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ON SOME 3-DIMENSIONAL CR SUBMANIFOLDS IN S^6

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Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

Abstract. We give two types of 3-dimensional CR-submanifolds of the 6-dimensional sphere. First we study whether there exists a 3-dimensinal CR-submanifold which is obtained as an orbit of a 3-dimensional simple Lie subgroup of G_2. There exists a unique (up to G_2) 3-dimensional CR-submanifold which is obtained as an orbit of reducible representations of $SU(2)$ on $\mathbf{R}^7$. As orbits of the subgroup which corresponds to the irreducible representation of $SU(2)$ on $\mathbf{R}^7$, we obtained 2-parameter family of 3-dimensional CR-submanifolds. Next we give a generalization of the example which was obtained by K. Sekigawa.

Introduction

Let $(M, J, \langle , \rangle)$ be an almost Hermitian manifold. For a submanifold N of M, we put $\mathcal{H}_x = T_xN \cap J(T_xN)$ $(x \in N)$ and denote by $\mathcal{H}_x^\perp$ the orthogonal complement of $\mathcal{H}_x$ in T_xN. If the dimension of $\mathcal{H}_x$ is constant and $J(\mathcal{H}_x^\perp) \subset T_x^\perp N$ for any $x \in N$, the submanifold N is called a *CR submanifold*. Especially if $\mathcal{H}_x = T_xN$, the submanifold N is said to be a *holomorphic* (or *invariant*) submanifold and if $\dim(\mathcal{H}_x) = 0$ and $J(T_xN) \subset T_x^\perp N$ for any $x \in N$, the submanifold N is said to be a *totally real submanifold*.

It is well-known that the 6-dimensional sphere S^6 admits an almost complex structure. On the existence of holomorphic or totally real submanifold of S^6, many results are obtained. A. Gray proved that there does not exist any 4-dimensional holomorphic submanifold ([7]) and R. Bryant proved that there exist infinitely many 2-dimensional holomorphic submanifolds ([1]). It was proved by Ejiri that any 3-dimensional totally real sub-

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maifold of S^6 is a minimal submanifold ([4]). He also proved that some tubes in the direction of the first and the second normal bundle of holomorphic curves are totally real submanifolds of S^6 ([5]). The second author classified 3-dimensional homogeneous minimal submanifolds of S^6 and determined all 3-dimensional homogeneous totally real submanifolds of S^6 ([11]).

Though there are many results on the existence of holomorphic submanifolds and totally real submanifolds of S^6, only one example is known about the existence of CR submanifold of S^6 ([13]).

The aim of this paper is to give many 3-dimensional CR submanifolds of S^6 with $\dim_{\mathbf{R}} \mathcal{H} = 2$. Second author proved that a 3-dimensional subspace V in $\mathbf{C}^3$ satisfies $\dim_{\mathbf{R}}(V \cap J(V)) = 2$ if and only if $\omega(V) = 0$, where J is the complex structure and ω is the Laglangean 3-form. The fact is also used in this paper.

§1. Preliminaries

1.1. Cayley algebra

Let $\mathbf{H}$ be the skew field of all quaternions. The Cayley algebra $\mathfrak{C}$ over $\mathbf{R}$ is $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}$ with the following multiplication;

$$(q, r) \cdot (s, t) = (qs - \overline{t}r, tq + r\overline{s}), \quad q, r, s, t \in \mathbf{H}$$

where "$\overline{}$" means the conjugation in $\mathbf{H}$. We define a conjugation in $\mathfrak{C}$ by $\overline{(q, r)} = (\overline{q}, -r), q, r \in \mathbf{H}$, and an inner product $\langle , \rangle$ by

$$\langle x, y \rangle = (x \cdot \overline{y} + y \cdot \overline{x})/2, \quad x, y \in \mathfrak{C}.$$

We put

$$\mathfrak{C}_0 = \{x \in \mathfrak{C} | x + \overline{x} = 0\}.$$

The Cayley algebra $\mathfrak{C}$ is neither commutative nor associative. But we have the following

(1) If $x, y \in \mathfrak{C}_0$, then $x \cdot y = -y \cdot x$.

(2) For any $x, y, z \in \mathfrak{C}$,

$$\overline{x} \cdot (x \cdot y) = (\overline{x} \cdot x) \cdot y, \quad \langle x \cdot y, x \cdot z \rangle = \langle x, x \rangle \langle y, z \rangle .$$

(3) If $x, y, z \in \mathfrak{C}$ are mutually orthogonal unit vectors,

$$x \cdot (y \cdot z) = y \cdot (z \cdot x) = z \cdot (x \cdot y).$$

The unit sphere $S^6 \subset \mathfrak{C}_0$ centered at the origin has an almost complex structure J defined by

$$J_p(X) = p \cdot X \quad p \in S^6,\ X \in T_pS^6.$$

We use the canonical orthonormal basis $e_0 = (1,0)$, $e_1 = (i,0)$, $e_2 = (j,0)$, $e_3 = (k,0)$, $e_4 = (0,1)$, $e_5 = (0,i)$, $e_6 = (0,j)$, $e_7 = (0,k)$ of the Cayely algebra, where $1, i, j, k$ is the standard orthonormal basis of $\mathbf{H}$. The vector e_0 is the unit element of $\mathfrak{C}$ and the product $e_i \cdot e_j$ is given in the following table;

$i\backslash j$	1	2	3	4	5	6	7
1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

1.2. Exceptional simple Lie group G_2

It is well-known that the group of all automorphisms of $\mathfrak{C}$ is a compact connected simple Lie group of type $\mathfrak{g}_2$ ([6]), which we denote by G_2. The group G_2 leaves the vector e_0 and the subspace $\mathfrak{C}_0 = \sum_{i=1}^{7} \mathbf{R}e_i$ invariant. Furthermore G_2 leaves the inner product $\langle , \rangle$ invariant. If we identify $\mathfrak{C}_0$ with the set of all 7-dimensional column vectors in a natural manner, then G_2 is a subgroup of $SO(7)$.

Lemma 1. *For a pair of mutually orthogonal unit vectors a_4, a_1 in $\mathfrak{C}_0$ put $a_5 = a_1 \cdot a_4$. Take a unit vector a_2, which is perpendicular to a_4, a_1 and a_5. If we put $a_3 = a_1 \cdot a_2$, $a_6 = a_2 \cdot a_4$ and $a_7 = a_3 \cdot a_4$ then the matrix*

$$g = (\, a_1, a_2, a_3, a_4, a_5, a_6, a_7\,) \in SO(7)$$

is an element of G_2 with $g \cdot e_4 = a_4$.

For the proof of Lemma 1, we refer to [8].

Let G_{ij} $(1 \leq i \neq j \leq 7)$ be the skew symmetric transformation on $\mathfrak{C}_0$ defined by

$$G_{ij}(e_k) = \begin{cases} e_i, & \text{if } k = j, \\ -e_j, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

The Lie algebra $\mathfrak{g}_2$ of G_2 is spanned by the following vectors in the Lie algebra $\mathfrak{so}(7)$ of $SO(7)$;

$$\begin{cases} aG_{23} + bG_{45} + cG_{76}, \\ aG_{31} + bG_{46} + cG_{57}, \\ aG_{12} + bG_{47} + cG_{65}, \\ aG_{51} + bG_{73} + cG_{62}, \\ aG_{14} + bG_{72} + cG_{36}, \\ aG_{17} + bG_{24} + cG_{53}, \\ aG_{61} + bG_{34} + cG_{25}, \end{cases}$$

where $a,\ b,\ c$ are real numbers with $a + b + c = 0$.

1.3. A criterion for a CR subspace

Let J be the standard complex structure on $\mathbf{C}^3$ with the standard Hermitian metric. Take an orthonormal basis e_1, e_2, e_3, $e_4 = J(e_1)$, $e_5 = J(e_2)$, $e_6 = J(e_3)$ of $\mathbf{C}^3$. We denote by $\omega_1, \cdots, \omega_6$ the orthonormal coframe on $\mathbf{C}^3$ dual to $e_1, \cdots, e_6$. Put

$$\omega = (\omega_1 + \sqrt{-1}\omega_4) \wedge (\omega_2 + \sqrt{-1}\omega_5) \wedge (\omega_3 + \sqrt{-1}\omega_6).$$

Remember that ω depends on the choise of the basis $e_1, \cdots, e_6$. For an element $g \in U(3)$ we have

$$g^*\omega = \det(g)\omega.$$

Proposition 2. *A 3-dimensional real subspace V of $\mathbf{C}^3$ satisfies $\dim_{\mathbf{R}}(V \cap J(V)) = 2$ if and only if $\omega(V) = 0$.*

If a 3-dimensional real subspace V of $\mathbf{C}^3$ satisfies $\dim_{\mathbf{R}}(V \cap J(V)) = 2$ then it also satisfies $J\left((V \cap JV)^{\perp} \cap V\right) \subset V^{\perp}$. For a 3-dimensional CR submanifold of a 6-dimensional almost complex manifold which is not a totally real submanifold we have $\dim_{\mathbf{R}}(T_xN \cap J(T_xN)) = 2$. Thus we have the following

Corollary 3. *Let M be a 6-dimensional almost complex manifold. A 3-dimensional submanifold N of M is a CR submanifold with $\dim \mathcal{H} = 2$ if and only if $\omega(T_xN) = 0$ for any $x \in N$.*

§2. Orbits of TDS in G_2

In this section, we study 3-dimensional CR submanifolds which are orbits of some 3-dimensional simple subgroup (abbreviated as TDS) of G_2.

2.1. Classification of TDS in G_2

Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Let $\mathfrak{u}$ be a simple 3-dimensional subalgebra of $\mathfrak{g}$. Take a basis X_1, X_2, X_3 of $\mathfrak{u}$ with

$$[X_1, X_2] = 2X_3,\ [X_2, X_3] = 2X_1,\ [X_3, X_1] = 2X_2 \tag{1}$$

and put

$$\begin{cases} H &= \sqrt{-1}X_1, \\ X_+ &= (1/\sqrt{2})(X_2 + \sqrt{-1}X_3), \\ X_- &= (1/\sqrt{2})(-X_2 + \sqrt{-1}X_3). \end{cases}$$

The bracket products of the basis H, X_+, X_- of $\mathfrak{u}^{\mathbf{C}}$ are

$$[H, X_+] = 2X_+,\ [H, X_-] = -2X_-,\ [X_+, X_-] = H. \tag{2}$$

We may assume that H is contained in $\sqrt{-1}\mathfrak{t}$. Hence $\alpha(H)$ is a real number for every root α of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$. Furthermore $\alpha(H) = 0, 1$ or 2 if α is a simple root ([3, p.166]). The weighted Dynkin diagram with weight $\alpha(H)$ added to each vertex α of the Dynkin diagram of $\mathfrak{g}^{\mathbf{C}}$ is called the *characteristic diagram* of $\mathfrak{u}$. Let $\mathfrak{u}$ and $\mathfrak{u}'$ be 3-dimensional simple Lie subalgebras of $\mathfrak{g}$. Then $\mathfrak{u}$ and $\mathfrak{u}'$ are mutually conjugate in $\mathfrak{g}$ if and only if $\mathfrak{u}^{\mathbf{C}}$ and $\mathfrak{u}'^{\mathbf{C}}$ have the same characteristic diagram.

Mal'cev [10] classified the 3-dimensional complex simple subalgebras of $\mathfrak{g}_2^{\mathbf{C}}$. From his classification, $\mathfrak{g}_2$ has 4 types of 3-dimensional simple subalgebras.

Type I $\overset{1}{\circ}\Rrightarrow\overset{0}{\circ}$ Type II $\overset{0}{\circ}\Rrightarrow\overset{1}{\circ}$

Type III $\overset{2}{\circ}\Rrightarrow\overset{0}{\circ}$ Type IV $\overset{2}{\circ}\Rrightarrow\overset{2}{\circ}$

We shall study 3 dimensional homogeneous CR submanifolds of S^6 which are orbits of 3 dimensional simple Lie subgroup of G_2. We denote by ω_i the orthogonal coframes on $\mathfrak{C}_0$ dual to e_i. We also denote by ω_i the restriction of ω_i to S^6. Since $J_{e_4}(e_1) = -e_5$, $J_{e_4}(e_2) = -e_6$ and $J_{e_4}(e_3) = -e_7$, we have

$$\omega_{|e_4} = (\omega_1 - \sqrt{-1}\omega_5) \wedge (\omega_2 - \sqrt{-1}\omega_6) \wedge (\omega_3 - \sqrt{-1}\omega_7).$$

2.2. Orbit of the TDS of type I

A basis of the subalgebra with (1) corresponding to the characteristic diagram of type I is as follows;

$$\begin{cases} X_1 &= -G_{45} + G_{76}, \\ X_2 &= -G_{46} + G_{57}, \\ X_3 &= -G_{47} + G_{65}. \end{cases}$$

We denote by U_1 the Lie subgroup of G_2 generated by the subalgebra. The subgroup U_1 is isomorphic to $Sp(1)$ and acts on $\mathfrak{C}_0$ as follows;

$$q \cdot (x, y) = (x, y\overline{q}), \quad q \in Sp(1).$$

In this case, $\mathbf{R}e_1$, $\mathbf{R}e_2$, $\mathbf{R}e_3$ and $\sum_{j=4}^{7} \mathbf{R}e_j$ are invariant irreducible subspaces so that each orbit is a small sphere or a great sphere.

2.3. Orbit of the TDS of type II

A basis satisfying (1) of the subalgebra corresponding to the characteristic diagram of type II is as follows;

$$\begin{cases} X_1 &= -2G_{23} + G_{45} + G_{76}, \\ X_2 &= -2G_{31} + G_{46} + G_{57}, \\ X_3 &= -2G_{12} + G_{47} + G_{65}. \end{cases}$$

We denote by U_2 the Lie subgroup of G_2 generated by the subalgebra. The subgroup U_2 is isomorphic to $Sp(1)$ and acts on $\mathfrak{C}_0$ as follows;

$$q \cdot (x, y) = (qx\overline{q}, y\overline{q}), \quad q \in Sp(1).$$

THEOREM 4. *Let N be the orbit of U_2 through the point $p_0 = (1/3)e_2 + (2\sqrt{2}/3)e_4$. Any 3 dimensional CR submanifold of S^6, which is an orbit of U_2 in S^6, is congruent to N under the action of G_2 on S^6.*

Proof. Take a point p on S^6 and consider the orbit $M = U_2 \cdot p$ of U_2 through p. Since the action of $Sp(1)$ on $S^3 \subset H$ by $y \to y\overline{q}$ $\quad (q \in Sp(1))$ is transitive, we may assume that p is of the form $p = \sum_{i=1}^{4} x_i e_i$. Put

$$g_t = \exp(t(X_3 - (G_{47} - G_{65}))) = \exp(-2t(G_{12} - G_{65}))$$

and consider the one parameter subgroup $Z = \{g_t : t \in \mathbf{R}\}$. Since $G_{47} - G_{65}$ commutes with X_1, X_2 and X_3 we have

$$U_2 \cdot g_t \cdot p = g_t \cdot M.$$

Namely the orbit M is congruent to the orbit through $p' = \sum_{i=2}^{4} x_i e_i$. If $x_4 = 0$ then we have $\dim(M) = 2$. Thus we assume $x_4 \neq 0$.

Put $a_4 = p'$, $a_1 = e_6$ and $a_5 = a_1 \cdot a_4$. The vector $a_2 = c(x_4 e_1 + x_2 e_7)$ $(c = 1/\sqrt{x_2^2 + x_4^2})$ is orthogonal to a_4, a_1 and a_5. Thus by Lemma 1, the matrix

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ cx_4 & 0 & 0 & 0 & 0 & 0 & cx_2 \\ -cx_2 & 0 & 0 & 0 & 0 & 0 & cx_4 \\ 0 & x_2 & x_3 & x_4 & 0 & 0 & 0 \\ 0 & -x_4 & 0 & x_2 & -x_3 & 0 & 0 \\ 0 & -cx_3x_4 & 0 & cx_2x_3 & 1/c & 0 & 0 \\ 0 & cx_2x_3 & -1/c & cx_3x_4 & 0 & 0 & 0 \end{pmatrix}$$

is an element of G_2 with $g \cdot p' = e_4$.

Substitute

$$\begin{aligned} v_1 &= g_*(X_1(p')) = (3x_3x_4)e_5 + cx_4(3x_3^2 - 1)e_6 - 2cx_2e_7, \\ v_2 &= g_*(X_2(p')) = -x_4e_1 + (2cx_3x_4)e_2 - (2cx_2x_3)e_3 \\ v_3 &= g_*(X_3(p')) = -3cx_2x_4e_2 + c(2x_2^2 - x_4^2)e_3, \end{aligned}$$

into $\omega|_{e_4}$, we have

$$\omega|_{e_4}(v_1, v_2, v_3) = \sqrt{-1}c^2x_4^2(8x_2^2 + x_4^2(9x_3^2 - 1)).$$

Thus the orbit $M = U_2(p')$ through the point $p' = x_2e_2 + x_3e_3 + x_4e_4$ is a 3-dimensional CR submanifold of S^6 if and only if

$$\begin{cases} x_4 \neq 0, \\ x_2^2 + x_3^2 + x_4^2 = 1, \\ 8x_2^2 + x_4^2(9x_3^2 - 1) = 0. \end{cases}$$

The solution of the above equations is as follows;

$$x_2^2 + x_3^2 = 1/9, \; x_4^2 = 8/9. \tag{3}$$

Every orbit through a point which satisfies (3) is congruent to N by $\exp(t(G_{23} - G_{76})) \in G_2$ for some $t \in \mathbf{R}$. ∎

2.4. Orbit of the TDS of type III

A basis satisfying (1) of the subalgebra corresponding to the characteristic diagram of type III is as follows;

$$\begin{cases} X_1 &= -2G_{21} - 2G_{65}, \\ X_2 &= -2G_{32} - 2G_{76}, \\ X_3 &= -2G_{31} - 2G_{75}. \end{cases}$$

We denote by U_3 the Lie subgroup of G_2 generated by the subalgebra. The subgroup U_3 is isomorphic to $SO(3)$ and the covering group $Sp(1)$ of U_3 acts on $\mathfrak{C}_0$ as follows;

$$q \cdot (x, y) = (qx\overline{q}, qy\overline{q}), \quad q \in Sp(1).$$

THEOREM 5. *There does not exist any* 3 *dimensional CR submanifold of* S^6 *which is an orbit of the subgroup* U_3.

Proof. Take a point p on S^6 and consider the orbit $M = U_3 \cdot p$ of U_3 through p. Since the action of $Sp(1)$ on S^2 by $x \to qx\overline{q} \quad (q \in Sp(1))$ is transitive, we may assume that p is of the form $p = x_1e_1 + x_4e_4 + x_5e_5 + x_6e_6$. Put $a_4 = p$, $a_1 = e_7$ and $a_5 = a_1 \cdot a_4$. If $x_1 = 0$ then we have $\dim(M) = 2$. Thus we assume $x_1 \neq 0$. The vector $a_2 = c(x_4e_1 + x_6e_3 - x_1e_4) \quad (c = 1/\sqrt{x_1^2 + x_4^2 + x_6^2})$ is orthogonal to a_4, a_1 and a_5. Thus by Lemma 1, the matrix

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ cx_4 & 0 & cx_6 & -cx_1 & 0 & 0 & 0 \\ 0 & 0 & cx_1 & cx_6 & 0 & -cx_4 & 0 \\ x_1 & 0 & 0 & x_4 & x_5 & x_6 & 0 \\ x_6 & -x_5 & -x_4 & 0 & 0 & -x_1 & 0 \\ -cx_1x_5 & 0 & 0 & -cx_4x_5 & 1/c & -cx_5x_6 & 0 \\ cx_5x_6 & 1/c & -cx_4x_5 & 0 & 0 & -cx_1x_5 & 0 \end{pmatrix}$$

is an element of G_2 with $g \cdot p = e_4$. Substitute

$$\begin{aligned} v_1 &= g_*(X_1(p)) = (2x_6, 0, 0, 0, 0, 0, 0), \\ v_2 &= g_*(X_2(p)) = (-2x_5, -2cx_1x_6, -2cx_1^2, 0, 2x_1x_4, 0, 2cx_1x_4x_5), \\ v_3 &= g_*(X_3(p)) = (0, 0, -2cx_4x_5, 0, -4x_1x_5, -2cx_6, 2cx_1(1 - 2x_5^2)), \end{aligned}$$

into $\omega|_{e_4}$, we have

$$\omega|_{e_4}(v_1, v_2, v_3) = 16c^2x_1^2x_6^2\sqrt{-1}(1 - x_5^2).$$

If we assume $\omega(v_1, v_2, v_3) = 0$, we have $x_1 = 0$, $x_6 = 0$ or $x_5 = \pm 1$. In any case, the dimension of the orbit is equal to 2. Thus there does not exist any 3 dimensional orbit which is a CR submanifold of S^6. ∎

2.5. Orbit of the TDS of type IV

A basis satisfying (1) of the subalgebra corresponding to the characteristic diagram of type IV is as follows;

$$\begin{cases} X_1 &= 4G_{32} + 2G_{54} + 6G_{76}, \\ X_2 &= \sqrt{6}(G_{37} + G_{26} - 2G_{15}) + \sqrt{10}(G_{42} - G_{35}), \\ X_3 &= \sqrt{6}(G_{63} + G_{27} - 2G_{41}) + \sqrt{10}(G_{25} - G_{34}). \end{cases}$$

We denote by U_4 the Lie subgroup of G_2 generated by the subalgebra. The subgroup U_4 is isomorphic to $SO(3)$.

From Lemma 1 in [2], the linear subspace $((\mathbf{R}X_1 + \mathbf{R}X_2 + \mathbf{R}X_3)e_7)^\perp$ meets every orbit of the action of U_4 on $\mathfrak{C}_0$. So the great sphere $S^3 = \{x_1e_1 + x_4e_4 + x_5e_5 + x_7e_7 : x_1, x_4, x_5, x_7 \in \mathbf{R}\} \cap S^6$ meets every orbit of the action of U_4 on S^6.

THEOREM 6. *If the dimension of the orbit $N = U_4 \cdot p$ through a point p of the great sphere*

$$\{x_1e_1 + x_4e_4 + x_5e_5 + x_7e_7 : x_1, x_4, x_5, x_7 \in \mathbf{R}\} \cap S^6$$

is 3, *then it is a CR-submanifold if and only if $f(x_1, x_4, x_5, x_7) = 0$ where*

$$\begin{aligned} f(x_1, x_4, x_5, x_7) = -5x_4^4 - 10x_4^2x_5^2 - 5x_5^4 + 42x_4^2x_7^2 + 72x_1^2x_7^2 + 42x_5^2x_7^2 \\ -9x_7^4 - 24\sqrt{15}x_4^2x_5x_7 + 8\sqrt{15}x_5^3x_7. \end{aligned}$$

Proof. Put $a_4 = x_1e_1 + x_4e_4 + x_5e_5 + x_7e_7$, $a_1 = e_2$ and $a_2 = c(x_5e_4 - x_4e_5 - x_7e_6)$ $(c = 1/\sqrt{x_5^2 + x_4^2 + x_7^2})$. From Lemma 1, we obtain an element

$$g = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 & 1/c & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 0 & 1/c \\ 0 & cx_5 & cx_7 & x_4 & 0 & -cx_1x_4 & 0 \\ 0 & -cx_4 & 0 & x_5 & -x_7 & cx_1x_5 & -cx_1x_7 \\ 0 & -cx_7 & cx_5 & 0 & x_4 & 0 & cx_1x_4 \\ 0 & 0 & -cx_4 & x_7 & x_5 & -cx_1x_7 & cx_1x_5 \end{pmatrix}$$

of G_2 with $g \cdot e_4 = a_4$. The vectors $v_i = g_*^{-1}(X_i(p))$ are given as follows;

$$\begin{aligned}
v_1 &= (0, 2c(-x_4^2 - x_5^2 + 3x_7^2), -8cx_5x_7, 0, -8x_4x_7, 0, -8cx_1x_4x_7),\\
v_2 &= (-\sqrt{10}x_4, -2\sqrt{6}cx_1x_4, 0, 0, \sqrt{10}x_1x_5 - 3\sqrt{6}x_1x_7,\\
&\qquad -2\sqrt{6}cx_5, -2\sqrt{6}cx_1^2x_7 + (1/c)(-\sqrt{10}x_5 + \sqrt{6}x_7)),\\
v_3 &= (\sqrt{10}x_5 + \sqrt{6}x_7, -2\sqrt{6}cx_1x_5, -2\sqrt{6}cx_1x_7, 0, \sqrt{10}x_1x_4,\\
&\qquad 2\sqrt{6}cx_4, -\sqrt{10}(1/c)x_4).
\end{aligned}$$

Using the Mathematica we obtained the following

$$\begin{aligned}
&\omega(v_1, v_2, v_3)\\
= \;& 24\sqrt{15}x_1x_4^3x_7 - 24\sqrt{15}c^2x_1x_4^3x_7 + 24\sqrt{15}c^2x_1^3x_4^3x_7 - 40\sqrt{15}x_1x_4x_5^2x_7\\
&+40\sqrt{15}c^2x_1x_4x_5^2x_7 - 40\sqrt{15}c^2x_1^3x_4x_5^2x_7 + 96x_1x_4x_5x_7^2 - 96c^2x_1x_4x_5x_7^2\\
&+96c^2x_1^3x_4x_5x_7^2 + 24\sqrt{15}x_1x_4x_7^3 - 24\sqrt{15}c^2x_1x_4x_7^3 + 24\sqrt{15}c^2x_1^3x_4x_7^3\\
&+ \sqrt{-1}\,(-20x_4^4 - 40x_4^2x_5^2 - 20x_5^4 - 64\sqrt{15}x_4^2x_5x_7 - 32\sqrt{15}c^2x_4^2x_5x_7\\
&+32\sqrt{15}c^2x_1^2x_4^2x_5x_7 + 32\sqrt{15}c^2x_5^3x_7 - 32\sqrt{15}c^2x_1^2x_5^3x_7 + 168x_4^2x_7^2\\
&+288c^2x_1^2x_4^2x_7^2 + 72x_5^2x_7^2 + 96c^2x_5^2x_7^2 + 192c^2x_1^2x_5^2x_7^2 - 36x_7^4 + 288c^2x_1^2x_7^4).
\end{aligned}$$

By a tedious calculation, we verified that the real part of the above vanishes and the imaginary part of the above reduces to $f(x_1, x_4, x_5, x_7)$. $\square$

Remark 7. Put $g(x_1, x_4, x_5, x_7) = x_1^2 + x_4^2 + x_5^2 + x_7^2 - 1$. It is easily verified that $f(x_1, x_4, x_5, x_7) = g(x_1, x_4, x_5, x_7) = 0$ hold at the point $(x_1, x_4, x_5, x_7) = (\pm 1/3, 0, 0, \pm 2\sqrt{2}/3)$ and the dimension of the orbit through $p = x_1e_1 + x_4e_4 + x_5e_5 + x_7e_7$ is 3. Furthermore, since the Jacobian $\partial(f, g)/\partial(x_1, x_7)$ is regular at the point (x_1, x_4, x_5, x_7), there exist a 2-parameter family of 3-dimensional CR submanifolds.

§3. Generalization of Sekigawa's example

3.1. Sekigawa's example and its generalization

In [13], Sekigawa obtained an example of 3-dimensional CR submanifold of S^6. His example was given as the image of the mapping of $S^2 \times S^1$ into S^6;

$$\begin{aligned}
\Psi(y, t) &= \Psi((y_2, y_4, y_6), e^{\sqrt{-1}t})\\
&= (y_2 \cos t)e_2 - (y_2 \sin t)e_3 + (y_4 \cos 2t)e_4 + (y_4 \sin 2t)e_5\\
&\quad +(y_6 \cos t)e_6 + (y_6 \sin t)e_7.
\end{aligned}$$

where $(y_2, y_4, y_6) \in S^2$ and $e^{\sqrt{-1}t} \in S^1$.

For a real triple $p = (p_1, p_2, p_3)$ with $p_1 + p_2 + p_3 = 0$ and $p_1\ p_2\ p_3 \neq 0$, define a mapping ψ_p of $S^2 \times R$ to $S^5 \subset S^6$ as follows;

$$\begin{aligned} &\psi_p(x_1, x_2, x_3, t) \\ = &\exp(t(p_1G_{51} + p_2G_{62} + p_3G_{73}))(x_1e_1 + x_2e_2 + x_3e_3) \\ = &\quad x_1(\cos(tp_1)e_1 + \sin(tp_1)e_5) + x_2(\cos(tp_2)e_2 + \sin(tp_2)e_6) \\ &+x_3(\cos(tp_3)e_3 + \sin(tp_3)e_7), \end{aligned}$$

where $(x_1)^2 + (x_2)^2 + (x_3)^2 = 1$ and $t \in \mathbf{R}$. We use another expression;

$$\psi_p(x_1, x_2, x_3, t) = (x_1, x_2, x_3)R_p(t),$$

where $R_p(t)$ is the $\mathfrak{C}$-valued $(3,1)$-matrix

$$R_p(t) = \begin{pmatrix} \cos(tp_1)e_1 + \sin(tp_1)e_5 \\ \cos(tp_2)e_2 + \sin(tp_2)e_6 \\ \cos(tp_3)e_3 + \sin(tp_3)e_7 \end{pmatrix}.$$

It is easily seen that there exists an element $g \in G_2$ with $\Psi = g \circ \psi_{(2,-1,-1)}$.

The tangent space $d\psi_{(p_1,p_2,p_3)}(T_xS^2 \oplus T_t\mathbf{R})$ is generated by

$$\begin{aligned} d\psi_p((v,\ 0)) &= (v_1, v_2, v_3)R_p(t), \\ d\psi_p((0, D_t)) &= (x_1p_1, x_2p_2, x_3p_3)R'_p(t), \end{aligned}$$

where $v = (v_1, v_2, v_3)$ is a tangent vector of S^2, $D_t = \partial/\partial t$ is a tangent vector of $\mathbf{R}$ and

$$R'_p(t) = \begin{pmatrix} -\sin(tp_1)e_1 + \cos(tp_1)e_5 \\ -\sin(tp_2)e_2 + \cos(tp_2)e_6 \\ -\sin(tp_3)e_3 + \cos(tp_3)e_7 \end{pmatrix}.$$

We can easily verify that

$$\left\{ \begin{array}{l} \langle XR_p(t), YR_p(t)\rangle = \left\langle XR'_p(t), YR'_p(t)\right\rangle = \langle X, Y\rangle, \\ \left\langle XR_p(t), YR'_p(t)\right\rangle = 0. \end{array} \right. \tag{4}$$

hold for any $X,\ Y \in \mathbf{R}^3$. By a direct calculation, we have the following

Lemma 8. *The induced metric $\tilde{g}$ on $S^2 \times \mathbf{R}$ is a warped product metric. Precisely*

$$\widetilde{g} = {\pi_1}^* g_0 + \left(\sum_{i=1}^{3} (x_ip_i)^2\right) {\pi_2}^* dt^2$$

where $\pi_1 : S^2 \times \mathbf{R} \to S^2$ and $\pi_2 : S^2 \times \mathbf{R} \to \mathbf{R}$ are natural projections and g_0 is the canonical Riemannian metric on S^2.

From (4), we have the following orthogonal direct sum decomposition

$$\mathfrak{C}_0 = V \oplus V' \oplus \mathbf{R}e_4$$

where we put

$$V = \{XR_p(t) : X \in \mathbf{R}^3\},\ V' = \{XR'_p(t) : X \in \mathbf{R}^3\}.$$

THEOREM 9. *Let $p = (p_1, p_2, p_3)$ be a real triple with $p_1 + p_2 + p_3 = 0$ and $p_1\ p_2\ p_3 \neq 0$. The image of the mapping*

$$\psi_p(x_1, x_2, x_3, t) : S^2 \times \mathbf{R} \to S^5 \subset S^6$$

is a 3-dimensional CR-submanifolds of S^6.

Proof. Let $x = (x_1, x_2, x_3)$ be an element of S^2 and $v = (v_1, v_2, v_3)$ be a tangent vector of S^2 at x. By direct calculation, we have

$$\begin{aligned}
& J\left(d\psi_p((v,\ 0))\right) \\
= \ & (v_3x_2 - v_2x_3)\cos(p_1t)e_1 + (-v_3x_1 + v_1x_3)\cos(p_2t)e_2 \\
& +(v_2x_1 - v_1x_2)\cos(p_3t)e_3 - (-v_3x_2 + v_2x_3)\sin(p_1t)e_5 \\
& -(v_3x_1 - v_1x_3)\sin(p_2t)e_6 - (-v_2x_1 + v_1x_2)\sin(p_3t)e_7 \\
= \ & (x \times v)R_p(t).
\end{aligned}$$

Thus we have $d\psi_p(T_xS^2 \oplus \{0\})$ is a J-invariant subspace. Since the image of the mapping ψ_p is 3-dimensional, we obtain the theorem. □

For a non zero constant k we can easily see

$$\psi_{(kp_1,kp_2,kp_3)}(x, t) = \psi_{(p_1,p_2,p_3)}(x, kt).$$

Thus we may assume that $p_3 = 1$.

Remark 10. (1) If p_1/p_2 is a rational number, then $\psi_{(p_1,p_2,p_3)}$ is an immersion but not injective, and its image is a compact manifold.
(2) If p_1/p_2 is an irrational number, then $\psi_{(p_1,p_2,p_3)}$ is an injective immersion but not an embedding.
(3) Let τ be a permutation of 3 characters and put $p' = \tau p$. There exists an element $g \in G_2$ such that $\psi_{p'} = g \circ \psi_p$.

Next we shall calculate the second fundamental form of the immersion $\psi_{(p_1,p_2,p_3)}$.

LEMMA 11. *For any $v, w \in T_xS^2$, $D_t \in T_t\mathbf{R}$ we have*

(1) $\sigma(v, w) = 0,$

(2) $\sigma(D_t, D_t) = 0,$

(3)

$$\begin{aligned}&\sigma(v,\xi)\\ &= \frac{1}{\sqrt{f(x)}}\left(v - \frac{1}{2}v(\log(f(x)) \cdot x\right)\begin{pmatrix} p_1(-\sin(tp_1)e_1 + \cos(tp_1)e_5) \\ p_2(-\sin(tp_2)e_2 + \cos(tp_2)e_6) \\ p_3(-\sin(tp_3)e_3 + \cos(tp_3)e_7)\end{pmatrix}\end{aligned}$$

where $f(x) = \sum_{i=1}^{3}(x_ip_i)^2$ *and* $\xi = (1/\sqrt{f(x)})D_t$.

Proof. (1) is trivial, since the restriction of ψ_p to $S^2 \times \{t\}$ is a totally geodesic immersion for any $t \in \mathbf{R}$.

Let $\tilde{D}$ be the canonical connection of $\mathbf{R}^7$. From

$$\tilde{D}_{D_t}\left(d\psi_{(p_1,p_2,p_3)}(0, D_t)\right) = -({x_1p_1}^2, {x_2p_2}^2, {x_3p_3}^2)R_p(t) \in V,$$

and $V = \mathbf{R}\psi(p,t) \oplus d\psi_p(T_xS^2 \oplus \{0\})$ we have (2).

For any tangent vector v of S^2, we have

$$\tilde{D}_v\left(d\psi_{(p_1,p_2,p_3)}(0, D_t)\right) = v\begin{pmatrix} p_1(-\sin(tp_1)e_1 + \cos(tp_1)e_5) \\ p_2(-\sin(tp_2)e_2 + \cos(tp_2)e_6) \\ p_3(-\sin(tp_3)e_3 + \cos(tp_3)e_7)\end{pmatrix}.$$

Taking the normal component, we get

$$\begin{aligned}&\sigma(v,\xi)\\ &= \left(\frac{1}{\sqrt{f(x)}}\right)\left\{\tilde{D}_v\left(d\psi_{(p_1,p_2,p_3)}(0, D_t)\right) - \left(\frac{v(f(x))}{2f(x)}\right)d\psi_{(p_1,p_2,p_3)}(0, D_t)\right\}.\end{aligned}$$

$\square$

From this proposition, we can calculate the trace and the square of the length of the second fundamental form.

Proposition 12.

(1) *Each immersion $\psi_{(p_1,p_2,p_3)}$ is a minimal immersion.*

(2)

$$|\sigma|^2 = \frac{2}{\left(\sum_{i=1}^3 (x_i p_i)^2\right)^2}\left\{\left(\sum_{i=1}^3 (p_i)^2\right)\cdot\left(\sum_{i=1}^3 ((x_i p_i)^2\right) - \left(\sum_{i=1}^3 (x_i)^2 (p_i)^4\right)\right\}$$

Since the scalar curvature τ $(= 6 - |\sigma|^2)$ is not constant, we have the following

Corollary 13. *The induced metric is neither homogeneous nor cyclic parallel.*

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