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A RECURRENCE FORMULA FOR q-BERNOULLI NUMBERS ATTACHED TO FORMAL GROUP

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Abstract. Kaneko [2] proved a new recurrence formula for the Bernoulli numbers and gave two proofs. One of them was due to Don Zagier. We shall apply Zagier's idea to the q-Bernoulli numbers attached to formal group.

1. Generalization of Kaneko's recurrence formula

Let $\mathfrak{B} = \mathfrak{B}(X)$ be the generating function of the Bernoulli numbers, i.e.,

$$\mathfrak{B} = \frac{X}{e^X - 1},$$

then it is anti-invariant under a map: $\mathfrak{B} \mapsto \mathfrak{B}e^X$, i.e.,

$$\mathfrak{B}(X)e^X = \mathfrak{B}(-X).$$

Zagier gave another proof of Kaneko's recurrence formula for the Bernoulli numbers by using this property [2]. On the other hand because of $\mathfrak{B}(-X) = \mathfrak{B}(X) + X$, we can see that $\mathfrak{B}$ is transformed to the sum of a polynomial and itself under the above map. We use the second property in order to generalize Kaneko's recurrence formula and prove a formula for the q-Bernoulli numbers attached to formal group.

First we suppose a power series B in X which satisfies the following:

Assumption 1.

$$Be^X = B + C\ ,$$

where C is a polynomial.

If $C = X$, then B is equal to $\mathfrak{B}$, and if $C = X^2$, then B is essentially equal to the generating function of $\tilde{B}_n$ which was defined in [2], i.e.,

$$\sum_{n \geq 0} \tilde{B}_n \frac{X^n}{n!} = \left(\frac{X^2}{e^X - 1} \right)'.$$

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The starting point of our argument is the following trivial lemma:

LEMMA 1. *For any power series A and non-negative integer n, we have*

$$A^{(n)}e^X = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (Ae^X)^{(i)},$$

where $A^{(n)}$ means the n-th derivative of A.

Proof. Because of $A = (Ae^X)e^{-X}$, we can get what we want. □

Set $A = B$ and compare the coefficient of $\frac{X^m}{m!}$ for any non-negative integer m, then we have a generalization of Kaneko's recurrence formula as follows:

PROPOSITION 1. *If B satisfies* Assumption 1, *then we have*

$$\sum_{i=0}^{m} \binom{m}{i} b_{n+i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (b_{m+i} + c_{m+i}),$$

where $B = \sum_{n\geq 0} b_n \frac{X^n}{n!}$ and $C = \sum_{n\geq 0} c_n \frac{X^n}{n!}$.

If $m > \deg C$, then we have

COROLLARY 1.

$$\sum_{i=0}^{m} \binom{m}{i} b_{n+i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_{m+i}.$$

Furthermore if $m = n$, then we have

COROLLARY 2.

$$\sum_{\substack{i=0 \\ i\not\equiv n \bmod 2}}^{n} \binom{n}{i} b_{n+i} = 0 \ .$$

If $C = X$, then there is no new information about the Bernoulli numbers. But if $C = X^2$, then this is equivalent to Kaneko's recurrence formula.

2. q-recurrence formula

In this section we shall extend results in the previous section for the q-Bernoulli numbers attached to formal group. Let q be an indeterminate and let $\mathfrak{o}$ be the formal power series ring in $q-1$ over some $\mathbb{Q}$-algebra. Furthermore let $\mathfrak{F}(X,Y)$ be a 1-dimensional commutative formal group defined over $\mathfrak{o}$ and let $\mathfrak{f}(X)$ be an isomorphism from the additive formal group $X+Y$ to $\mathfrak{F}(X,Y)$. We note that there exists a unique isomorphism $\mathfrak{f}_{\mathfrak{F}}(X)$ from $X+Y$ to $\mathfrak{F}(X,Y)$ defined over $\mathfrak{o}$ such that $\mathfrak{f}'_{\mathfrak{F}}(0)=1$. And $\mathfrak{f}(X)$ is equal to $\mathfrak{f}_{\mathfrak{F}}(cX)$ for some invertible element $c \in \mathfrak{o}^{\times}$. Conversely for any $c \in \mathfrak{o}^{\times}$, $\mathfrak{f}_{\mathfrak{F}}(cX)$ is an isomorphism from $X+Y$ to $\mathfrak{F}(X,Y)$. Throughout this paper we assume that

Assumption 2.

$$\operatorname{ord}_{q-1} \mathfrak{f}_{\mathfrak{F}}^{(n)}(0) \geq n-1 \quad \textit{for all} \;\; n \geq 1.$$

We note that by this assumption $\mathfrak{F}_n(A,B)$ (see Definition 1 below) and $\mathfrak{f}(a)$ are convergent in $\mathfrak{o}$ for any $A, B \in \mathfrak{o}[[X]]$ and $a \in \mathfrak{o}$ as formal power series (see [6, Remark 3]).

Definition 1. For each non-negative integer n, we denote the expansion of $\mathfrak{F}(X,Y)^n$ by

$$\mathfrak{F}(X,Y)^n = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} X^i Y^j,$$

and we set

$$\mathfrak{F}_n(A,B) = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} a_i b_j$$

for any power series $A = \sum_{n \geq 0} a_n \frac{X^n}{n!}$ and $B = \sum_{n \geq 0} b_n \frac{X^n}{n!}$ in $\mathfrak{o}[[X]]$. Then we define the $*_{\mathfrak{F}}$-product by

$$A *_{\mathfrak{F}} B = \sum_{n \geq 0} \mathfrak{F}_n(A,B) \frac{X^n}{n!}.$$

We can prove $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$ is an $\mathfrak{o}$-algebra (see [6, Proposition 1]). Next we extend the following map:

$$X^n \mapsto c^n \underbrace{X *_{\mathfrak{F}} \cdots *_{\mathfrak{F}} X}_{n \text{ times}}$$

$\mathfrak{o}$-linearly. Hence we can get a natural homomorphism from $(\mathfrak{o}[[X]], +, \cdot)$ to $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$. We call this map q-operator and denote the image of $A \in \mathfrak{o}[[X]]$ under the q-operator by $A_{\mathfrak{F},c}$. We define a q-analogue of power series A attached to $\mathfrak{F}$ and c by $A_{\mathfrak{F},c}$. The following proposition is essential for our theory of q-analogue (see [6, Theorem 1 and Proposition 2]).

Proposition 2. *For any $a, b \in \mathfrak{o}$, we have*

(i) $(e^{aX})_{\mathfrak{F},c} = e^{\mathfrak{f}(a)X}$,

(ii) $e^{\mathfrak{f}(a)X} *_{\mathfrak{F}} e^{\mathfrak{f}(b)X} = e^{\mathfrak{f}(a+b)X}$.

We define the q-Bernoulli numbers $\beta_n(\mathfrak{F}, c)$ attached to $\mathfrak{F}$ and c as follows:

Definition 2. For each non-negative integer n, we define the n-th q-Bernoulli number $\beta_n(\mathfrak{F}, c)$ attached to $\mathfrak{F}(X, Y) \in \mathfrak{o}[[X, Y]]$ and $c \in \mathfrak{o}^\times$ by the coefficient of $\frac{X^n}{n!}$ in $\mathfrak{B}_{\mathfrak{F},c} = \left(\frac{X}{e^X - 1}\right)_{\mathfrak{F},c}$.

We note that if $\mathfrak{F} = X + Y + (q-1)XY$ and $c = \frac{\log q}{q-1}$, then $\mathfrak{f}(X) = \frac{q^X - 1}{q-1}$ and $\beta_n(\mathfrak{F}, c)$ satisfies the following recurrence formula:

$$\beta_0(\mathfrak{F}, c) = 1, \quad (q\beta(\mathfrak{F}, c) + 1)^n - \beta_n(\mathfrak{F}, c) = \begin{cases} \frac{\log q}{q-1} & \text{for} \quad n = 1, \\ 0 & \text{for} \quad n > 1, \end{cases}$$

where we use the usual convention about replacing $\beta(\mathfrak{F}, c)^i$ by $\beta_i(\mathfrak{F}, c)$ for each non-negative integer i.

Proof. Apply the q-operator to $\mathfrak{B}e^X - \mathfrak{B} = X$, then we have $\mathfrak{B}_{\mathfrak{F},c} *_{\mathfrak{F}} e^X - \mathfrak{B}_{\mathfrak{F},c} = cX$ and $\mathfrak{F}_n(\mathfrak{B}_{\mathfrak{F},c}, e^X) = (q\beta(\mathfrak{F}, c) + 1)^n$. Hence the above recurrence formula holds. ∎

Now we may get a q-analogue of Proposition 1 by applying the q-operator to Lemma 1, but it is unknown the commutativity of the q-operator and the derivative on $\mathfrak{o}[[X]]$. So we need to take another approach to get a q-analogue of Lemma 1.

LEMMA 2. *For any power series A, B and non-negative integer n, we have*

$$(A *_{\mathfrak{F}} B)^{(n)} = \sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} A^{(i)} *_{\mathfrak{F}} B^{(j)}. \tag{1}$$

Proof. For any non-negative integer m, the coefficient of $\frac{X^m}{m!}$ in the left hand side of (1) is equal to

$$\mathfrak{F}_{m+n}(A, B) = \sum_{i,j\geq 0} \binom{m+n}{i,j}_{\mathfrak{F}} a_i b_j \ .$$

On the other hand that in the right hand of (1) is equal to

$$\sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} \mathfrak{F}(A^{(i)}, B^{(j)}) = \sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} \sum_{k,l\geq 0} \binom{m}{k,l}_{\mathfrak{F}} a_{i+k} b_{j+l}.$$

Hence it is sufficient to show that

$$\binom{m+n}{i,j}_{\mathfrak{F}} = \sum_{\substack{0\leq k\leq i\\ 0\leq l\leq j}} \binom{m}{k,l}_{\mathfrak{F}} \binom{n}{i-k, j-l}_{\mathfrak{F}}$$

for all $i \geq 0$ and $j \geq 0$. Because of $\mathfrak{F}(X,Y)^{m+n} = \mathfrak{F}(X,Y)^m \mathfrak{F}(X,Y)^n$, we can get what we want. ∎

Apply this lemma to $A *_{\mathfrak{F}} e^{\mathfrak{f}(1)X}$ and $e^{\mathfrak{f}(-1)X}$, then we have

LEMMA 3.

$$A^{(n)} *_{\mathfrak{F}} e^{\mathfrak{f}(1)X} = \sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} \mathfrak{f}(-1)^j (A *_{\mathfrak{F}} e^{\mathfrak{f}(1)X})^{(i)}.$$

This is a q-analogue of Lemma 1. If B satisfies Assumption 1, by applying the q-operator, we have

$$B_{\mathfrak{F},c} *_{\mathfrak{F}} e^{\mathfrak{f}(1)X} = B_{\mathfrak{F},c} + C_{\mathfrak{F},c}.$$

If C is a polynomial, then $C_{\mathfrak{F},c}$ is also a polynomial and $\deg C = \deg C_{\mathfrak{F},c}$ (see [6, Lemma 2]). Hence we have the following:

PROPOSITION 3. *If B satisfies* Assumption 1, *then we have*

$$\sum_{i,j\geq 0}\binom{m}{i,j}_{\mathfrak{F}}\mathfrak{f}(1)^j\beta_{n+i}=\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}}\mathfrak{f}(-1)^j(\beta_{m+i}+\gamma_{m+i})$$

for any non-negative integer m and n, where $B_{\mathfrak{F},c}=\sum_{n\geq 0}\beta_n\frac{X^n}{n!}$ and $C_{\mathfrak{F},c}=\sum_{n\geq 0}\gamma_n\frac{X^n}{n!}$.

If $m>\deg C$, then we have

COROLLARY 3.

$$\sum_{i,j\geq 0}\binom{m}{i,j}_{\mathfrak{F}}\mathfrak{f}(1)^j\beta_{n+i}=\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}}\mathfrak{f}(-1)^j\beta_{m+i}.$$

Furthermore if $m=n$, then we have

COROLLARY 4.

$$\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}}\{\mathfrak{f}(1)^j-\mathfrak{f}(-1)^j\}\beta_{n+i}=0\ .$$

Hence if $C=X$, then we get a recurrence formula for $\beta_n=\beta_n(\mathfrak{F},c)$. On the other hand if $C=X^2$, then β_n is the coefficient of $\frac{X^n}{n!}$ in

$$\left(\frac{X^2}{e^X-1}\right)_{\mathfrak{F},c}=cX *_{\mathfrak{F}}\mathfrak{B}_{\mathfrak{F},c}=cXd_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}),$$

where $d_{\mathfrak{F}}=\sum_{i\geq 0}\binom{1}{i,1}_{\mathfrak{F}}\frac{d^i}{dX^i}$ (see [6, Lemma 1]). Furthermore if $\mathfrak{F}(X,Y)=X+Y+(q-1)XY$ and $c=\frac{\log q}{q-1}$, then $\frac{1}{c}d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c})$ is equal to the generating function of Carlitz's q-Bernoulli numbers (see the next section). This means that we get a Kaneko's type of recurrence formula for Carlitz's q-Bernoulli numbers.

3. $X+Y+(q-1)XY$

If $\mathfrak{F}(X,Y)=X+Y+(q-1)XY$ and $c=\frac{\log q}{q-1}$, then we can state results in the previous section as follows:

COROLLARY 5. *If B satisfies* Assumption 1, *then we have*

(i) $\sum_{i=0}^{m} \binom{m}{i} q^{n+i}\beta_{n+i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}(\beta_{m+i}+\gamma_{m+i}),$

(ii) $\sum_{i=0}^{m} \binom{m}{i} q^{n+i}\beta_{n+i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}\beta_{m+i} \quad$ *if* $\quad m > \deg C,$

(iii) $\sum_{i=0}^{n} \binom{n}{i} \{q^{n+i} - (-1)^{n-i}\}\beta_{n+i} = 0 \qquad$ *if* $\quad m = n > \deg C,$

where $B_{\mathfrak{F},c} = \sum_{n\geq 0} \beta_n \frac{X^n}{n!}$ *and* $C_{\mathfrak{F},c} = \sum_{n\geq 0} \gamma_n \frac{X^n}{n!}$.

Proof. If $\mathfrak{F}(X,Y) = X + Y + (q-1)XY$, then, by the definition of $\binom{m}{i,j}_{\mathfrak{F}}$, we have

$$\binom{m}{i,j}_{\mathfrak{F}} = \binom{m}{i}\binom{i}{i+j-m}(q-1)^{i+j-m}. \tag{2}$$

Hence for any $a \in \mathfrak{o}$, we have

$$\begin{aligned}
&\sum_{i,j\geq 0} \binom{m}{i,j}_{\mathfrak{F}} \mathfrak{f}(a)^j \beta_{n+i} \\
&= \sum_{i,j\geq 0} \binom{m}{i}\binom{i}{i+j-m}(q-1)^{i+j-m}\mathfrak{f}(a)^j\beta_{n+i} \\
&= \sum_{i=0}^{m} \binom{m}{i}\mathfrak{f}(a)^{n-i}\beta_{n+i} \sum_{j=m-i}^{m} \binom{i}{i+j-m}\{(q-1)\mathfrak{f}(a)\}^{i+j-m} \\
&= \sum_{i=0}^{m} \binom{m}{i}\mathfrak{f}(a)^{n-i}\beta_{n+i}q^{ai}.
\end{aligned}$$

Hence we can get what we want. □

Let $\bar{\beta}_n(\mathfrak{F},c)$ be the coefficient of $\frac{X^n}{n!}$ in $\frac{1}{c}d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c})$, i.e., that of $c^2\frac{X^{n+1}}{n!}$ in $\left(\frac{X^2}{e^X-1}\right)_{\mathfrak{F},c}$, then $\bar{\beta}_n(\mathfrak{F},c)$ satisfies the following Carlitz's recurrence formula ([1]):

$$\bar{\beta}_0(\mathfrak{F},c) = 1, \quad q(q\bar{\beta}(\mathfrak{F},c)+1)^n - \bar{\beta}_n(\mathfrak{F},c) = \begin{cases} 1 & \text{for} \quad n = 1, \\ 0 & \text{for} \quad n > 1. \end{cases}$$

Hence $\bar{\beta}_n(\mathfrak{F}, c)$ is equal to the n-th Carlitz's q-Bernoulli number. To prove this we need the following:

Lemma 4. *If $\mathfrak{F}(X, Y) = X + Y + (q-1)XY$, then $d_{\mathfrak{F}}$ is a homomorphism on $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$.*

Proof. In this case we can write

$$d_{\mathfrak{F}}(A) = A + (q-1)A'$$

for any power series $A \in \mathfrak{o}[[X]]$. Hence by Lemma 2 we have

$$\begin{aligned} d_{\mathfrak{F}}(A *_{\mathfrak{F}} B) &= A *_{\mathfrak{F}} B + (q-1)(A *_{\mathfrak{F}} B)' \\ &= A *_{\mathfrak{F}} B + (q-1)\{A' *_{\mathfrak{F}} B + A *_{\mathfrak{F}} B' + (q-1)A' *_{\mathfrak{F}} B'\} \\ &= (A + (q-1)A') *_{\mathfrak{F}} (B + (q-1)B') \\ &= d_{\mathfrak{F}}(A) *_{\mathfrak{F}} d_{\mathfrak{F}}(B) \end{aligned}$$

for any A and B in $\mathfrak{o}[[X]]$. Hence $d_{\mathfrak{F}}$ is a homomorphism on $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$. ∎

Proof (Carlitz's recurrence formula). Apply $d_{\mathfrak{F}}$ to $\mathfrak{B}_{\mathfrak{F},c} *_{\mathfrak{F}} e^X - \mathfrak{B}_{\mathfrak{F},c} = cX$, then we have

$$d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}) *_{\mathfrak{F}} qe^X - d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}) = cX - \log q\ .$$

Hence $\bar{\beta}_n(\mathfrak{F}, c)$ satisfies Carlitz's recurrence formula. ∎

Finally we give another proof of Corollary 5.

Lemma 5. *If $\mathfrak{F}(X, Y) = X + Y + (q-1)XY$, then the $*_{\mathfrak{F}}$-product is written by*

$$A *_{\mathfrak{F}} X^n = d_{\mathfrak{F}}^n(A)X^n$$

for any non-negative integer n.

Proof. It is sufficient to prove for $\frac{X^i}{i!} *_{\mathfrak{F}} \frac{X^j}{j!}$ ($i \geq 0$ and $j \geq 0$). By the definition of $*_{\mathfrak{F}}$ and (2), we have

$$\begin{aligned} \frac{X^i}{i!} *_{\mathfrak{F}} \frac{X^j}{j!} &= \sum_{m \geq 0} \binom{m}{i, j}_{\mathfrak{F}} \frac{X^m}{m!} \\ &= \sum_{m=j}^{i+j} \binom{m}{i} \binom{i}{i+j-m} (q-1)^{i+j-m} \frac{X^m}{m!}. \end{aligned}$$

On the other hand

$$d^i_{\mathfrak{F}}\left(\frac{X^j}{j!}\right)\frac{X^i}{i!} = \sum_{m=0}^{i}\binom{i}{m}(q-1)^m\frac{d^m}{dX^m}\left(\frac{X^j}{j!}\right)\frac{X^i}{i!}$$
$$= \sum_{m=0}^{i}\binom{i}{m}\binom{i+j-m}{i}(q-1)^m\frac{X^{i+j-m}}{(i+j-m)!}.$$

Hence we have what we want. □

Remark 1. If $\mathfrak{F} = X + Y + (q-1)XY$ and $c = \frac{\log q}{q-1}$, then, by Lemma 5, we can get

$$A(X) *_{\mathfrak{F}} e^{\mathfrak{f}(a)} = A(q^a X)e^{\mathfrak{f}(a)}$$

for any power series $A \in \mathfrak{o}[[X]]$ and $a \in \mathfrak{o}$. By this we can get Corollary 5 from Lemma 1 not using Lemma 2.

Remark 2. Lemma 4 and Lemma 5 hold only for $\mathfrak{F}(X,Y) = X + Y + (q-1)XY$ (see [6, Lemma 1 and Proposition 4]).

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