$. We choose

$$\lambda_2 = \log^{-2\alpha(1-2t_0)+2m}(2+\delta)2^{-nl}2^{-t_0 l}.$$

Again by the estimate (7) and (12), we have

$$\begin{aligned}&\frac{1}{\lambda_2}\frac{1}{|I|}\int_I |T^*_{\delta,l}f(x)|^2\log^{2m}\Big(2+\frac{|T^*_{\delta,l}f(x)|^2}{\lambda_2}\Big)dx\\ &\le \frac{C}{\lambda_2}2^{-nl}2^{-2t_0 l}\log^{-2\alpha(1-t_0)}(2+\delta)\log^{2m}\Big(2+\frac{2^{nl}\delta^{2n}}{\lambda_2}\Big)\\ &\le C2^{-t_0 l}\log^{-2m-2\alpha t_0}(2+\delta)\log^{2m}\Big(2+\frac{2^{nl}\delta^{2n}}{\lambda_2}\Big)\le C.\end{aligned}$$

The desired estimate (11) follows directly.

Now we turn our attention to the $L^2(\mathbb{R}^n)$ estimate for $T_{\delta,l;b,k}$. Write $\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} I_j$, where each I_j is a cube with side length 2^l, and these cubes have disjoint interiors. Let f_j be the restriction of f on I_j. Then

$$f(x) = \sum_{j\in\mathbb{Z}} f_j(x), \ \text{a.e. } x\in\mathbb{R}^n.$$

Observe that $\operatorname{supp} K_{\delta,l} \subset \{|x|\le 2^{l+2}\}$, it is obvious that the support of $T_{\delta,l;b,k}$ is contained in a fixed multiple of I_j, and that the supports of various terms $T_{\delta,l;b,k}f_j$ have bounded overlaps. So we have

$$\|T_{\delta,l;b,k}f\|_2^2 \le \sum_{j\in\mathbb{Z}}\sum_{j'=-M}^{M}\|T_{\delta,l;b,k}f_{j+j'}\|_2^2 = C\sum_{j\in\mathbb{Z}}\|T_{\delta,l;b,k}f_j\|_2^2,$$

where M is a positive integer which is independent of j. Thus we may assume that $\operatorname{supp} f\subset I$ for a cube I with side length 2^l. We also assume $\|f\|_2 = 1$. Set $I^* = 10nI$, $I^{**} = 20nI$. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$, $0\le\varphi\le 1$, φ is identically one on I^* and vanishes outside I^{**}. Let $\bar b(x) = (b(x)-m_{I^*}(b))\varphi(x)$, where $m_{I^*}(b)$ denotes the mean value of b on I^*. Obviously,

$$|T_{\delta,l;b,k}f(x)| \le \sum_{m=0}^{k} C_k^m|\bar b^m(x)T_{\delta,l}(\bar b^{k-m}f)(x)|.$$

Note that $\operatorname{supp} T_{\delta,l}(\bar{b}^{k-m}f) \subset I^*$. Recall that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. For each fixed integer m, $0 \le m \le k$, by the generalized Hölder inequality,

$$\begin{aligned}\|\bar{b}^m T_{\delta,l}(\bar{b}^{k-m}f)\|_2^2 &\le |I^*| \|\bar{b}^{2m}\|_{\exp(L)^{1/(2m)},\, I^*} \|(T_{\delta,l}(\bar{b}^{k-m}f))^2\|_{L(\log L)^{2m},\, I^*} \\ &\le C|I| \|(T_{\delta,l}(\bar{b}^{k-m}f))^2\|_{L(\log L)^{2m},\, I^*}.\end{aligned}$$

The last inequality follows from the well-known John-Nirenberg inequality which states that for positive constants $C_1,\ C_2$ depending only on n,

$$\frac{1}{|I^*|}\int_{I^*} \exp\Big(\frac{|b(x) - m_{I^*}(b)|}{C_1\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big) dx \le C_2.$$

To computate $\|(T_{\delta,l}(\bar{b}^{k-m}f))^2\|_{L(\log L)^{2m},\, I^*}$, we first observe that by (6),

$$\|T_{\delta,l}(\bar{b}^{k-m}f)\|_\infty \le C\delta^n \|\bar{b}^{k-m}f\|_1 \le C\delta^n 2^{nl/2}, \tag{13}$$

where we have invoked the corollary of the John-Nirenberg inequality (see [9, page 144]) and the fact that $|m_{I^*}(b) - m_I(b)| \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. A standard duality argument gives us that

$$\begin{aligned}\|T_{\delta,l}(\bar{b}^{k-m}f)\|_2 &= \sup_{\operatorname{supp} h \subset I^*,\, \|h\|_2 \le 1} \Big|\int_{I^*} T_{\delta,l}(\bar{b}^{k-m}f)(x)h(x)dx\Big| \\ &= \sup_{\operatorname{supp} h \subset I^*,\, \|h\|_2 \le 1} \Big|\int_{I^*} T^*_{\delta,l}h(x)\bar{b}^{k-m}(x)f(x)dx\Big| \\ &\le \sup_{\operatorname{supp} h \subset I^*,\, \|h\|_2 \le 1} \Big(\int_{I^*} |T^*_{\delta,l}h(x)|^2 |\bar{b}^{2(k-m)}(x)| dx\Big)^{1/2} \\ &\le C2^{nl/2} \sup_{\operatorname{supp} h \subset I^*,\, \|h\|_2 \le 1} \|\bar{b}^{2(k-m)}\|^{1/2}_{\exp(L)^{1/2(k-m)},\, I^*} \\ &\qquad \times \|(T^*_{\delta,l}h)^2\|^{1/2}_{L(\log L)^{2(k-m)},\, I^*} \\ &\le C2^{nl/2} \sup_{\operatorname{supp} h \subset I^*,\, \|h\|_2 \le 1} \|(T^*_{\delta,l}h)^2\|^{1/2}_{L(\log L)^{2(k-m)},\, I^*}.\end{aligned} \tag{14}$$

If $\delta < 10/\sqrt{A}$, it follows from the inequality (10) that

$$\|T_{\delta,l}(\bar{b}^{k-m}f)\|_2^2 \le C(A\delta)^{2(1-2t_0)} 2^{-t_0 l} \log^{2(k-m)}\big(\frac{1}{A}\big).$$

Set $\lambda_3 = (A\delta)^{2(1-3t_0)} 2^{-nl} 2^{-t_0 l/2} \log^{2k}(\frac{1}{A})$. The last inequality together with the estimate (13) shows that

$$\int_{I^*} |T_{\delta,l}(\bar{b}^{k-m}f)(x)|^2 \log^{2m}\Big(2 + \frac{|T_{\delta,l}(\bar{b}^{k-m}f)(x)|^2}{\lambda_3}\Big)dx$$

$$\leq C(A\delta)^{2(1-2t_0)}2^{-t_0 l}\log^{2(k-m)}(\frac{1}{A})\log^{2m}\Big(\frac{2^{(2n+1)l}}{A^n(A\delta)^{2(1-3t_0)}}\Big)$$
$$\leq C(A\delta)^{2(1-3t_0)}2^{-t_0 l/2}\log^{2k}(\frac{1}{A}),\ \text{if } \delta < 10/\sqrt{A}.$$

This in turn implies that

$$\|\bar{b}^m T_{\delta,l}(\bar{b}^{k-m}f)\|_2^2$$
$$\leq C2^{nl}\|(T_{\delta,l}(\bar{b}^{k-m}f))^2\|_{L(\log L)^{2m},I^*}$$
$$\leq C2^{nl}\lambda_3 = C(A\delta)^{2(1-3t_0)}2^{-t_0 l/2}\log^{2k}(\frac{1}{A}),\ \text{if } \delta < 10/\sqrt{A},$$

and the estimate (8) follows. On the other hand, if $\delta > 1/\sqrt{A}$, set $\lambda_4 = 2^{-nl}2^{-t_0 l/2}\log^{-2\alpha(1-3t_0)+2k}(2+\delta)$. The same argument involving the inequalities (13) and (14) as above yields that

$$\|\bar{b}^m T_{\delta,l}(\bar{b}^{k-m}f)\|_2^2 \leq C2^{nl}\lambda_4 = 2^{-t_0 l/2}\log^{-2\alpha(1-3t_0)+2k}(2+\delta).$$

This leads to the inequality (9).

Proof of Theorem 1. Choose radial function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$, $\operatorname{supp}\phi \subset \{1/4 \leq |\xi| \leq 4\}$ and

$$\sum_{l\in\mathbb{Z}}\phi^3(2^{-l}\xi) = 1,\ |\xi| \neq 0.$$

Define the multiplier operator S_l by

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi).$$

Set $m_j(\xi) = \widehat{K_j}(\xi)$, $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$ and

$$\widehat{T_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Obviously, $\operatorname{supp} m_j^l(2^{-j}\xi) \subset \{|\xi| \leq 2^{l+2}\}$ and

$$\|m_j^l(2^{-j}\cdot)\|_\infty \leq C\min\{A2^l, \log^{-\alpha}(2+2^l)\},\ \|\nabla m_j^l(2^{-j}\cdot)\|_\infty \leq C. \tag{15}$$

Let

$$U_l f(x) = \sum_{j\in\mathbb{Z}}\Big((S_{l-j}T_j^l S_{l-j})_{b,k}f\Big)(x).$$

We claim that for $f,\, h \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x) T_{b,\,k} f(x)dx = \int_{\mathbb{R}^n} h(x) \sum_{l\in\mathbb{Z}} U_l f(x)dx, \tag{16}$$

and that

$$\|U_l f\|_2 \le C \sum_{m=0}^{k} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k-m} \Big\| \Big(\sum_{j\in\mathbb{Z}} |(T_j^l S_{l-j})_{b,\,m} f|^2 \Big)^{1/2} \Big\|_2. \tag{17}$$

Both of these had been proved in [7, page 365], but for the reader's convenience and for the sake of self-containment, we give their proof here. To prove (16), let $B = B(O,\, R)$ be the ball centered at the origin and large enough radius R such that supp f, supp $h \subset B(O,\, R)$. Denote by b_B the mean value of b on B. Define the operator T by

$$T\widetilde{f}(x) = \sum_{j\in\mathbb{Z}} K_j * \widetilde{f}(x).$$

Write

$$\int_{\mathbb{R}^n} h(x) T_{b,\,k} f(x)dx$$
$$= \sum_{i=0}^{k} C_k^i \int_{\mathbb{R}^n} (b(x) - b_B)^i h(x) T\Big((b_B - b(\cdot))^{k-i} f \Big)(x)dx.$$

Note that $(b(x) - b_B)^i h(x)$ and $(b_B - b(x))^{k-i} f(x)$ belong to the space $L^2(\mathbb{R}^n)$. Thus, as in [5, page 545], it follows that

$$\int_{\mathbb{R}^n} h(x) T_{b,\,k} f(x)dx$$
$$= \sum_{i=0}^{k} C_k^i \int_{\mathbb{R}^n} (b(x) - b_B)^i h(x) \sum_{j\in\mathbb{Z}} K_j * \Big(\sum_{l\in\mathbb{Z}} S_{l-j}^3 ((b_B - b(\cdot))^{k-i} f) \Big)(x)dx$$
$$= \sum_{i=0}^{k} C_k^i \int_{\mathbb{R}^n} (b(x) - b_B)^i h(x) \sum_{l\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Big((S_{l-j} T_j^l S_{l-j} ((b_B - b(\cdot))^{k-i} f) \Big)(x)dx$$
$$= \int_{\mathbb{R}^n} h(x) \sum_{l\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Big((S_{l-j} T_j^l S_{l-j})_{b,\,k} f \Big)(x)dx.$$

This establishes (16). With the aid of the formula

$$(b(x)-b(y))^k=\sum_{i=0}^{k} C_k^i (b(x)-b(z))^i (b(z)-b(y))^{k-i},\ x,\,y,\,z\in\mathbb{R}^n,$$

the Fubini theorem and trivial computation leads to that for $f,\,h\subset C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x)(S_{l-j}T_j^l S_{l-j})_{b,\,k} f(x)dx$$
$$=\sum_{i=0}^{k} C_k^i \int_{\mathbb{R}^n} h(x) S_{l-j;\,b,\,k-i}((T_j^l S_{l-j})_{b,\,i} f)(x)dx,$$

which via Lemma 1 yields the estimate (17).

We first consider $\sum_{l\le[\log(\frac{1}{\sqrt{A}})]+1}\|U_l f\|_2$, where we use $[a]$ to denote the integral part of the real number a. Let $\widetilde{T_j^l}$ be the operator defined by

$$\widehat{\widetilde{T_j^l} f}(\xi)=m_j^l(2^{-j}\xi)\widehat{f}(\xi).$$

The inequality (15) via Lemma 2 (with $\varepsilon=1-\nu$) says that for positive integer i,

$$\|\widetilde{T}_{j;\,b,\,i}^l f\|_2\le C\log^i(\frac{1}{A})(A2^l)^\nu\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^i\|f\|_2,\ l\le[\log(\frac{1}{\sqrt{A}})]+1.$$

Note that if $b\in\mathrm{BMO}(\mathbb{R}^n)$, then for any $t>0$, $b_t(x)=b(tx)\in\mathrm{BMO}(\mathbb{R}^n)$ and $\|b_t\|_{\mathrm{BMO}(\mathbb{R}^n)}=\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. By dilation-invariance,

$$(18)\quad \|T_{j;\,b,\,i}^l f\|_2\le C\log^i(\frac{1}{A})(A2^l)^\nu\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^i\|f\|_2,\ l\le[\log(\frac{1}{\sqrt{A}})]+1.$$

On the other hand, since $|m_j^l(\xi)|\le C\min\{A2^l,\,1\}\le C(A2^l)^\nu$, the Plancherel theorem states that the estimate (18) is also true for $i=0$, that is,

$$(19)\qquad \|T_j^l f\|_2\le C(A2^l)^\nu\|f\|_2.$$

Observe that for $f,\,h\in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x)(T_j^l S_{l-j})_{b,\,m} f(x)dx=\sum_{i=0}^{m} C_m^i\int_{\mathbb{R}^n} h(x) T_{j;\,b,\,i}^l (S_{l-j;\,b,\,m-i} f)(x)dx,$$

It follows from the estimates (18), (19) and Lemma 1 that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|(T_j^lS_{l-j})_{b,\,m}f\Big|^2\Big)^{1/2}\Big\|_2^2\\&\le C(A2^l)^{2\nu}\log^{2k}(\frac{1}{A})\sum_{i=0}^{m}\|b\|^{2i}_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{j\in\mathbb{Z}}\|S_{l-j;\,b,\,m-i}f\|_2^2\\&\le C(A2^l)^{2\nu}\log^{2k}(\frac{1}{A})\|b\|^{2m}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2^2,\ f\in C_0^\infty(\mathbb{R}^n).\end{aligned}$$

This via the estimate (17) in turn implies

$$\|U_lf\|_2\le C(A2^l)^{\nu}\log^{k}(\frac{1}{A})\|b\|^{k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2,\ \ l\le[\log(\frac{1}{\sqrt{A}})]+1,$$

and

$$\begin{aligned}\sum_{l\le[\log(\frac{1}{\sqrt{A}})]+1}\|U_lf\|_2&\le C\log^{k}(\frac{1}{A})A^{(1-\log 2/2)\nu}\|b\|^{k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2\\&\le C\log^{-\alpha\nu+k+1}(\frac{1}{A})\|b\|^{k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.\end{aligned}$$

Now we consider $\sum_{l>[\log(\frac{1}{\sqrt{A}})]+1}\|U_lf\|_2$. Again by Lemma 2 and (15), we have

$$\|U_lf\|_2\le C\log^{-\alpha\nu+k}(2+2^l)\|b\|^{k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2,\ \ l>[\log(\frac{1}{\sqrt{A}})]+1.$$

Recall that $\alpha\nu>k+1$. Therefore,

$$\sum_{l>[\log(\frac{1}{\sqrt{A}})]+1}\|U_lf\|_2\le C\log^{-\alpha\nu+k+1}(\frac{1}{A})\|b\|^{k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Set

$$K_j(x)=\frac{\Omega(x)}{|x|^n}\chi_{\{2^j\le|x|<2^{j+1}\}}(x).$$

By the integrablity of Ω, it is easy to verify that K_j satisfies the estimate (3). On the other hand, Grafakos and Stefanov [6] proved that if Ω satisfies (5), then K_j satisfies the estimate (4) for $A=1/2$. Theorem 2 follows easily from Theorem 1.

Proof of Theorem 3. At first, we claim that if $\int_{S^{n-1}} |\Omega(x')| \log^\alpha(2 + |\Omega(x')|)dx' < \infty$, then for each positive integer l, there exists Ω_l on S^{n-1} such that $\Omega_l \in L^\infty(S^{n-1})$, and

$$\|\Omega_l\|_{L^\infty(S^{n-1})} \le C2^l, \ \|\Omega - \Omega_l\|_{L^1(S^{n-1})} \le Cl^{-\alpha}.$$

In fact, for given Ω as above, set $E_0 = \{x' \in S^{n-1} : |\Omega(x')| \le 2\}$, and $E_d = \{x' \in S^{n-1} : 2^d < |\Omega(x')| \le 2^{d+1}\}$ for $d \ge 1$. Denote by Ω_d the restriction of Ω on E_d $(d \ge 0)$. For positive integer l, let

$$\Omega_l(x') = \sum_{d=0}^{l-1} \Omega_d(x').$$

It is easy to show that

$$\begin{aligned}\|\Omega_l - \Omega\|_{L^1(S^{n-1})} &\le \sum_{d\ge l} \|\Omega_d\|_{L^1(S^{n-1})} \\ &\le C\sum_{d\ge l} 2^d|E_d| \le Cl^{-\alpha}\sum_{d\ge l} d^\alpha 2^d|E_d| \le Cl^{-\alpha}.\end{aligned}$$

Let l be a positive integer which will be chosen later. For each fixed $j \in \mathbb{Z}$, set

$$K_j^l(x) = h(x)\frac{\Omega_l(x)}{|x|^n}\chi_{\{2^j \le |x| < 2^{j+1}\}}(x),$$

where Ω_l be the function on S^{n-1} such that $\|\Omega_l\|_{L^\infty(S^{n-1})} \le 2^l$ and $\|\Omega_l - \Omega\|_{L^1(S^{n-1})} \le l^{-\alpha}$. Let $\widetilde{s} = \max\{2, s'\}$. We will use a preliminary Fourier transform estimate for K_j^l, that is, for for each $0 < \gamma < 1$, there exists a positive constant $C = C(n, \gamma)$ such that

$$|\widehat{K_j^l}(\xi)| \le C\|\Omega_l\|_\infty |2^j\xi|^{-\gamma/\tilde{s}}. \tag{20}$$

In fact, if $s > 2$, then

$$\sup_{R>0} \int_R^{2R} |h(r)|^2 \frac{dr}{r} < \infty,$$

and the estimate (20) is an easy corollary of the familiar Fourier transform estimate due to Duoandikoetxea and Rubio de Francia (see [5, page 551]). On the other hand, if $s < 2$, set

$$I_r^l(\xi) = \int_{S^{n-1}} e^{-2\pi i r\xi\theta}\Omega_l(\theta)d\theta.$$

Invoking the Hölder inequality and the fact that $\|I_r^l\|_\infty \leq C\|\Omega_l\|_\infty$, we get that

$$\begin{aligned}|\widehat{K_j^l}(\xi)| &\leq \Big(\int_{2^j}^{2^{j+1}} |h(r)|^s \frac{dr}{r}\Big)^{1/s} \Big(\int_{2^j}^{2^{j+1}} |I_r^l(\xi)|^{s'} \frac{dr}{r}\Big)^{1/s'} \\ &\leq C\|\Omega_l\|_\infty^{1-2/s'} \Big(\int_{2^j}^{2^{j+1}} |I_r^l(\xi)|^2 \frac{dr}{r}\Big)^{1/s'} \\ &\leq C\|\Omega_l\|_\infty |2^j\xi|^{-\gamma/s'},\end{aligned}$$

where in the last inequality, we again employed the Fourier transform estimate due to Duoandikoetxea and Rubio de Francia.

We can now conclude the proof of Theorem 3. Let

$$K_j(x) = h(x)\frac{\Omega(x)}{|x|^n}\chi_{\{2^j \leq |x| < 2^{j+1}\}}(x).$$

Obviously, K_j satisfies (3), and by the vanishing moment of Ω,

$$|\widehat{K_j}(\xi)| \leq C|2^j\xi|.$$

For each $\xi \in \mathbb{R}^n$ such that $|2^j\xi| > 2$, let l be the positive integer such that $2^l < |2^j\xi|^{\gamma/(2\tilde{s})} \leq 2^{l+1}$. We finally obtain

$$\begin{aligned}|\widehat{K_j}(\xi)| &\leq |\widehat{K_j^l}(\xi)| + \|\Omega - \Omega_l\|_{L^1(S^{n-1})} \\ &\leq |2^j\xi|^{-\gamma/\tilde{s}}\|\Omega_l\|_{L^\infty(S^{n-1})} + \|\Omega - \Omega_l\|_{L^1(S^{n-1})} \\ &\leq C|2^j\xi|^{-\gamma/(2\tilde{s})} + C\log^{-\alpha}(|2^j\xi|) \leq C\log^{-\alpha}(|2^j\xi|),\ |2^j\xi| > 2.\end{aligned}$$

Combining the estimates above, we see that K_j satisfies (4) for $A = 1/2$. This via Theorem 1 establishes Theorem 3.

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