

DETERMINANT FORMULAS FOR THE τ -FUNCTIONS OF THE PAINLEVÉ EQUATIONS OF TYPE A

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Abstract. Explicit determinant formulas are presented for the τ -functions of the generalized Painlevé equations of type A . This result allows an interpretation of the τ -functions as the Plücker coordinates of the universal Grassmann manifold.

§1. Introduction

For each generalized Cartan matrix of affine type $A = (a_{ij})_{ij \in I}$, we introduced a representation of the Weyl group $W(A)$ on the rational functions of variables α_i, f_i, τ_i , ($i \in I$) [1]. The representation is characterized by the action of the generator s_i , ($i \in I$), such that

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}, \quad s_i(\tau_j) = \tau_j (f_j \prod_{k \in I} \tau_k^{-a_{kj}})^{\delta_{ij}},$$

under certain conditions on the constants $(u_{ij})_{ij \in I}$.

This representation is a generalization of the Bäcklund transformations of the Painlevé equations P_{IV}, P_V and P_{VI} . As to the root systems of type $A_l^{(1)}$, it is also known that there exist Painlevé type differential (or difference) system which has the $W(A)$ symmetry [1][2]. In the context of the Painlevé equations, the variables α_i, f_i and τ_i play the role of parameters (or the discrete time variables), dependent variables and the τ -functions, respectively. In [1], it is conjectured that the τ -functions have strong regularity. This regularity is crucial for the differential (or difference) systems, since it should be closely related to the Painlevé (or the singularity confinement) properties.

In this paper, we prove the regularity conjecture in the case of affine Weyl groups of type $A_l^{(1)}$ and A_∞ , by constructing explicit determinant formulas for the τ -functions. The determinant formulas allow an interpretation of the τ -functions as Plücker coordinates of the universal Grassmann manifold as in the theory of KP hierarchy [3].

§2. The τ -functions

We formulate our results in the case of A_∞ . For the reduction to $A_t^{(1)}$ case, see Remark 1 in the last section.

Let $K = \mathbf{C}(\alpha; f; \tau)$ be the field of rational functions in infinitely many variables $\alpha = (\alpha_i)$, $f = (f_i)$ and $\tau = (\tau_i)$ where $i \in \mathbf{Z}$. For each $i \in \mathbf{Z}$, define an automorphism s_i of the field K by

$$\begin{aligned} s_i(\alpha_i) &= -\alpha_i, & s_i(\alpha_{i\pm 1}) &= \alpha_{i\pm 1} + \alpha_i, & s_i(\alpha_j) &= \alpha_j, \quad (j \neq i, i \pm 1), \\ s_i(f_i) &= f_i, & s_i(f_{i\pm 1}) &= f_{i\pm 1} \pm \frac{\alpha_i}{f_i}, & s_i(f_j) &= f_j, \quad (j \neq i, i \pm 1), \end{aligned}$$

and

$$s_i(\tau_i) = f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad s_i(\tau_j) = \tau_j, \quad (j \neq i).$$

These automorphisms satisfy the relations

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad s_i s_j = s_j s_i, \quad (j \neq i \pm 1),$$

and define a representation of the affine Weyl group $W = W(A_\infty)$ of type A_∞ . One can extend this representation W to the extended affine Weyl group \widetilde{W} by adding the automorphism π defined as

$$\pi(\alpha_i) = \alpha_{i+1}, \quad \pi(f_i) = f_{i+1}, \quad \pi(\tau_i) = \tau_{i+1}.$$

The diagram shift π satisfies the relation $\pi s_i = s_{i+1} \pi$.

Let Λ_i ($i \in \mathbf{Z}$) be the fundamental weights of A_∞ on which the Weyl group \widetilde{W} acts as

$$s_i(\Lambda_i) = \Lambda_{i+1} + \Lambda_{i-1} - \Lambda_i, \quad s_i(\Lambda_j) = \Lambda_j, \quad (j \neq i), \quad \pi(\Lambda_i) = \Lambda_{i+1}.$$

We also use the notation $v_i = \Lambda_i - \Lambda_{i-1}$. Then $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} = v_i - v_{i+1}$, and we can put formally $\Lambda_i = \sum_{j \leq i} v_j$. The action of s_i on v_j is given by the permutation

$$s_i(v_i) = v_{i+1}, \quad s_i(v_{i+1}) = v_i, \quad s_i(v_j) = v_j, \quad (j \neq i, i+1).$$

Furthermore, the Weyl group W is identified with the infinite symmetric group S_∞ which permutes v_j .

There exist a family of rational functions $\phi_w(\Lambda_j) \in \mathbf{C}(\alpha; f)$ for $w \in \widetilde{W}$ and $j \in \mathbf{Z}$ such that

$$w(\tau_j) = \phi_w(\Lambda_j) \prod_{i \in \mathbf{Z}} \tau_i^{m_i},$$

where $m_i = \langle \alpha_i, w\Lambda_j \rangle \in \{0, \pm 1\}$ is the coefficient of Λ_i in $w\Lambda_j$. The conjecture in [1] states that the function $\phi_w(\Lambda_j)$ is a polynomial in α and f with integral coefficients. We will prove this conjecture by constructing an explicit determinant formula for $\phi_w(\Lambda_j)$. Since any function $\phi_w(\Lambda_j)$ is easily obtained from the case $j = 0$ by the shift $\phi_w(\Lambda_j) = \pi^j(\phi_{\pi^{-j}w\pi^j}(\Lambda_0))$, we will concentrate on the functions $\phi_w(\Lambda_0)$.

LEMMA 1. *If $w_1\Lambda_0 = w_2\Lambda_0$ for $w_1, w_2 \in W$, then $w_1(\tau_0) = w_2(\tau_0)$.*

Proof. The condition $w_1(\Lambda_0) = w_2(\Lambda_0)$ means that the element $w = w_2^{-1}w_1 \in W$ is in the stabilizer W_0 of Λ_0 . An element in W belongs to W_0 if and only if the corresponding permutation of $\{v_i \mid i \in \mathbf{Z}\} \simeq \mathbf{Z}$ preserves the subset $\mathbf{Z}_{\leq 0}$. Then W_0 is a product of permutations of $\mathbf{Z}_{\leq 0}$ and $\mathbf{Z}_{\geq 1}$. Hence $w \in W_0 = \langle s_j, (j \in \mathbf{Z}_{<0}) \rangle \times \langle s_j, (j \in \mathbf{Z}_{\geq 1}) \rangle$ and $w(\tau_0) = \tau_0$. \square

By this Lemma, the functions $\phi_w(\Lambda_0)$ are parametrized by $w\Lambda_0$. We will prepare some notations to describe the orbit $W\Lambda_0 = \{w\Lambda_0 \mid w \in W\}$ whose elements will be parametrized by Young diagrams [4].

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0)$ be a partition of length $l = l(\lambda)$. The corresponding Young diagram Y is defined by

$$Y = \{s = (i, j) \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}.$$

The transposition $\lambda' = Y'$ is defined by $(i, j) \in Y'$ if and only if $(j, i) \in Y$. For a partition λ , the corresponding Frobenius symbol

$$(I, J) = (\{i_1 > i_2 > \dots > i_k > 0\}, \{j_1 > j_2 > \dots > j_k \geq 0\})$$

is defined by

$$i_n = \lambda_n - n + 1, \quad j_n = \lambda'_n - n,$$

where $k = \max\{n \mid \lambda_n \geq n\}$. We always identify the three notions, partition λ , Young diagram Y and Frobenius symbol (I, J) by these correspondence.

For $w \in W$, the element $w(\Lambda_0)$ can be parametrized by the partitions $\lambda = Y = (I, J)$ as follows.

Recall that $\Lambda_0 = \sum_{j \leq 0} v_j$. Then we have $w(\Lambda_0) = \sum_{i \in M} v_i$, where M is a subset of \mathbf{Z} different from $\mathbf{Z}_{\leq 0}$ only by finite elements. Such subset M (called Maya diagram) corresponds to a Young diagram Y with Frobenius symbol (I, J) by the rule

$$M \cup (-J) = \mathbf{Z}_{\leq 0} \cup I.$$

In terms of the Maya diagram M , the coefficient m_i of Λ_i in $w(\Lambda_0) = (\sum_{i \in M} - \sum_{i+1 \in M})\Lambda_i$ is given by

- $m_i = 1$, if $i \in M$ and $i + 1 \notin M$,
- $m_i = -1$, if $i \notin M$ and $i + 1 \in M$,
- $m_i = 0$, if $i, i + 1 \in M$ or $i, i + 1 \notin M$.

Translating this into the language of the Young diagrams, we have

LEMMA 2. *Let $w(\Lambda_0) = \sum_{i \in \mathbf{Z}} m_i \Lambda_i$ and let Y be the corresponding Young diagram, then the Young diagram $s_i(Y)$ corresponding to $s_i w(\Lambda_0)$ is given as follows,*

- $s_i(Y)$ is obtained by adding the node with color i , if $m_i = 1$,
- $s_i(Y)$ is obtained by removing the node with color i , if $m_i = -1$,
- $s_i(Y) = Y$, if $m_i = 0$,

where the color k of the (i, j) -th node is given by $k = j - i$.

In summary, we have

PROPOSITION 1. *Any element in $W.\Lambda_0$ can be obtained from Λ_0 by the action of $s_{i_p} \cdots s_{i_1}$ with $m_{i_k} = \langle \alpha_{i_k}, s_{i_{k-1}} \cdots s_{i_1}(\Lambda_0) \rangle = 1$, ($k = 1, \dots, p$). Hence, the functions $\phi_w(\Lambda_0)$ are uniquely determined by the cocycle condition*

$$\phi_{s_i w}(\Lambda_0) = s_i(\phi_w(\Lambda_0))f_i, \quad (m_i = 1),$$

with the initial condition $\phi_{id}(\Lambda_0) = 1$.

It is convenient to introduce another normalization $\tilde{\phi}_w(\Lambda_0)$ defined by

$$\tilde{\phi}_w(\Lambda_0) = \frac{1}{N_w} \phi_w(\Lambda_0).$$

Here the normalization factor N_w is a polynomial in α , which is defined, in terms of corresponding Young diagram $w(\Lambda_0) \leftrightarrow \lambda = Y$, as

$$N_w = \prod_{s=(i,j) \in Y} h(s, \alpha), \quad h((i, j), \alpha) = v_{j-\lambda'_j} - v_{\lambda_i-i+1}.$$

Note that when specialized to $\alpha_i = 1$, $h(s, 1)$ is nothing but the hook-length of the node $s \in Y$.

LEMMA 3. *The normalization factor N_w satisfies the relation*

$$N_{s_k w} = \alpha_k s_k(N_w), \quad (m_k = 1).$$

Proof. From the Lemma 2 and the condition $m_k = 1$ the Young diagram $s_k(Y)$ is obtained from Y by adding one node, say (i_0, j_0) -th node, with color $k = j_0 - i_0$. Since $\lambda_{i_0} = j_0 - 1$ and $\lambda'_{j_0} = i_0 - 1$, the hook length $h(s, \alpha) = v_{j-\lambda'_j} - v_{\lambda_i-i+1}$ for $s = (i, j) \in Y$ contains v_k or v_{k+1} if and only if

- $s = (i_0, *)$, with $h(s, \alpha) = v_* - v_k$,
- $s = (*, j_0)$, with $h(s, \alpha) = v_{k+1} - v_*$.

Under the action of s_k , these factors are replaced with the hook length of the same node s in the new diagram $s_k(Y)$. Multiplying $s_k(N_w)$ by the extra factor $v_k - v_{k+1} = \alpha_k$ corresponding to the added node (i_0, j_0) , we get $N_{s_k w}$. \square

As a Corollary of this Lemma 3 and Proposition 1 we have

PROPOSITION 2. *The normalized functions $\tilde{\phi}_w(\Lambda_0)$ are determined by the cocycle condition*

$$\tilde{\phi}_{s_i w}(\Lambda_0) = s_i \left(\tilde{\phi}_w(\Lambda_0) \right) \frac{f_i}{\alpha_i}, \quad (m_i = 1),$$

with the initial condition $\tilde{\phi}_{id}(\Lambda_0) = 1$.

EXAMPLE 1. For the partitions $(0), (1), (2), (1, 1), (2, 1)$ and $(2, 2)$, the corresponding normalized functions $\tilde{\phi}_w(\Lambda_0)$ are given as follows.

$$\begin{aligned} a_{12} &= \tilde{\phi}_1(\Lambda_0) = 1 \leftrightarrow (0), \\ a_{13} &= \tilde{\phi}_{s_0}(\Lambda_0) = \frac{f_0}{\alpha_0} \leftrightarrow (1), \\ a_{14} &= \tilde{\phi}_{s_1 s_0}(\Lambda_0) = \frac{f_0 f_1 - \alpha_1}{(\alpha_0 + \alpha_1) \alpha_1} \leftrightarrow (2), \\ a_{23} &= \tilde{\phi}_{s_{-1} s_0}(\Lambda_0) = \frac{f_{-1} f_0 + \alpha_{-1}}{(\alpha_{-1} + \alpha_0) \alpha_{-1}} \leftrightarrow (1, 1), \end{aligned}$$

$$a_{24} = \tilde{\phi}_{s_{-1}s_1s_0}(\Lambda_0) = \frac{f_{-1}f_0f_1 + \alpha_{-1}f_1 - \alpha_1f_{-1}}{(\alpha_{-1} + \alpha_0 + \alpha_1)\alpha_{-1}\alpha_1} \leftrightarrow (2, 1),$$

and for the partition (2, 2), we have

$$a_{34} = \tilde{\phi}_{s_0s_{-1}s_1s_0}(\Lambda_0) = \frac{f_{-1}f_0^2f_1 + \alpha_{-1}f_0f_1 - \alpha_1f_{-1}f_0 + \alpha_0(\alpha_{-1} + \alpha_0 + \alpha_1)}{(\alpha_{-1} + \alpha_0 + \alpha_1)(\alpha_0 + \alpha_1)(\alpha_{-1} + \alpha_0)\alpha_0}.$$

It is interesting to note the relations

$$(1) \quad a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0,$$

$$(2) \quad a_{23} = \pi^{-1}(a_{13})a_{13} - \pi^{-1}(a_{14})a_{12}.$$

Each of the equations plays a fundamental role in the proof of Theorem 1 or 2 respectively.

§3. The determinant formulas

For integers $p \geq 1$ and $q \geq 0$, put

$$X_{p,q} = \det \begin{bmatrix} f_{-q} & 1 & & & 0 \\ \beta_{-q+1} & f_{-q+1} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{p-2} & f_{p-2} & 1 \\ 0 & & & \beta_{p-1} & f_{p-1} \end{bmatrix},$$

where β_j ($-q + 1 \leq j \leq p - 1$) is given by

$$\beta_j = \sum_{i=1}^{p-1} \alpha_i = v_j - v_p, \quad (j > 0), \quad \beta_j = - \sum_{i=-q}^{j-1} \alpha_i = v_j - v_{-q+1}, \quad (j \leq 0).$$

LEMMA 4. Put $w_{p,q} = (s_q \cdots s_{-1})(s_{p-1} \cdots s_0)$, which corresponds to the hook diagram $w_{p,q}(\Lambda_0) \leftrightarrow \lambda = (\{p\}, \{q\})$, then

$$\phi_{p,q} := \phi_{w_{p,q}}(\Lambda_0) = X_{p,q}.$$

Proof. For $p = 1$ and $q = 0$, the formula is trivially satisfied, $X_{1,0} = f_0$. We need to check the following relations for $p \geq 1$ and $q \geq 0$.

$$(3) \quad X_{p+1,q} = s_p(X_{p,q})f_p, \quad X_{p,q+1} = s_{-(q+1)}(X_{p,q})f_{-(q+1)}.$$

Let us prove the first one. By definition

$$X_{p+1,q} = \det \begin{bmatrix} \ddots & & \ddots & & \\ \ddots & & f_{p-2} & & 1 \\ & \alpha_{p-1} + \alpha_p & f_{p-1} & & 1 \\ & & \alpha_p & & f_p \end{bmatrix},$$

expanding this with respect to the last row, we have

$$= \det \begin{bmatrix} \ddots & & \ddots & & \\ \ddots & & f_{p-2} & & 1 \\ & s_p(\alpha_{p-1}) & f_{p-1} & & \\ & & & & \end{bmatrix} f_p - \det \begin{bmatrix} \ddots & & \ddots & & \\ \ddots & & f_{p-2} & & \\ & s_p(\alpha_{p-1}) & & & 1 \end{bmatrix} \alpha_p,$$

which is nothing but

$$s_p(X_{p,q}) = f_p \det \begin{bmatrix} \ddots & & \ddots & & \\ \ddots & & f_{p-2} & & 1 \\ & s_p(\alpha_{p-1}) & f_{p-1} & & -\frac{\alpha_p}{f_p} \end{bmatrix}.$$

Thus the first relation is proved. The second one is similar. □

Using the notations above, we can state our main result as follows.

THEOREM 1. *For any $w \in W$, the normalized function $\tilde{\phi}_w(\Lambda_0)$ is given by the following determinant*

$$\tilde{\phi}_w(\Lambda_0) = \det \left(\tilde{\phi}_{p,q} \right)_{p \in I, q \in J},$$

where (I, J) is the Frobenius symbol of $\lambda = Y$ corresponding to $w(\Lambda_0)$.

Equivalently, we also have the following Jacobi-Trudi type ([4]) formulas

THEOREM 2. *For any $w \in W$, the normalized function $\tilde{\phi}_w(\Lambda_0)$ is given by the following determinant*

$$\tilde{\phi}_w(\Lambda_0) = \det \left(h_{\lambda_i - i + j}^{(1-j)} \right)_{1 \leq i, j \leq l(\lambda)},$$

where λ is the partition corresponds to $w(\Lambda_0)$ and $h_k^{(j)} = \pi^j(\tilde{\phi}_{k,0})$ is the normalized function for the single row $\lambda = (k)$.

The proofs of these theorems are given in the next section.

COROLLARY. *For any $w \in W$, the function $\phi_w(\Lambda_0)$ is a polynomial in α and f with integral coefficients. The leading term with respect to f is $\prod_{i \in \mathbf{Z}} f_i^{\nu_i}$, where ν_i is the number of nodes with color i in $Y \leftrightarrow w(\Lambda_0)$.*

Proof. We will prove by induction on the length of $w \in W$. $\phi_{id}(\Lambda_0) = 1$ is in $\mathbf{Z}[\alpha; f]$. Assume that $R(\alpha; f) = \phi_w(\Lambda_0) \in \mathbf{Z}[\alpha; f]$. By Proposition 1, the function $S(\alpha; f) = \phi_{s_i w}(\Lambda_0)$ for $m_i = 1$ is given by

$$S(\dots, \alpha_{i-1} + \alpha_i, -\alpha_i, \alpha_{i+1} + \alpha_i, \dots; \dots, f_{i-1} - \frac{\alpha_i}{f_i}, f_i, f_{i+1} + \frac{\alpha_i}{f_i}, \dots) f_i,$$

which belongs to $\mathbf{Z}[\alpha; f; \frac{1}{f_i}]$. The condition on the leading term of S follows from that on R . On the other hand, from Theorem 1, we see that $\phi_{s_i w}(\Lambda_0) = N_{s_i w} \tilde{\phi}_{s_i w}(\Lambda_0)$ is a polynomial in f , hence the function $S(\alpha; f)$ also belongs to $\mathbf{Z}[\alpha; f]$. \square

§4. Proof of the Theorems

LEMMA 5. *We have*

$$z^{n-1} \det \begin{bmatrix} (a_{ij})_{1 \leq i, j \leq n} & (x_i)_{1 \leq i \leq n} \\ (y_j)_{1 \leq j \leq n} & z \end{bmatrix} = \det \left[(a_{ij}z - x_i y_j)_{1 \leq i, j \leq n} \right].$$

Proof. Let e_i , ($0 \leq i \leq n$) be a basis in \mathbf{C}^{n+1} . For $1 \leq i \leq n$, put

$$a_i = \eta_i + x_i e_0, \quad \eta_i = \sum_{j=1}^n a_{ij} e_j, \quad a_0 = \xi + z e_0, \quad \xi = \sum_{j=1}^n y_j e_j,$$

$$\text{and } b_i = \sum_{j=1}^n (z a_{ij} - x_i y_j) e_j = z \eta_i - x_i \xi.$$

Then the both hand sides of the identity are the coefficients of $e_0 \wedge \dots \wedge e_n$ in

$$(\text{LHS}) = z^{n-1} a_0 \wedge a_1 \wedge \dots \wedge a_n, \quad (\text{RHS}) = e_0 \wedge b_1 \wedge \dots \wedge b_n.$$

It is easy to see that these two are the same and equal to

$$z^n e_0 \wedge \eta_1 \wedge \dots \wedge \eta_n - z^{n-1} \sum_{i=1}^n e_0 \wedge \eta_1 \wedge \dots \wedge x_i \xi \wedge \dots \wedge \eta_n.$$

\square

Proof of the Theorem 1. To prove the theorem, it is enough to check that the determinant satisfy the transformation properties in Proposition 2.

For the actions of s_i , $i \neq 0$, the transformation properties directly follows from eq.(3) for the hook $X_{p,q}$ in Lemma 4, since the s_i acts only on the single row ($i > 0$) or single column ($i < 0$).

For the s_0 action, we need some computation because in this case the size of the determinant changes. For $w(\Lambda_0) \leftrightarrow (I, J)$ such that $m_0 = 1$, we have $s_0 w(\Lambda_0) \leftrightarrow (I \cup \{1\}, J \cup \{0\})$. Then

$$\tilde{\phi}_{s_0 w} = \det \begin{bmatrix} \left(\tilde{\phi}_{p,q} \right)_{p \in I, q \in J} & \left(\tilde{\phi}_{1,q} \right)_{q \in J} \\ \left(\tilde{\phi}_{p,0} \right)_{p \in I} & \tilde{\phi}_{1,0} \end{bmatrix}.$$

We shall prove that this is equal to $s_0(\tilde{\phi}_w) \frac{f_0}{\alpha_0}$.

As is shown in eq.(1), we have

$$s_0(\tilde{\phi}_{2,1})\tilde{\phi}_{1,0} = \tilde{\phi}_{2,1}\tilde{\phi}_{1,0} - \tilde{\phi}_{2,0}\tilde{\phi}_{1,1}.$$

Applying the actions s_k , $k \geq 2$ or $k \leq -2$ repeatedly, one get

$$s_0(\tilde{\phi}_{p,q})\tilde{\phi}_{1,0} = \tilde{\phi}_{p,q}\tilde{\phi}_{1,0} - \tilde{\phi}_{p,0}\tilde{\phi}_{1,q},$$

for any $p \geq 2$ and $q \geq 1$. By using this and $\tilde{\phi}_{1,0} = f_0/\alpha_0$, we get

$$\frac{f_0}{\alpha_0} s_0 \left(\tilde{\phi}_w(\Lambda_0) \right) = \tilde{\phi}_{1,0}^{1-|I|} \det \left[\tilde{\phi}_{p,q}\tilde{\phi}_{1,0} - \tilde{\phi}_{p,0}\tilde{\phi}_{1,q} \right]_{p \in I, q \in J},$$

and this is indeed equal to $\tilde{\phi}_{s_0 w}(\Lambda_0)$, because of the identity in Lemma 5. \square

Proof of the Theorem 2. The proof for the general $\tilde{\phi}_w(\Lambda_0)$ cases can be reduced to the single column cases by the action of s_k , ($k > 0$) which preserves the size of the determinant.

For the single column case $\lambda = (1^{q+1})$, the desired formula is

$$(4) \quad \tilde{\phi}_{1,q}(\Lambda_0) = \det \begin{bmatrix} h_1^{(-q)} & h_2^{(-q)} & \cdots & h_{q+1}^{(-q)} \\ 1 & h_1^{(-q+1)} & \cdots & h_q^{(-q+1)} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_2^{(-1)} \\ & & & 1 & h_1^{(0)} \end{bmatrix}.$$

The case of $q = 1$ follows from the relation in eq.(2). More generally, we have

$$(5) \quad \tilde{\phi}_{i,1} = \det \begin{bmatrix} h_1^{(-1)} & h_{i+1}^{(-1)} \\ 1 & h_i^{(0)} \end{bmatrix}.$$

To prove the case $q > 1$, we will check the condition in Proposition 2,

$$(6) \quad \frac{f_{-(q+1)}}{\alpha_{-(q+1)}} s_{-(q+1)}(\tilde{\phi}_{1,q}(\Lambda_0)) = \tilde{\phi}_{1,q+1}(\Lambda_0).$$

Using the eq.(5), the left hand side of this relation (6) can be written as the same determinant as $\tilde{\phi}_{1,q}(\Lambda_0)$ in (4), with the first row $h_i^{(-q)}$ replaced by $h_i'^{(-q)}$ such as

$$h_i'^{(-q)} = \frac{f_{-(q+1)}}{\alpha_{-(q+1)}} s_{-(q+1)} \pi^{-q}(\tilde{\phi}_{i,0}) = \pi^{-q}(\tilde{\phi}_{i,1}) = \det \begin{bmatrix} h_1^{(-q-1)} & h_{i+1}^{(-q-1)} \\ 1 & h_i^{(-q)} \end{bmatrix}.$$

Then the left hand side of the eq.(6) is equal to the right hand side

$$\tilde{\phi}_{1,q+1}(\Lambda_0) = \det \begin{bmatrix} h_1^{(-q-1)} & h_2^{(-q-1)} & \cdots & h_{q+2}^{(-q-1)} \\ 1 & h_1^{(-q)} & \cdots & h_{q+1}^{(-q)} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_1^{(-1)} \\ & & & h_2^{(-1)} \\ & & & 1 & h_1^{(0)} \end{bmatrix},$$

because of the identity

$$\det \begin{bmatrix} a_{00} & \begin{pmatrix} a_{0j} \\ a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \\ a_{10} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}_{1 \leq j \leq n} = \det \begin{bmatrix} \left(\det \begin{bmatrix} a_{00} & a_{0j} \\ a_{10} & a_{1j} \\ a_{20} & a_{2j} \\ \vdots & \\ a_{n0} & a_{nj} \end{bmatrix} \right) \\ a_{10} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}_{1 \leq j \leq n}.$$

□

§5. Remarks

Remark 1. The reduction to the finite rank cases $A_{N-1}^{(1)}$ is given by the N -reduced condition “ $\pi^N = 1$ ”. On the variables α, f, τ , this reduction is simply realized by the specialization

$$\alpha_{i+N} = \alpha_i, \quad f_{i+N} = f_i, \quad \tau_{i+N} = \tau_i.$$

By putting $\bar{s}_i = \prod_{n \in \mathbf{Z}} s_{i+nN}$, ($i \in \mathbf{Z}/N\mathbf{Z}$), the representation $W(A_\infty)$ reduces to that of $W = W(A_{N-1}^{(1)})$ on the field $\mathbf{C}(\alpha_i; f_i; \tau_i, (i \in \mathbf{Z}/N\mathbf{Z}))$. The τ -functions for $A_{N-1}^{(1)}$ case is nothing but the specialization of τ for A_∞ and have the same determinant formulas. Note that only the N -reduced Young diagrams can be generated by the actions of \bar{s}_i .

Remark 2. The determinant formulas of the τ -function provide explicit solutions for initial value problem of the associated discrete dynamical system introduced in [1]. The polynomiality of the τ -functions and the multiplicative formula of f -variables in terms of the τ -functions

$$w(f_i) = \frac{\phi_w(\Lambda_i)\phi_{ws_i}(\Lambda_i)}{\phi_w(\Lambda_{i-1})\phi_w(\Lambda_{i+1})},$$

give a strong support for the singularity confinement property which the discrete dynamical system expected to have.

Remark 3. The representation of affine Weyl groups $W(A_l^{(1)})$ has an interpretation as the Bäcklund transformations for the Painlevé equations P_{IV} (for $l = 2$), P_V (for $l = 3$) and their generalizations for $l \geq 4$ [2]. Under this interpretation, the polynomials $\phi_w(\Lambda_j)$ are the far-reaching generalization of the “special polynomials” arising in Painlevé equations in the sense of Umemura et. al. [7][8]. When specialized to certain “initial solutions”, we obtain explicit determinant formulas for the Okamoto polynomials (for P_{IV} [5]), the Umemura polynomials (for P_V [6]) and their generalizations.

Remark 4. The normalized functions $\tilde{\phi}_w(\Lambda_0)$ can be represented as the minor determinants of the following frame X of the universal Grassmann manifold,

$$X = \begin{bmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \dots \end{bmatrix}.$$

Where $h_i^{(j)} = \pi^j(\tilde{\phi}_{i,0})$. In this picture, the Weyl group W is nothing but the Weyl group $W(GL_\infty)$. It would be interesting if the space of the initial values of Painlevé equations can be realized as a natural sub-manifolds of the universal Grassmann manifold.

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