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UNIQUENESS PROBLEM WITH TRUNCATED MULTIPLICITIES IN VALUE DISTRIBUTION THEORY, II

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Abstract. Let $H_1, H_2, \dots, H_q$ be hyperplanes in $P^N(\mathbb{C})$ in general position. Previously, the author proved that, in the case where $q \geq 2N+3$, the condition $\nu(f, H_j) = \nu(g, H_j)$ imply $f = g$ for algebraically nondegenerate meromorphic maps $f, g : \mathbb{C}^n \to P^N(\mathbb{C})$, where $\nu(f, H_j)$ denote the pull-backs of H_j through f considered as divisors. In this connection, it is shown that, for $q \geq 2N+2$, there is some integer ℓ_0 such that, for any two nondegenerate meromorphic maps $f, g : \mathbb{C}^n \to P^N(\mathbb{C})$ with $\min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0)$ the map $f \times g$ into $P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is algebraically degenerate. He also shows that, for $N = 2$ and $q = 7$, there is some ℓ_0 such that the conditions $\min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0)$ imply $f = g$ for any two nondegenerate meromorphic maps f, g into $P^2(\mathbb{C})$ and seven generic hyperplanes H_j's.

§1. Introduction

In [2]–[4], the author gave several types of generalizations of the classical Nevanlinna's uniqueness theorem for meromorphic functions to the case of meromorphic maps of $\mathbb{C}^n$ into $P^N(\mathbb{C})$. He considered two (linearly) nondegenerate meromorphic maps f and g of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ satisfying the condition that $\nu(f, H_j) = \nu(g, H_j)$ for q hyperplanes $H_1, H_2, \cdots, H_q$ in $P^N(\mathbb{C})$ located in general position, where we denote by $\nu(f, H)$ the map of $\mathbb{C}^n$ into $\mathbb{Z}$ whose values at each point $z \in \mathbb{C}^n$ is given by the intersection multiplicity of $f(\mathbb{C}^n)$ and a hyperplane H at $f(z)$. He showed that, if $q \geq 3N+2$ then $f = g$, and if $q = 3N+1$ then there is a projective linear transformation L of $P^N(\mathbb{C})$ onto $P^N(\mathbb{C})$ itself such that $g = L \cdot f$. Moreover, he proved that, if either f or g is algebraically nondegenerate and $q \geq 2N+3$, then $f = g$. In connection with these results, it is an interesting problem to ask whether these results remain valid if the assumption concerning multiplicity is weaken. In this paper, we will try to get some partial answers to this problem.

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Take q hyperplanes $H_1, H_2, \ldots, H_q$ in $P^N(\mathbb{C})$ located in general position, a nondegenerate meromorphic map $g : \mathbb{C}^n \to P^N(\mathbb{C})$ and a positive integer ℓ_0. We consider the family $\mathcal{G}(H_1, \ldots, H_q; g; \ell_0)$ of all nondegenerate meromorphic maps $f : \mathbb{C}^n \to P^N(\mathbb{C})$ satisfying the condition

$$\text{(H)} \qquad \min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0) \quad (1 \leq j \leq q).$$

Here, for $\ell_0 = 1$, the condition (H) means that $f^{-1}(H_j) = g^{-1}(H_j)$ $(1 \leq j \leq q)$. The purpose of this paper is to give some degeneracy and uniqueness theorems of maps in $\mathcal{G}(H_j; g; \ell_0)$ for a sufficiently large ℓ_0.

There are some results related to this study which concern the family $\mathcal{F}(H_1, \ldots, H_q; g; \ell_0)$ of all maps f in $\mathcal{G}(H_j; g; \ell_0)$ satisfying the additional conditions;

(a) $\dim \bigcup_{i<j} f^{-1}(H_i \cap H_j) \leq n - 2$,

(b) $f = g$ on $\bigcup_{j=1}^{q} g^{-1}(H_j)$.

For the case $\ell_0 = 1$, the following results were given by L. Smiley and S. Ji:

Theorem 1.1. ([10]) *If $q > 3N + 1$, then $\mathcal{F}(H_j; g; 1) = \{g\}$.*

Theorem 1.2. ([8]) *Assume that $q = 3N + 1$. Then, for three maps $f^1, f^2, f^3 \in \mathcal{F}(H_j; g; 1)$, the map $F = f^1 \times f^2 \times f^3 : \mathbb{C}^n \to P^N(\mathbb{C}) \times P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is algebraically degenerate, namely, $\{(f^1(z), f^2(z), f^3(z))\ ;\ z \in \mathbb{C}^n\}$ is included in a proper algebraic subset of $P^N(\mathbb{C}) \times P^N(\mathbb{C}) \times P^N(\mathbb{C})$.*

In the previous paper ([7]), the author considered the family $\mathcal{F}(H_j; g; \ell_0)$ for the case $\ell_0 > 1$ and gave the following results:

Theorem 1.3. *Suppose that $q \geq 2N+2$ and take $N+2$ maps $f^1, \ldots, f^{N+2}$ in $\mathcal{F}(H_j; g; N(N+1)/2+N)$. Then, suitably chosen $N+1$ hyperplanes among H_j's, say $H_1, H_2, \ldots, H_{N+1}$, satisfy the following:*

If we take homogeneous coordinates $(w_1 : \cdots : w_{N+1})$ on $P^N(\mathbb{C})$ with $H_j = \{w_j = 0\}$ $(1 \leq j \leq N+1)$ and write $f^k = (f^k_1 : \cdots : f^k_{N+1})$ with nonzero holomorphic functions f^k_j, then

$$\frac{f^1_i}{f^1_j} - \frac{f^{N+2}_i}{f^{N+2}_j}, \frac{f^2_i}{f^2_j} - \frac{f^{N+2}_i}{f^{N+2}_j}, \ldots, \frac{f^{N+1}_i}{f^{N+1}_j} - \frac{f^{N+2}_i}{f^{N+2}_j}$$

are linearly dependent over $\mathbb{C}$ for $1 \leq i, j \leq N+1$.

THEOREM 1.4. *If $q = 3N + 1$, then $\#\mathcal{F}(H_j; g; 2) \leq 2$, where $\#A$ denotes the number of elements of the set A.*

In this paper, we prove the following result for the family $\mathcal{G}(H_j; g; \ell_0)$:

THEOREM 1.5. *Assume that $q \geq 2N + 2$. Then, there exists some positive integer ℓ_0 depending only on N such that, for any two maps f^1 and f^2 in $\mathcal{G}(H_1, \dots, H_q; g; \ell_0)$, the map $F := f^1 \times f^2 : \mathbb{C}^n \to P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is algebraically degenerate.*

For the particular case $N = 2$, we can show the following uniqueness theorem:

THEOREM 1.6. *Assume that $N = 2$ and $q = 7$. Then, there exist some positive integer ℓ_0 and a proper algebraic set V in the cartesian product of seven copies of the space $P^2(\mathbb{C})^*$ of all hyperplanes in $P^2(\mathbb{C})$ such that, for an arbitrary set $(H_1, H_2, \dots, H_7) \notin V$, $\mathcal{G}(H_1, \cdots, H_7; g; \ell_0) = \{g\}$.*

We have several open problems related to the above results. We have not got yet any uniqueness theorem for maps in $\mathcal{G}(H_1, \dots, H_q; g; \ell_0)$ in case $N > 2$. We do not know the best possible number ℓ_0. We cannot answer to the question whether Theorem 1.6 remains valid under the only assumption that H_j's are in general position or not.

In §2, we give some combinatorial lemmas which are improvements of the results given in [2] and, in §3, a representation theorem of meromorphic mappings as an application of Borel's method. After these preparations, we give a proof of Theorem 1.5 in §§4 and 5. Theorem 1.6 is proved in §6.

§2. Combinatorial lemmas

Set $\mathcal{I} := \{1, 2, \dots, q\}$. For $1 \leq s \leq q$ we denote by $\mathcal{I}_{q,s}$ the set of all combinations of s elements in $\mathcal{I}$, namely,

$$\mathcal{I}_{q,s} := \{(i_1, i_2, \dots, i_s)\ ;\ 1 \leq i_1 < i_2 < \cdots < i_s \leq q\}.$$

Consider a relation $\overset{R}{\sim}$ between two elements in $\mathcal{I}_{q,s}$ satisfying the conditions;

(i) $I \overset{R}{\sim} I$ for all elements I in $\mathcal{I}_{q,s}$,

(ii) if $I \overset{R}{\sim} J$, then $J \overset{R}{\sim} I$.

In the following, we call such a relation a pre-quivalence relation.

To give some properties of $\overset{R}{\sim}$, we consider $\mathbb{Z}$-module $\mathbb{Z}^q$. With each pair of $I = (i_1, \dots, i_s)$ and $J = (j_1, \dots, j_s)$ in $\mathcal{I}_{q,s}$ we associate the element

$$R_{I,J} = \delta_{i_1} + \cdots + \delta_{i_s} - (\delta_{j_1} + \cdots + \delta_{j_s}) \in \mathbb{Z}^q,$$

where $\delta_i := (0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0) \in \mathbb{Z}^q$ $(1 \leq i \leq q)$. By $\mathcal{R}$ we denote $\mathbb{Z}$-submodule of $\mathbb{Z}^q$ generated by all elements $R_{I,J}$ associated with I and J in $\mathcal{I}_{q,s}$ with $I \overset{R}{\sim} J$. In the following, we assume $\mathcal{R} \neq \{(0, \dots, 0)\}$.

Every element $L = (\ell_1, \ell_2, \dots, \ell_q) \in \mathcal{R}$ can be represented as

$$L = m_1 R_{I_1,J_1} + m_2 R_{I_2,J_2} + \cdots + m_k R_{I_k,J_k},$$

where m_ℓ are integers and I_ℓ, J_ℓ are elements in $\mathcal{I}_{q,s}$ with $I_\ell \overset{R}{\sim} J_\ell$. This implies the following:

(2.1) *If $L = (\ell_1, \ell_2, \dots, \ell_q) \in \mathcal{R}$, then $\ell_1 + \ell_2 + \cdots + \ell_q = 0$.*

Definition 2.2. For two elements I and J in $\mathcal{I}_{q,s}$ by the notation $I \sim J$ we mean that there is a positive integer m such that $mR_{I,J} \in \mathcal{R}$.

We can easily show that

(2.3)
(i) *if $I \overset{R}{\sim} J$, then $I \sim J$,*
(ii) *the relation $\sim$ is an equivalence relation.*

Now, we prove the following:

Proposition 2.4. *There are q real numbers $p_1, p_2, \dots, p_q$ satisfying the following conditions*;

(i) *for $I = (i_1, \dots, i_s), J = (j_1, \dots, j_s) \in \mathcal{I}_{q,s}$, $p_{i_1} + \cdots + p_{i_s} = p_{j_1} + \cdots + p_{j_s}$ if and only if $I \sim J$,*

(ii) *for $1 \leq i < j \leq q$, $p_i = p_j$ if and only if there is a nonzero integer m_0 such that*

$$(0, \dots, 0, \overset{i\text{-th}}{m_0}, 0, \dots, 0, \overset{j\text{-th}}{-m_0}, 0, \dots, 0) \in \mathcal{R}. \tag{2.5}$$

Proof. Take a system of generators $L_i := (\ell_{i1}, \ell_{i2}, \dots, \ell_{iq})$ $(1 \leq i \leq K)$ of $\mathcal{R}$ and define a matrix

$$L := (\ell_{ij}\,;\, 1 \leq i \leq K,\, 1 \leq j \leq q)\,.$$

We change L by the following operations:

T0. Two columns are exchanged for each other.

T1. One row is multiplied by a nonzero integer and

T2. A nonzero integer multiple of one row is added to the another row.

As is easily seen, by repeating these changes suitably, we obtain a new matrix $\tilde{L}$ of the form

$$\tilde{L} = \begin{pmatrix} \tilde{\ell}_{11} & & & & \tilde{\ell}_{1R+1} & \cdots & \tilde{\ell}_{1q} \\ & \tilde{\ell}_{22} & & & \tilde{\ell}_{2R+1} & \cdots & \tilde{\ell}_{2q} \\ & & \ddots & & & \cdots & \\ & & & \tilde{\ell}_{RR} & \tilde{\ell}_{RR+1} & \cdots & \tilde{\ell}_{Rq} \\ & 0 & & & & 0 & \end{pmatrix},$$

where $\tilde{\ell}_{ii}$ $(1 \leq i \leq R)$ are positive integers and $R \leq q-1$ by (2.1). We denote the first R rows of $\tilde{L}$ by $L_1^*, L_2^*, \dots, L_R^*$. Then, after suitable changes of indices $1, 2, \dots, q$, every $L \in \mathcal{R}$ is represented as

$$m_0 L = m_1 L_1^* + \cdots + m_R L_R^*$$

for some integers m_i $(0 \leq i \leq R)$, where $m_0 > 0$. Moreover, the vector $(m_1/m_0, \dots, m_R/m_0)$ of rational numbers are uniquely determined. In fact, the above-mentioned operations T1 and T2 are invertible up to multiplications of nonzero integers and so $L_1^*, L_2^*, \dots, L_R^*$ give a basis of $\mathcal{R}$ over $\mathbb{Q}$.

For each I and J in $\mathcal{I}_{q,s}$, we can write $R_{I,J}$ as

$$R_{I,J} = \sum_{\ell=1}^{R} r_\ell^{IJ} L_\ell^* + (0, \dots, 0, r_{R+1}^{IJ}, \dots, r_q^{IJ})$$

with rational numbers r_ℓ^{IJ}. Here, $I \sim J$ if and only if $r_{R+1}^{IJ} = \cdots = r_q^{IJ} = 0$.

Now, take real numbers $p_{R+1}, p_{R+2}, \dots, p_q$ which are linearly independent over $\mathbb{Q}$ and set

$$(2.6)\quad p_j := -(\tilde{\ell}_{jR+1} p_{R+1} + \tilde{\ell}_{jR+2} p_{R+2} + \cdots + \tilde{\ell}_{jq} p_q)/\tilde{\ell}_{jj} \quad (1 \leq j \leq R).$$

Then, the numbers p_i $(R+1 \leq i \leq q)$ satisfy the condition that, for all I and J with $I \not\sim J$,

$$r_{R+1}^{IJ} p_{R+1} + \cdots + r_q^{IJ} p_q \neq 0,$$

and, for $1 \leq i < j \leq R$, $p_i = p_j$ if and only if

$$\frac{1}{\tilde{\ell}_{ii}} (\tilde{\ell}_{iR+1}, \dots, \tilde{\ell}_{iq}) = \frac{1}{\tilde{\ell}_{jj}} (\tilde{\ell}_{jR+1}, \dots, \tilde{\ell}_{jq}).$$

By (L_1, L_2) we denote the inner product of L_1 and L_2, namely, $(L_1, L_2) = \sum_{j=1}^{q} \ell_j \ell'_j$ for $L_1 = (\ell_1, \ldots, \ell_q), L_2 = (\ell'_1, \ldots, \ell'_q) \in \mathbb{Z}^q$. Since $(L_j^*, (p_1, \ldots, p_q)) = 0$ $(1 \leq j \leq R)$ by (2.6), we have

$$\begin{aligned} p_{i_1} + \cdots + p_{i_s} - (p_{j_1} + \cdots + p_{j_s}) &= (R_{I,J}, (p_1, \ldots, p_q)) \\ &= r_{R+1}^{IJ} p_{R+1} + \cdots + r_q^{IJ} p_q, \end{aligned}$$

This identity vanishes if and only if $I \sim J$, which shows that these p_j's satisfy the condition (i) of Proposition 2.4. On the other hand, if (2.5) holds for some m_0, then $m_0 p_i - m_0 p_j = 0$ and so $p_i = p_j$. Conversely, assume that $p_i = p_j$ for $1 \leq i < j \leq q$. Then, $1 \leq i \leq R$. If $1 \leq i \leq R < j \leq q$, then (2.5) holds for $m_0 := \tilde{\ell}_{ii}$ and, if $1 \leq i < j \leq R$, then $(1/\tilde{\ell}_{ii})(\tilde{\ell}_{iR+1}, \ldots, \tilde{\ell}_{iq}) = (1/\tilde{\ell}_{jj})(\tilde{\ell}_{jR+1}, \ldots, \tilde{\ell}_{jq})$, which gives also (2.5) for some nonzero integer m_0. This completes the proof of Proposition 2.4.

Proposition 2.7. *Take real numbers $p_1, p_2, \ldots, p_q$ satisfying the conditions of Proposition* 2.4 *and q elements $g_1, \ldots, g_q$ in a torsion free abelian group $\mathcal{G}$. If $p_i = p_j$ for some i, j with $1 \leq i < j \leq q$, then there are some positive integer m_0 and $I_1, J_1, \ldots, I_{k_0}, J_{k_0} \in \mathcal{I}_{q,s}$ with $I_\ell \overset{R}{\sim} J_\ell$ $(1 \leq \ell \leq k_0)$ such that*

$$(g_i/g_j)^{m_0} = \prod_{\ell=1}^{k_0} G_{I_\ell}/G_{J_\ell},$$

where $G_I := g_{i_1} g_{i_2} \cdots g_{i_s}$ for $I = (i_1, \ldots, i_s) \in \mathcal{I}_{q,s}$ and the number k_0 is taken so as to be bounded by a constant depending only on q.

Proof. By Proposition 2.4, there is a nonzero integer m_0 satisfying (2.5). Since $\mathcal{R}$ is generated by $R_{I,J}$ with $I \overset{R}{\sim} J$, this implies that

$$(g_i/g_j)^{m_0} = \prod_{\ell=1}^{k_0} G_{I_\ell}/G_{J_\ell}$$

for $I_\ell, J_\ell \in \mathcal{I}_{q,s}$ with $I_\ell \overset{R}{\sim} J_\ell$, Moreover, the number k_0 can be taken so as to be bounded above by a constant depending only on q, because there are only finitely many possible cases in these combinatorial considerations.

Definition 2.8. Let $\overset{R}{\sim}$ be a pre-equivalence relation among the elements in $\mathcal{I}_{q,s}$. For $1 \leq s < r \leq q$, we say that the relation $\overset{R}{\sim}$ have the property $(P_{r,s})$ if any chosen r distinct elements $\iota(1), \iota(2), \ldots, \iota(r)$ in $\mathcal{I}$ satisfy the condition that, for any given $i_1, \dots, i_s$ $(1 \leq i_1 < \dots < i_s \leq r)$ there exist some other $j_1, \dots, j_s$ $(1 \leq j_1 < \dots < j_s \leq r,\ \{i_1, \dots, i_s\} \neq \{j_1, \dots, j_s\})$ such that

$$(\iota(i_1), \iota(i_2), \dots, \iota(i_s)) \overset{R}{\sim} (\iota(j_1), \iota(j_2), \dots, \iota(j_s)).$$

Now, take a pre-equivalence relation $\overset{R}{\sim}$ among the elements in $\mathcal{I}_{q,s}$ with the property $(P_{r,s})$ and choose real numbers $p_1, p_2, \dots, p_q$ satisfying the conditions in Proposition 2.4. Changing labels $1, 2, \dots, q$, we assume that

$$p_1 \leq p_2 \leq \dots \leq p_q. \tag{2.9}$$

PROPOSITION 2.10. (i) $p_s = p_{s+1} = \dots = p_{s+u}$ *for some* $u \geq q-r+1$.

(ii) *Choose* r *distinct elements in* $\{1, 2, \dots, q\}$ *arbitrarily, say* $1, 2, \dots, r$. *Assume that*

$$p_1 \leq \dots \leq p_{t-1} < p_t = \dots = p_{s+v} < p_{s+v+1} \leq \dots \leq p_r$$

for some t *and* v *with* $1 \leq t \leq s$, $1 \leq v \leq r-s$. *If for some* $i_1, \dots, i_s$ *with* $1 \leq i_1 < \dots < i_s \leq r$

$$p_1 + \dots + p_s = p_{i_1} + \dots + p_{i_s},$$

then $(i_1, \dots, i_{t-1}) = (1, \dots, t-1)$ *and* $t \leq i_j \leq s+v$ *for* $t \leq j \leq s$.

Proof. Take the number v with $0 \leq v \leq q-s$ such that

$$p_1 \leq \dots \leq p_s = \dots = p_{s+v} < p_{s+v+1} \leq \dots \leq p_q,$$

and assume that $0 \leq v < q-r+1$. In $\mathcal{I}$, we choose r elements $\iota(1) := 1$, $\dots, \iota(s) := s$, $\iota(s+1) := q-r+s+1, \dots, \iota(r) := q$. By the assumption, for $I = (1, 2, \dots, s) \in \mathcal{I}_{r,s}$, we can take some other $J = (j_1, \dots, j_s) \in \mathcal{I}_{r,s}$ such that $(\iota(1), \iota(2), \dots, \iota(s)) \overset{R}{\sim} (\iota(j_1), \iota(j_2), \dots, \iota(j_s))$. This gives

$$p_1 + \dots + p_s = p_{\iota(j_1)} + \dots + p_{\iota(j_s)},$$

and so

$$(p_{\iota(j_1)} - p_1) + (p_{\iota(j_2)} - p_2) + \dots + (p_{\iota(j_s)} - p_s) = 0.$$

On the other hand, we see easily $i \leq j_i$ and so $p_{\iota(j_i)} - p_i \geq 0$ for $1 \leq i \leq s$. This implies that

$$p_1 = p_{\iota(j_1)},\ p_2 = p_{\iota(j_2)}, \dots,\ p_s = p_{\iota(j_s)}.$$

By the assumption, $p_i < p_{\iota(i')}$ for any i, i' with $1 \leq i \leq s$, $s+1 \leq i' \leq r$. We have necessarily $j_i = i$ for $1 \leq i \leq s$. This is a contradiction. We conclude $v \geq q - r + 1$. This completes the proof of Proposition 2.10 (i).

The proof of (ii) is similar to the above. Under the assumption of Proposition 2.10 (ii), we have

$$p_1 = p_{\iota(i_1)},\, p_2 = p_{\iota(i_2)}, \ldots, p_s = p_{\iota(i_s)},$$

whence we get $(i_1, \ldots, i_{t-1}) = (1, \ldots, t-1)$ and $t \leq i_j \leq s+v$ for $t \leq j \leq s$. The proof of Proposition 2.10 is completed.

§3. An application of Borel's method

Let f be a nonzero meromorphic function on a domain in $\mathbb{C}^n$. For a set $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and define $D^\alpha f := (\partial^{|\alpha|} f)/(\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n)$. Consider a vector-valued meromorphic function $F = (f_1, \ldots, f_p)$ on $\mathbb{C}^n$. For each $a \in \mathbb{C}^n$, we denote by $\mathcal{M}_a$ the set of all germs of meromorphic functions at a and, for each $\kappa \geq 0$, by $\mathcal{F}^\kappa$ the $\mathcal{M}_a$-submodule of $\mathcal{M}_a^p$ generated by $\{D^\alpha F := (D^\alpha f_1, \ldots, D^\alpha f_p)\ ;\ |\alpha| \leq \kappa\}$. Set $\ell_F(\kappa) := \operatorname{rank}_{\mathcal{M}_a} \mathcal{F}^\kappa$.

Definition 3.1. Assume that meromorphic functions $f_1, \ldots, f_p$ are linearly independent over $\mathbb{C}$. For p vectors $\alpha^i := (\alpha_{i1}, \ldots, \alpha_{in})$ $(1 \leq i \leq p)$ composed of nonnegative integers α_{ij}, we call a set $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^p)$ an admissible set for $F = (f_1, \ldots, f_p)$ if $\{D^{\alpha^1} F, \ldots, D^{\alpha^{\ell_F(\kappa)}} F\}$ is a basis of $\mathcal{F}^\kappa$ for each $\kappa = 1, 2, \ldots, \kappa_0 := \min\{\kappa'\ ;\ \ell_F(\kappa') = p\}$.

By definition, for an admissible set $(\alpha^1, \alpha^2, \ldots, \alpha^p)$ we have

$$\det\bigl(D^{\alpha^1} F, \ldots, D^{\alpha^p} F\bigr) \not\equiv 0.$$

As was shown in [5], we have the following:

Proposition 3.2. ([5, Proposition 4.5]) *For arbitrarily given linearly independent meromorphic functions $f_1, \ldots, f_p$ on $\mathbb{C}^n$, there exists an admissible set $\alpha = (\alpha^1, \ldots, \alpha^p)$ with $|\alpha^\ell| \leq \ell - 1$.*

Proposition 3.3. ([5, Proposition 4.9]) *Let $\alpha = (\alpha^1, \ldots, \alpha^p)$ be an admissible set for $F = (f_1, \ldots, f_p)$ and let h be a holomorphic function. Then,*

$$\det\bigl(D^{\alpha^1}(hF), \ldots, D^{\alpha^p}(hF)\bigr) = h^p \det\bigl(D^{\alpha^1} F, \ldots, D^{\alpha^p} F\bigr).$$

We say that a polynomial $Q(\dots, X_j^\alpha, \dots)$ in variables $\dots, X_j^\alpha, \dots$, where $j = 1, 2, \dots$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integers α_ℓ, is of weight d if

$$\tilde{Q}(t_1, t_2, \dots) := Q(\dots, t_j^{|\alpha|}, \dots)$$

is of degree d as a polynomial in $t_1, t_2, \dots$, where a polynomial of weight 0 means a constant function.

Let $h_1, h_2, \dots$ be finitely many nonzero meromorphic functions on $\mathbb{C}^n$. By a rational function of weight $\leq d$ in logarithmic derivatives of h_j's we mean a nonzero meromorphic function φ on $\mathbb{C}^n$ which is represented as

$$\varphi = \frac{P\big(\dots, D^\alpha h_j/h_j, \dots\big)}{Q\big(\dots, D^\alpha h_j/h_j, \dots\big)}$$

with polynomials $P(\dots, X_j^\alpha, \dots)$ and $Q(\dots, X_j^\alpha, \dots)$ in variables $\dots, X_j^\alpha, \dots$ of weight $\leq d$. Particularly, if we can take $Q = 1$ in the above representation, φ is called a polynomial of weight $\leq d$ in logarithmic derivatives of h_j's.

PROPOSITION 3.4. *Let $h_1, h_2, \dots, h_p$ and $a_1, a_2, \dots, a_p$ be nonzero meromorphic functions on $\mathbb{C}^n$ such that each a_i $(1 \leq i \leq p)$ is a rational function of weight $\leq d$ in logarithmic derivatives of h_j's. Assume that*

$$a_1 h_1 + a_2 h_2 + \cdots + a_p h_p = 0$$

for some $p \geq 2$. Then, the set $\{1, 2, \dots, p\}$ of indices has a partition

$$\{1, 2, \dots, p\} = J_1 \cup J_2 \cup \cdots \cup J_k, \ \ \#J_\alpha \geq 2 \text{ for all } \alpha, \ J_\alpha \cap J_\beta = \emptyset \text{ for } \alpha \neq \beta$$

such that, for each α,

(i) $\sum_{i \in J_\alpha} a_i h_i = 0$,

(ii) *$h_{i'}/h_i$ $(i, i' \in J_\alpha)$ are rational functions in logarithmic derivatives of h_j's with weights bounded by a constant $D(d, p)$ depending only on d and p.*

For the proof, we first give the following:

LEMMA 3.5. *If a nonzero meromorphic function a on $\mathbb{C}^n$ can be written as a polynomial in logarithmic derivatives of h_j's with weight d, then $D^\alpha a$ is also written as a polynomial in logarithmic derivatives of h_j's with weight $\leq d + |\alpha|$.*

Proof. It suffices to show Lemma 3.5 for the case $m := |\alpha| = 1$, because general cases are shown by induction on m. Assume that a is written as

$$a = P\Big(\dots, \frac{D^\alpha h_j}{h_j}, \dots\Big)$$

with a polynomial $P(\dots, X_j^\alpha, \dots)$ with weight d. Then, for $D_i = \partial/\partial z_i$ $(1 \leq i \leq n)$ we get

$$D_i a = \sum_{j,\alpha} \frac{\partial P}{\partial X_j^\alpha} D_i\Big(\frac{D^\alpha h_j}{h_j}\Big).$$

On the other hand, it is easily seen that $\partial P/\partial X_j^\alpha$ is a polynomial of weight $\leq d - |\alpha|$ and $D_i(D^\alpha h_j/h_j)$ is represented as a polynomial of weight $\leq |\alpha|+1$ in logarithmic derivatives of h_j. These give Lemma 3.5.

Proof of Proposition 3.4. This is proved by induction on p. For the case $p = 2$, we have nothing to prove, because $h_1/h_2 = -a_2/a_1$. Assume that $p \geq 3$.

We first show that there are some indices $i_1 := 1, i_2, \dots, i_{p_0}$, where $p_0 \geq 2$, such that h_{i_ℓ}/h_1 $(2 \leq \ell \leq p_0)$ can be written as a rational function of logarithmic derivatives of h_j's whose weight is bounded by a constant depending only on d and p_0. To this end, we take a subset J of $\mathcal{I}_p := \{1, 2, \dots, p\}$ such that $\#J$ takes the minimum among all subsets J' of $\mathcal{I}_q$ satisfying the condition that $1 \in J'$ and $\sum_{i \in J'} c_i a_i h_i = 0$ for some nonzero constants $c_i \in \mathbb{C}$. Changing indices if necessary, we assume that $J = \{1, 2, \dots, p_0\}$, where $p_0 \geq 2$ because of $a_i h_i \neq 0$ for each i. By definition of p_0, there are some nonzero constants c_i such that

$$c_1 a_1 h_1 + c_2 a_2 h_2 + \dots + c_{p_0} a_{p_0} h_{p_0} = 0. \tag{3.6}$$

Moreover, $a_1 h_1, a_2 h_2, \dots, a_{p_0-1} h_{p_0-1}$ are linearly independent over $\mathbb{C}$. In fact, if there is a nonzero vector $(d_1, \dots, d_{p_0-1})$ with

$$d_1 a_1 h_1 + d_2 a_2 h_2 + \dots + d_{p_0-1} a_{p_0-1} h_{p_0-1} = 0,$$

we can easily construct the identity of the form (3.6) with less than p_0 terms, which contradicts the property of J. We set $\varphi_i := c_i a_i h_i$ for $1 \leq i \leq p_0$. By the use of Proposition 3.2, we can choose an admissible set $\alpha = (\alpha^1, \dots, \alpha^{p_0-1})$ with $|\alpha| := |\alpha^1| + \dots + |\alpha^{p_0-1}| \leq (p_0 - 2)(p_0 - 1)/2$

for the functions $\varphi_1, \varphi_2, \ldots, \varphi_{p_0-1}$, where $\alpha^1 = (0, \ldots, 0)$. We differentiate both sides of (3.6) and get

$$D^{\alpha^\ell}\varphi_1 + \cdots + D^{\alpha^\ell}\varphi_{p_0} = \frac{D^{\alpha^\ell}\varphi_1}{h_1}h_1 + \cdots + \frac{D^{\alpha^\ell}\varphi_{p_0}}{h_{p_0}}h_{p_0} = 0$$

for $\ell = 1, 2, \ldots, p_0 - 1$. We regard these identities as a simultaneous system of linear equations with unknowns $h_1, \ldots, h_{p_0}$, and obtain

$$(h_1 : h_2 : \cdots : h_{p_0}) = (\Delta_1 : -\Delta_2 : \cdots : (-1)^{p_0-1}\Delta_{p_0}),$$

where

$$\Delta_i := \begin{vmatrix} \frac{\varphi_1}{h_1} & \cdots & \frac{\varphi_{i-1}}{h_{i-1}} & \frac{\varphi_{i+1}}{h_{i+1}} & \cdots & \frac{\varphi_{p_0}}{h_{p_0}} \\ \frac{D^{\alpha^2}\varphi_1}{h_1} & \cdots & \frac{D^{\alpha^2}\varphi_{i-1}}{h_{i-1}} & \frac{D^{\alpha^2}\varphi_{i+1}}{h_{i+1}} & \cdots & \frac{D^{\alpha^2}\varphi_{p_0}}{h_{p_0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{D^{\alpha^{p_0-1}}\varphi_1}{h_1} & \cdots & \frac{D^{\alpha^{p_0-1}}\varphi_{i-1}}{h_{i-1}} & \frac{D^{\alpha^{p_0-1}}\varphi_{i+1}}{h_{i+1}} & \cdots & \frac{D^{\alpha^{p_0-1}}\varphi_{p_0}}{h_{p_0}} \end{vmatrix}.$$

On the other hand, as is easily seen, $D^{\alpha^\ell}\varphi_i/h_i$ can be represented as a polynomial in functions $D^\beta a_i$ and $D^\beta h_i/h_i$ with $|\beta| \le |\alpha^\ell|$. Therefore, by the use of Lemma 3.5, each Δ_i can be represented as a polynomial in logarithmic derivatives of h_j's with uniformly bounded weight. These conclude that each h_i/h_1 $(2 \le i \le p_0)$ is represented as a rational function of logarithmic derivatives of h_j's with uniformly bounded weight. For the case $p = p_0$, we have Proposition 3.4. In fact, we may take $k = 1, J_1 = 1$ in Proposition 3.4. In the following, we assume $p_0 < p$.

Now, we set

$$\tilde{a} := a_1 + a_2\frac{h_2}{h_1} + \cdots + a_{p_0}\frac{h_{p_0}}{h_1}.$$

As was shown above, $\tilde{a}$ is a rational function of logarithmic derivatives of h_j's with uniformly bounded weight. On the other hand, the assumption implies that

$$\tilde{a}h_1 + a_{p_0+1}h_{p_0+1} + \cdots + a_p h_p = 0.$$

If $\tilde{a} = 0$, then we easily obtain the desired conclusion by applying the induction hypothesis for the case $\le p - 1$. On the other hand, for the case $\tilde{a} \ne 0$, we can also apply the induction assumption to get the desired conclusion, because $1 + (p - p_0) < p$. Since there are only finitely many possible cases where the indices $1, 2, \ldots, p_0$ are chosen, all weights appearing in the above discussion are bounded by a constant depending only on d and p. The proof of Proposition 3.4 is completed.

§4. Relations among the pull-backs of hyperplanes

Let f and g be nondegenerate meromorphic maps of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ with reduced representations $f = (f_1 : \cdots : f_{N+1})$, $g = (g_1 : \cdots : g_{N+1})$ respectively. Here, a reduced representation $f = (f_1 : \cdots : f_{N+1})$ means that f_j are holomorphic functions on $\mathbb{C}^n$ with $\dim\{f_1 = \cdots = f_{N+1} = 0\} \leq n-2$.

Let

$$H_j : a_{j1}w_1 + \cdots + a_{jN+1}w_{N+1} = 0 \quad (1 \leq j \leq q)$$

be hyperplanes in general position, where $q \geq 2N+2$. We define meromorphic functions h_j $(1 \leq j \leq q)$ on $\mathbb{C}^n$ by

$$h_j := \frac{a_{j1}g_1 + \cdots + a_{jN+1}g_{N+1}}{a_{j1}f_1 + \cdots + a_{jN+1}f_{N+1}}. \tag{4.1}$$

Consider the set $\mathcal{I} := \{1, 2, \ldots, q\}$ as in §2 and $\mathcal{I}_{q,N+1}$ of all combinations of $N+1$ elements in $\mathcal{I}$.

Definition 4.2. Two combinations $I = (i_1, \ldots, i_{N+1})$ and $J = (j_1, \ldots, j_{N+1}) \in \mathcal{I}_{q,N+1}$ are said to be R-related and indicated as $I \overset{R}{\sim} J$ if we have a representation

$$\frac{h_{i_1}h_{i_2}\cdots h_{i_{N+1}}}{h_{j_1}h_{j_2}\cdots h_{j_{N+1}}} = \frac{Q_1(\ldots, D^\alpha h_j/h_j, \ldots)}{Q_2(\ldots, D^\alpha h_j/h_j, \ldots)}$$

with polynomials $Q_1(\ldots, X_j^\alpha, \ldots)$ and $Q_2(\ldots, X_j^\alpha, \ldots)$ with weights bounded from above by the constant $D(0, p)$ given in Proposition 3.4, where $p := \binom{2N+2}{N+1}$.

Obviously, the relation $\overset{R}{\sim}$ is a pre-equivalence relation.

In this situation, we can prove the following:

Proposition 4.3. *The relation* $\overset{R}{\sim}$ *have the property* $(P_{2N+2,N+1})$.

Proof. Choose arbitrary $2N+2$ distinct indices $\iota(i)$ among $\{1, 2, \dots, q\}$, say $\iota(1) = 1, \iota(2) = 2, \dots, \iota(2N+2) = 2N+2$. By the definition of h_j's, we have

$$a_{i1}g_1 + \cdots + a_{iN+1}g_{N+1} - h_i a_{i1} f_1 - \cdots - h_i a_{iN+1} f_{N+1} = 0 \qquad (1 \leq i \leq 2N+2)$$

From these $2N+2$ identities eliminating $2N+2$ functions $f_1, \dots, f_{N+1}, g_1, \dots, g_{N+1}$, we get

$$\Psi := \det(a_{i1}, \dots, a_{iN+1}, h_i a_{i1}, \dots, h_i a_{iN+1} \,;\, 1 \leq i \leq 2N+2) = 0. \tag{4.4}$$

With each combination $I = (i_1, i_2, \dots, i_{N+1})$ $(1 \leq i_1 < \cdots < i_{N+1} \leq 2N+2)$ we associate $J = (j_1, j_2, \dots, j_{N+1})$ $(1 \leq j_1 < \cdots < j_{N+1} \leq 2N+2)$ such that

$$\{i_1, i_2, \dots, i_{N+1}, j_1, j_2, \dots, j_{N+1}\} = \{1, 2, \dots, 2N+2\},$$

and set

$$A_I = (-1)^{i_1 + \cdots + i_{N+1} + (N+1)(N+2)/2} \det(a_{i_r s} \,;\, 1 \leq r, s \leq N+1) \times \det(a_{j_r s} \,;\, 1 \leq r, s \leq N+1),$$

where $A_I \neq 0$ because H_j's are assumed to be in general position. Then, by the Laplace expansion formula,

$$\Psi = \sum_{I \in \mathcal{I}_{2N+2, N+1}} A_I h_I = 0,$$

where $h_I := h_{i_1} h_{i_2} \cdots h_{i_{N+1}}$ for $I = (i_1, i_2, \dots, i_{N+1})$. We now apply Proposition 3.4 to show that $\mathcal{I}_{2N+2,N+1}$ is divided as

$$\mathcal{I}_{2N+2,N+1} = J_1 \cup J_2 \cup \cdots \cup J_k, \ \ \#J_\alpha \geq 2 \text{ for all } \alpha, \ J_\alpha \cap J_\beta = \emptyset \text{ for } \alpha \neq \beta$$

such that, for each α, $\sum_{I \in J_\alpha} A_I h_I = 0$ and each $h_I / h_{I'}$ $(I, I' \in J_\alpha)$ is a rational function in the logarithmic derivatives of h_j's whose weight is bounded above by $D(0, p)$. This concludes that, for any given $i_1, \dots, i_{N+1}$ $(1 \leq i_1 < \cdots < i_{N+1} \leq 2N+2)$ there exists some other $j_1, \dots, j_{N+1}$ $(1 \leq j_1 < \cdots < j_{N+1} \leq 2N+2,\ \{i_1, \dots, i_{N+1}\} \neq \{j_1, \dots, j_{N+1}\})$ such that $(i_1, i_2, \dots, i_{N+1}) \overset{R}{\sim} (j_1, j_2, \dots, j_{N+1})$, because each J_α contains at least two elements. This completes the proof of Proposition 4.3.

PROPOSITION 4.5. *In the above situation, assume that $q = 2N + 2$. Then, one of the following two cases occurs*;

(i) *There is some positive integer m such that, for $1 \leq i < i' \leq 2N+2$, $(h_{i'}/h_i)^m$ are rational functions in logarithmic derivatives of h_j's whose weights divided by m are bounded by a constant depending only on N.*

(ii) *$2N+1$ functions among $h_1, \dots, h_{2N+2}$ are algebraically dependent.*

Proof. We take real numbers $p_1, p_2, \dots, p_{2N+2}$ satisfying the conditions in Proposition 2.4. Changing indices, we assume that

$$p_1 \leq p_2 \leq \cdots \leq p_{2N+2}.$$

If $p_1 = \cdots = p_{2N+2}$, then we can apply Proposition 2.7 to the torsion-free abelian group $\mathcal{G}$ of all nonzero meromorphic functions on $\mathbb{C}^n$ and $g_j := h_j \in \mathcal{G}$ $(1 \leq j \leq 2N+2)$ to get the case (i) of Proposition 4.5.

Assume that $p_{i_0} < p_{i_0+1}$ for some i_0, where $i_0 \neq N+1$ by Proposition 2.10. Replacing each p_i by $-p_i$ if necessary, we may assume that $1 \leq i_0 \leq N$. We now observe a combination $(j_1, \dots, j_{N+1}) \in \mathcal{I}_{2N+2,N+1}$ such that

$$p_1 + p_2 + \cdots + p_{N+1} = p_{j_1} + p_{j_2} + \cdots + p_{j_{N+1}}.$$

Proposition 2.10 implies that

$$j_1 = 1, \dots, j_{i_0} = i_0, \quad i_0 + 1 \leq j_{i_0+1} < \cdots < j_{N+1} \leq 2N+2. \tag{4.6}$$

Therefore, the set

$$\mathcal{J} := \{J \in \mathcal{I}_{2N+2,N+1} \,;\, (1, 2, \dots, N+1) \sim J\}$$

consists of combinations satisfying the above condition (4.6), where $\sim$ means the eqivalence relation defined by Definition 2.2 associated with the relation $\overset{R}{\sim}$. As a consequence of Proposition 3.4, we have a nontrivial relation

$$\begin{aligned}
&\sum_{J=(j_1,\dots,j_{N+1})\in\mathcal{J}} A_J h_{j_1} \cdots h_{j_{N+1}} \\
&\qquad = h_1 \cdots h_{i_0} \left(\sum_{J=(1,\dots,i_0,j_{i_0+1},\dots,j_{N+1})\in\mathcal{J}} A_J h_{j_{i_0+1}} \cdots h_{j_{N+1}} \right) = 0
\end{aligned}$$

for the nonzero constants A_J. This shows that there is a nontrivial algebraic relation among $h_{i_0+1}, h_{i_0+2}, \dots, h_{2N+2}$.

For a nonzero meromorphic function F on $\mathbb{C}^n$ and $a \in \mathbb{C}$, we define the divisor $\nu_F^a : \mathbb{C}^n \to \mathbb{Z}$ of F by setting

$$\nu_F^a(z) := \text{the vanishing order of } F - a \text{ at each point } z \in \mathbb{C}^n.$$

We also define $\nu_F^\infty = \nu_{1/F}^0$, $\nu_F := \nu_F^0 - \nu_F^\infty$ and $\bar{\nu} := \min(\nu, 1)$. For a hyperplane

$$H : a_1 w_1 + \cdots + a_{N+1} w_{N+1} = 0$$

and a nondegenerate meromorphic map $f : \mathbb{C}^n \to P^N(\mathbb{C})$ with a reduced representation $f = (f_1 : \cdots : f_{N+1})$, we define $\nu(f, H) := \nu_{a_1 f_1 + \cdots + a_{N+1} f_{N+1}}$. Choose an admissible set $\alpha = (\alpha^1, \dots, \alpha^{N+1})$ for $(f_1, \dots, f_{N+1})$ and define the generalized Wronskian W_f^α by

$$W_f^\alpha := \det\Big(D^{\alpha^\ell} f_1, D^{\alpha^\ell} f_2, \dots, D^{\alpha^\ell} f_{N+1}\,;\, 1 \le \ell \le N+1\Big).$$

Although W_f^α depends on a choice of reduced representations, the divisor $\nu_{W_f^\alpha}^0$ depends only on f.

PROPOSITION 4.7. *For a nondegenerate meromorphic map $f : \mathbb{C}^n \to P^N(\mathbb{C})$ and hyperplanes $H_1, H_2, \dots, H_q$ in general position,*

$$\sum_{j=1}^{q} (\nu(f, H_j) - N)^+ \le \nu_{W_f^\alpha}^0 \tag{4.8}$$

outside a set of dimension $\le n-2$, where $\nu^+ = \max(\nu, 0)$.

Proof. Let $A := \{f_1 = f_2 = \cdots = f_{N+1} = 0\}$. Since $\dim A \le n-2$, it suffices to show (4.8) at every point $a \in \mathbb{C}^n - A$. Since H_j's are in general position, we have $\#S \le N$ for the set $S := \{j \,;\, \nu(f, H_j)(a) > N\}$. We may assume $S \ne \emptyset$. For, otherwise, (4.8) is obvious. Changing indices and homogeneous coordinates $(w_1 : \cdots : w_{N+1})$ on $P^N(\mathbb{C})$, we may assume that $S = \{1, 2, \dots, k\}$, where $1 \le k \le N$ and $H_j := \{w_j = 0\}$ for $1 \le j \le k$. For an admissible set $\alpha = (\alpha^1, \dots, \alpha^{N+1})$, we have

$$W_f^\alpha = \sum_{(i_1, \dots, i_{N+1}) \in S_{N+1}} \operatorname{sgn}\begin{pmatrix} 1 & \cdots & N+1 \\ i_1 & \cdots & i_{N+1} \end{pmatrix}$$
$$\times D^{\alpha^{i_1}} f_1 \cdots D^{\alpha^{i_k}} f_k D^{\alpha^{i_{k+1}}} f_{k+1} \cdots D^{\alpha^{i_{N+1}}} f_{N+1},$$

where S_{N+1} denotes all permutations of $\{1, 2, \dots, N+1\}$. Since we may assume that $|\alpha^\ell| \leq N$ by Proposition 3.2, $\nu^0_{D^{\alpha^i} f_i}(a) \geq (\nu^0_{f_i}(a) - N)^+$ outside the union of all singularities of the analytic sets $\{f_i = 0\}$, and so we have

$$\nu_G(a) \geq \sum_{i=1}^{k} (\nu_{f_i}(a) - N)^+ = \sum_{j=1}^{q} (\nu(f, H_j)(a) - N)^+$$

outside an analytic set of dimension $\leq n-2$ for each $G := D^{\alpha^{i_1}} f_1 \cdots D^{\alpha^{i_k}} f_k$. This yields the desired conclusion.

Now, we assume that

$$\min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0) \quad (1 \leq j \leq q)$$

for a positive number ℓ_0.

Take admissible sets α and β for the maps f and g respectively. Then, we have the following:

Proposition 4.9.
(i) $\displaystyle\sum_{j=1}^{q} (\ell_0 - N)\bar{\nu}^\infty_{h_j} \leq \sum_{j=1}^{q} (\nu(f, H_j) - N)^+ \leq \nu^0_{W^\alpha_f}$.
(ii) $\displaystyle\sum_{j=1}^{q} (\ell_0 - N)\bar{\nu}^0_{h_j} \leq \sum_{j=1}^{q} (\nu(g, H_j) - N)^+ \leq \nu^0_{W^\beta_g}$.

Proof. By the assumption, $\{z\,;\, \nu(f, H_j)(z) \geq \ell_0\} = \{z\,;\, \nu(g, H_j)(z) \geq \ell_0\}$, which we denote by A. We have $\nu^0_{h_j}(z) = \nu^\infty_{h_j}(z) = 0$ for each point $z \notin A$, because $\nu(f, H_j)(z) = \nu(g, H_j)(z)$. Take a pole a of h_j. Then, we have $a \in A$. Therefore, $(\ell_0 - N)\bar{\nu}^\infty_{h_j}(a) \leq (\nu(f, H_j)(a) - N)^+$, which gives the first inequality of (i). The second inequality of (i) is due to Proposition 4.7. The proof of the assertion (ii) is similar to the proof of (i).

§5. A degeneracy theorem for two meromorphic maps

In this section, we give the following degeneracy theorem for two meromorphic maps into $P^N(\mathbb{C})$, which is a restatement of Theorem 1.5.

Theorem 5.1. *Let $f, g : \mathbb{C}^n \to P^N(\mathbb{C})$ be nondegenerate meromorphic maps and let $H_1, \dots, H_{2N+2}$ be hyperplanes in general position. For a sufficiently large integer ℓ_0 depending only on N, if*

$$\min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0) \quad (1 \leq j \leq 2N+2), \tag{5.2}$$

then the map $f \times g : \mathbb{C}^n \to P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is algebraically degenerate.

For the proof of Theorem 5.1, we recall some results from value distribution theory for meromorphic maps into $P^N(\mathbb{C})$.

As usual, we set $\|z\| := \left(\sum_{j=1}^{n} |z_j|^2\right)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $B(r) := \{z \,;\, \|z\| < r\}$, $S(r) := \{z \,;\, \|z\| = r\}$ and

$$d^c := \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial), \quad v := (dd^c\|z\|^2)^{n-1},$$
$$\sigma := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}.$$

For a meromorphic map $f : \mathbb{C}^n \to P^N(\mathbb{C})$ with a reduced representation $f = (f_1 : \cdots : f_{N+1})$, set $\|f\| := \left(\sum_{j=1}^{N+1} |f_j|^2\right)^{1/2}$. We define the order function of f by

$$T(r, f) = \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma,$$

and the counting function of a divisor $\nu : \mathbb{C}^n \to \mathbb{Z}$ by

$$N(r, \nu) := \int_1^r \frac{n(t)}{t}\, dt \quad (1 < r < +\infty),$$

where $n(t) := t^{2-2n} \int_{|\nu| \cap B(t)} \nu v$ for $n \geq 2$ and $n(t) := \sum_{|z| \leq t} \nu(z)$ for $n = 1$.

We have the following Jensen's formula:

PROPOSITION 5.3. *Let φ be a nonzero meromorphic function on $\mathbb{C}^n$. Then,*

$$N(r, \nu_\varphi) = \int_{S(r)} \log |\varphi| \sigma - \int_{S(1)} \log |\varphi| \sigma.$$

For the proof, see [11, p. 248].

Let $f : \mathbb{C}^n \to P^N(\mathbb{C})$ be a meromorphic map, H a hyperplane with $f(\mathbb{C}^n) \not\subset H$ and m a positive integer or $+\infty$. The (truncated) counting function of H for f by

$$N_m(r, H) \equiv N_m^f(r, H) := N(r, \min(\nu(f, H), m)).$$

For brevity, we set $N(r, H) := N_{+\infty}^f(r, H)$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^n$, which are occationally regarded as a meromorphic map into $P^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r; \varphi) := \int_{S(r)} \log \max(|\varphi|, 1) \sigma.$$

Take two distinct hyperplanes $H_k = \left\{\sum_{j=1}^{N+1} a_{kj} w_j = 0\right\}$ with $f(\mathbb{C}^n) \not\subset H_k$ $(k = 1, 2)$ and consider a meromorphic function

$$\varphi_f^{H_1,H_2} := \frac{\sum_{j=1}^{N+1} a_{1j} f_j}{\sum_{j=1}^{N+1} a_{2j} f_j}.$$

We can easily prove

$$\begin{aligned} T\Big(r, \varphi_f^{H_1,H_2}\Big) &= N\Big(r, \nu^{\infty}_{\varphi_f^{H_1,H_2}}\Big) + m\Big(r; \varphi_f^{H_1,H_2}\Big) + O(1) \\ &\leq T(r, f) + O(1). \end{aligned} \tag{5.4}$$

As usual, by the notation "$\|$ P" we mean the assertion P holds for all $r \in [0, +\infty)$ excluding a Borel subset E of the interval $[0, +\infty)$ with $\int_E dr < +\infty$. The following so-called logarithmic derivative lemma acts essential roles in Nevanlinna theory.

THEOREM 5.5. *For any $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have*

$$\| \;\; m\left(r; \frac{D^{\alpha}\Big(\varphi_f^{H_1,H_2}\Big)}{\varphi_f^{H_1,H_2}}\right) = o(T(r, f)).$$

For the proof, refer to [5] and [9, Lemma 3.11].

PROPOSITION 5.6. *Let $f : \mathbb{C}^n \to P^N(\mathbb{C})$ be a nondegenerate meromorphic map which is represented as $f = (\varphi_1 : \cdots : \varphi_{N+1})$ with nonzero meromorphic functions φ_i on $\mathbb{C}^n$. Take a nonzero holomorphic function h on $\mathbb{C}^n$ such that $h\varphi_i$ are holomorphic for $1 \leq i \leq N+1$. Then,*

$$T(r, f) \leq N(r, \nu_h^0) + \sum_{j=1}^{N+1} m(r; \varphi_j) + O(1).$$

Proof. If we take a reduced representation $f = (f_1 : \cdots : f_{N+1})$, then we can find a nonzero holomorphic function $\tilde{h}$ such that $h\varphi_i = \tilde{h} f_i$ $(1 \leq i \leq N+1)$. By the use of Proposition 5.3, we have

$$T(r, f) \leq T(r, f) + N(r, \nu_{\tilde{h}}^0) = \int_{S(r)} \log(|\tilde{h}| \|f\|)\sigma + O(1)$$

$$= \int_{S(r)} \log |h|\sigma + \int_{S(r)} \log \Big(\sum_{j=1}^{N+1} |\varphi_j|^2 \Big)^{1/2} \sigma + O(1)$$
$$\leq N(r, \nu_h^0) + \sum_{j=1}^{N+1} m(r; \varphi_j) + O(1).$$

This gives Proposition 5.6.

COROLLARY 5.7. ([12]) *Let $f : \mathbb{C}^n \to P^N(\mathbb{C})$ be a meromorphic map with a reduced representation $f = (f_1 : \cdots : f_{N+1})$, where we assume $f_{N+1} \not\equiv 0$. Then, for $\varphi_i := f_i/f_{N+1}$ $(1 \leq i \leq N)$,*

$$T(r, f) \leq \sum_{i=1}^{N} T(r, \varphi_i) + O(1).$$

Proof. For every zero z_0 of f_{N+1} outside $\{f_1 = \cdots = f_{N+1} = 0\}$, there is some j_0 with $f_{j_0}(z_0) \neq 0$, whence $\nu^0_{f_{N+1}} \leq \sum_{i=1}^N \nu^\infty_{\varphi_i}$ outside an analytic set of dimension $\leq n-2$. It follows from Proposition 5.6 and (5.4) that

$$T(r, f) \leq N(r, \nu^0_{f_{N+1}}) + \sum_{i=1}^{N} m(r, \varphi_i) + O(1)$$
$$\leq \sum_{i=1}^{N} \big(N(r, \nu^\infty_{\varphi_i}) + m(r; \varphi_i) \big) + O(1) = \sum_{i=1}^{N} T(r, \varphi_i) + O(1).$$

For our purpose, we need another algebraic lemma. Let

$$H_i : a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{iN+1}w_{N+1} = 0 \quad (1 \leq i \leq 2N+2)$$

be hyperplanes in general position. Choose arbitrary $2N+1$ indices among $1, 2, \ldots, 2N+2$, say, $1, 2, \ldots, 2N+1$, and consider the rational map $\Phi : P^N(\mathbb{C}) \times P^N(\mathbb{C}) \to P^{2N}(\mathbb{C})$ defined as follows:

For $v = (v_1 : \cdots : v_{N+1}), w = (w_1 : \cdots : w_{N+1}) \in P^N(\mathbb{C})$, we define the value $\Phi(w, v) = (u_1 : u_2 : \cdots : u_{2N+1}) \in P^{2N}(\mathbb{C})$ by

$$u_i := \frac{a_{i1}w_1 + \cdots + a_{iN+1}w_{N+1}}{a_{i1}v_1 + \cdots + a_{iN+1}v_{N+1}} \quad (1 \leq i \leq 2N+1). \tag{5.8}$$

PROPOSITION 5.9. *The map Φ is a birational map of $P^N(\mathbb{C}) \times P^N(\mathbb{C})$ onto $P^{2N}(\mathbb{C})$.*

Proof. By (5.8), we have the identities

$$a_{i1}u_iv_1 + \cdots + a_{iN+1}u_iv_{N+1} = a_{i1}w_1 + \cdots + a_{iN+1}w_{N+1} \quad (1 \le i \le 2N+1).$$

We regard these identities as a simultaneous system of linear equations in unknown variables $v_1, \ldots, v_{N+1}, w_1, \ldots, w_{N+1}$ whose coefficients are functions in $u_1, \ldots, u_{2N+1}$. Since we have

$$\mathrm{rank}\big(a_{i1}, \ldots, a_{iN+1}, u_ia_{i1}, \ldots, u_ia_{iN+1}\,;\, 1 \le i \le 2N+1\big) = 2N+1,$$

we can solve these equations and obtain the rational map $\Psi : P^{2N}(\mathbb{C}) \to P^N(\mathbb{C}) \times P^N(\mathbb{C})$ such that $\Psi \cdot \Phi$ and $\Phi \cdot \Psi$ are the identity maps. Therefore, Φ is a birational map.

Now, we go back to the proof of Theorem 5.1. The assumption of Theorem 5.1 enable us to apply the results given in §4. We have one of the cases (i) and (ii) as in Proposition 4.5. If the case (ii) occurs, then the map $f \times g : \mathbb{C}^n \to P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is obviously algebraically degenerate by virtue of Proposition 5.9. Therefore, after suitable changes of indices, we may assume the following:

(5.10) *There is some positive integer m such that, for $1 \le i < i' \le q := 2N+2$, $(h_i/h_{i'})^m$ are rational functions in logarithmic derivatives of h_k's whose weights divided by m are bounded by a constant depending only on N.*

We now choose homogeneous coordinates $(w_1 : \cdots : w_{N+1})$ on $P^N(\mathbb{C})$ such that the given hyperplanes are written as

$$\begin{array}{ll} H_i : w_i = 0 & (1 \le i \le N+1) \\ H_i : a_{i1}w_1 + \cdots + a_{iN+1}w_{N+1} = 0 & (N+2 \le i \le q), \end{array}$$

where any minor of the matrix $(a_{ij}\ ;\ N+2 \le i \le q, 1 \le j \le N+1)$ of order $\le N+1$ does not vanish because H_j's are in general position. In this representation, for a matrix

$$Q := (a_{i1}(h_1 - h_i), a_{i2}(h_2 - h_i), \ldots, a_{iN+1}(h_{N+1} - h_i)\ ;\ N+2 \le i \le 2N+2),$$

the identity (4.4) is rewritten as $\Psi := \det Q = 0$. Set $r := \mathrm{rank} Q$, where $r \le N$. Assume that $r < N$. Then, any minor of Q of order N vanishes identically. Therefore, there is a nontrivial algebraic relation among

$h_1, h_2, \dots, h_{2N+1}$. By substituting $h_i = \sum_j a_{ij} g_j / \sum_j a_{ij} f_j$ $(1 \leq i \leq q)$ into this relation, we have non-trivial algebraic relations among the functions $f_1, f_2, \dots, f_{N+1}, g_1, g_2, \dots, g_{N+1}$ by virtue of Proposition 5.9. This shows that the map $f \times g : \mathbb{C}^n \to P^N(\mathbb{C}) \times P^N(\mathbb{C})$ is algebraically degenerate in this case.

It remains to study the case $r = N$. To complete the proof for this case, it suffices to show the following:

PROPOSITION 5.11. *There is some ℓ_0 depending only on N such that, for the maps f, g satisfying the condition* (5.2), *the case*

$$\operatorname{rank}\bigl(a_{ij}(h_i - h_j)\,;\, 1 \leq i \leq N+1,\ N+2 \leq j \leq 2N+1\bigr) = N$$

is impossible.

Proof. We regard the identities

$$\sum_{j=1}^{N+1} a_{ij}(h_i - h_j) f_j = 0 \quad (N+2 \leq i \leq 2N+1)$$

as a simultaneous system of equations in unknown variables $f_1, \dots, f_{N+1}$ and solve these to obtain the identity

$$f = (f_1 : f_2 : \cdots : f_{N+1}) = (\Phi_1 : \Phi_2 : \cdots : \Phi_{N+1})$$

outside the set of all poles of Φ_i $(1 \leq i \leq N+1)$, where each Φ_i is a homogeneous polynomial of degree N in variables $h_1, h_2, \dots, h_{2N+1}$. We set $\tilde{\Phi}_i := \Phi_i / h_{2N+1}^N$, which are polynomials of degree $\leq N$ in variables $\varphi_j := h_j / h_{2N+1}$ $(1 \leq j \leq 2N)$. Using (5.4), we easily have

$$T(r, \tilde{\Phi}_i) \leq N \sum_{j=1}^{2N} T(r, \varphi_j) + O(1) \quad (1 \leq i \leq N+1).$$

Then, by Corollary 5.7, we have

$$T(r, f) \leq \sum_{i=1}^{N+1} T(r, \tilde{\Phi}_i) + O(1) \leq N(N+1) \sum_{j=1}^{2N} T(r, \varphi_j) + O(1).$$

On the other hand, by (5.10), there are some positive integer m and polynomials Q_1^i, Q_2^i of logarithmic derivatives of h_j $(1 \leq j \leq 2N+2)$ such that $\varphi_i^m = Q_1^i/Q_2^i$, where the weights of Q_k^i divided by m are bounded by a constant $d_1(N)$ depending only on N. We easily show that $T(r, \varphi_i) = (1/m)T(r, \varphi_i^m) + O(1)$ and

$$\begin{aligned} T(r, \varphi_i^m) &\leq T(r, Q_1^i) + T(r, Q_2^i) + O(1) \\ &\leq \sum_{k=1}^{2} \big(N(r, \nu_{Q_k^i}^\infty) + m(r; Q_k^i) \big) + O(1). \end{aligned}$$

Moreover, by the use of Theorem 5.5 and the fact that all poles of Q_k^i are zeros or poles of some h_j and of order at most $md_1(N)$, we can find a constant $d_2(N)$ depending only on N such that

$$\begin{aligned} \| \ T(r, \varphi_i) &\leq d_2(N) \sum_{j=1}^{2N+1} \big(N(r, \bar\nu_{h_j}^0) + N(r, \bar\nu_{h_j}^\infty) \big) + o\Big(\sum_j T(r, h_j) \Big) \\ &\leq d_2(N) \sum_{j=1}^{2N+1} \big(N(r, \bar\nu_{h_j}^0) + N(r, \bar\nu_{h_j}^\infty) \big) + o(T(r, f) + T(r, g)). \end{aligned}$$

From these facts, we can conclude that there exists a positive constant $d_3(N)$ depending only on N such that

$$\| \ T(r, f) \leq d_3(N) \sum_{j=1}^{2N+1} \big(N(r, \bar\nu_{h_j}^0) + N(r, \bar\nu_{h_j}^\infty) \big) + o(T(r, f) + T(r, g)).$$

On the other hand, by Proposition 4.9 we have

$$\sum_{j=1}^{2N+1} (\ell_0 - N) \big(N(r, \bar\nu_{h_j}^\infty) + N(r, \bar\nu_{h_j}^0) \big) \leq N(r, \nu_{W_f^\alpha}^0) + N(r, \nu_{W_g^\beta}^0)$$

for some admissible sets α and β. Moreover, since W_f^α is represented as

$$W_f^\alpha = f_1^{N+1} \chi$$

with a polynomial χ of logarithmic derivatives of the functions $\psi_j := f_j/f_1$ $(2 \leq j \leq N+1)$, we have

$$\begin{aligned} \| \ N(r, \nu_{W_f^\alpha}^0) &= \int_{S(r)} \log |W_f^\alpha| \sigma + O(1) \\ &\leq (N+1) \int_{S(r)} \log \|f\| \sigma + m(r; \chi) \leq (N+1) T(r, f) + o(T(r, f)) \end{aligned}$$

by the use of Proposition 5.3 and Theorem 5.5. Similarly, we have

$$\| \; N(r, \nu^0_{W_g^\beta}) \leq (N+1)T(r,g) + o(T(r,g)).$$

Consequently, we obtain

$$\begin{aligned}\| \; T(r,f) &\leq \frac{d_3(N)}{\ell_0 - N} \sum_{j=1}^{2N+1} (\ell_0 - N)\big(N(r, \bar{\nu}^\infty_{h_j}) + N(r, \bar{\nu}^0_{h_j})\big) \\ &\leq \frac{d_3(N)(N+1)}{\ell_0 - N}(T(r,f) + T(r,g)) + o(T(r,f) + T(r,g)).\end{aligned}$$

By adding this to the similar inequality for g, we get

$$\| \; T(r,f)+T(r,g) \leq \frac{2d_3(N)(N+1)}{\ell_0 - N}(T(r,f)+T(r,g))+o(T(r,f)+T(r,g)).$$

Divide both sides of this by $T(r,f) + T(r,g)$ and let r tend to $+\infty$ outside a set of finite measure. Then, we have necessarily

$$\ell_0 \leq 2d_3(N)(N+1) + N.$$

If the number ℓ_0 were chosen so as to satisfy the condition

$$\ell_0 > 2d_3(N)(N+1) + N$$

from the beginning, this is a contradiction. This shows that the case $r = N$ is impossible. The proof of Proposition 5.11 is completed.

§6. A uniqueness theorem for meromorphic maps into $P^2(\mathbb{C})$

In this section, we shall give a proof for the following theorem which is stated in §1:

THEOREM 6.1. *There are a positive integer ℓ_0 and a proper algebraic subset V of $(P^2(\mathbb{C})^*)^7$ with the following properties*:

For nondegenerate meromorphic maps $f, g : \mathbb{C}^n \to P^2(\mathbb{C})$ and seven hyperplanes H_j's in general position with $(H_1, \ldots, H_7) \notin V$, if

$$\min(\nu(f, H_j), \ell_0) = \min(\nu(g, H_j), \ell_0),$$

then $f \equiv g$.

Proof. As in §4, we consider the meromorphic functions h_j $(1 \leq j \leq 7)$ defined by (4.1) for the given hyperplanes H_j's in general position and maps f and g. Let $\overset{R}{\sim}$ be the pre-equivalence relation defined by Definition 4.2, where $q = 7$ and $N = 2$, and take real numbers $p_1, \dots, p_7$ with the properties of Proposition 2.4.

We first study the case where all numbers p_j's except one coincide with others. Changing indices, we assume that $p_1 = p_2 = \cdots = p_6$. In this case, there is some positive integer m such that $(h_i/h_{i'})^m$ are rational functions in logarithmic derivatives of h_j's with uniformly bounded weights for $1 \leq i < i' \leq 6$. Since H_j's are assumed to be in general position, we can choose homogeneous coordinates $(w_1 : w_2 : w_3)$ on $P^2(\mathbb{C})$ such that

$$\begin{aligned} &H_j : w_j = 0 \quad (j = 1, 2, 3), \\ &H_4 : w_1 + a w_2 + b w_3 = 0, \\ &H_5 : w_1 + c w_2 + d w_3 = 0, \\ &H_6 : w_1 + w_2 + w_3 = 0, \end{aligned}$$

where every minor of the matrix $\begin{pmatrix} 1 & a & b \\ 1 & c & d \\ 1 & 1 & 1 \end{pmatrix}$ of order ≤ 3 does not vanish. As in the previous section, we set

$$r := \operatorname{rank} \begin{pmatrix} h_1 - h_4 & a(h_2 - h_4) & b(h_3 - h_4) \\ h_1 - h_5 & c(h_2 - h_5) & d(h_3 - h_5) \\ h_1 - h_6 & h_2 - h_6 & h_3 - h_6 \end{pmatrix}$$

Here, the case $r = 2$ is impossible for a sufficiently large ℓ_0 because of Proposition 5.11. Assume that $r < 2$.

Set $k_j := h_j - h_6$ $(j = 1, 2, \dots, 5)$. We may assume that $(k_1, k_2, k_3) \neq (0, 0, 0)$. For, otherwise, $h_1 = h_2 = h_3 = h_6$, whence we have $f = g$. By assumption, there are meromorphic functions φ and ψ such that

$$\begin{aligned} &k_1 - k_4 = \varphi k_1, \quad a(k_2 - k_4) = \varphi k_2, \quad b(k_3 - k_4) = \varphi k_3, \\ &k_1 - k_5 = \psi k_1, \quad c(k_2 - k_5) = \psi k_2, \quad d(k_3 - k_5) = \psi k_3. \end{aligned}$$

These implies that

$$\begin{aligned} &(\varphi - 1)k_1 + k_4 = 0, \quad (\varphi - a)k_2 + a k_4 = 0, \quad (\varphi - b)k_3 + b k_4 = 0, \\ &(\psi - 1)k_1 + k_5 = 0, \quad (\psi - c)k_2 + c k_5 = 0, \quad (\psi - d)k_3 + d k_5 = 0. \end{aligned}$$

If $\varphi = 1$, then $k_4 = k_2 = k_3 = 0$ and hence $h_2 = h_3 = h_4 = h_6$, which gives $f = g$. For the case $\varphi = a, \varphi = b, \psi = 1, \psi = c$ or $\psi = d$, we have also the same conclusion $f = g$ similarly. Otherwise, we have

$$\frac{k_4}{k_5} = \frac{\varphi - 1}{\psi - 1} = \frac{\varphi/a - 1}{\psi/c - 1} = \frac{\varphi/b - 1}{\psi/d - 1},$$

and hence

$$\Big(\frac{1}{c} - \frac{1}{a}\Big)\varphi\psi - \Big(\frac{1}{c} - 1\Big)\psi + \Big(\frac{1}{a} - 1\Big)\varphi = 0,$$
$$\Big(\frac{1}{d} - \frac{1}{b}\Big)\varphi\psi - \Big(\frac{1}{d} - 1\Big)\psi + \Big(\frac{1}{b} - 1\Big)\varphi = 0.$$

These give

$$\bigg(\Big(\frac{1}{d} - \frac{1}{b}\Big)\Big(\frac{1}{c} - 1\Big) - \Big(\frac{1}{c} - \frac{1}{a}\Big)\Big(\frac{1}{d} - 1\Big)\bigg)\psi$$
$$- \bigg(\Big(\frac{1}{d} - \frac{1}{b}\Big)\Big(\frac{1}{a} - 1\Big) - \Big(\frac{1}{c} - \frac{1}{a}\Big)\Big(\frac{1}{b} - 1\Big)\bigg)\varphi = 0,$$

and we can conclude that $\varphi = \psi$ or

$$\chi := \frac{1}{ad} - \frac{1}{bc} + \frac{1}{c} - \frac{1}{d} + \frac{1}{b} - \frac{1}{a} = 0.$$

Therefore, if the given hyperplanes H_j's satisfy the condition $\chi \neq 0$, then $\varphi = \psi$ and hence $k_2 = k_3 = k_4 = k_5$, which gives $f = g$. This shows that Theorem 6.1 is true in this case.

To see the remaining cases, we may assume that

$$p_1 \leq p_2 \leq \cdots \leq p_7.$$

Then, by Proposition 2.10 (i) we have $p_3 = p_4 = p_5$.

Assume that

$$p_1 \leq p_2 < p_3 = p_4 = p_5 < p_6 \leq p_7.$$

Take two indices i, j with $3 \leq i < j \leq 5$ and choose

$$\iota(1) := 1,\ \ \iota(2) := 2,\ \ \iota(3) := i,\ \ \iota(4) := j,\ \ \iota(5) := 6,\ \ \iota(6) := 7,$$

and consider combinations $J := (i_1, i_2, i_3) \in \mathcal{I}_{6,3}$ with

$$p_{\iota(1)} + p_{\iota(2)} + p_{\iota(3)} = p_{\iota(i_1)} + p_{\iota(i_2)} + p_{\iota(i_3)}.$$

By Proposition 2.10 (ii), $J = (1,2,3)$ or $J = (1,2,4)$. Applying Proposition 3.4 to the products of three functions among $h_{\iota(1)}, \dots, h_{\iota(6)}$, we get

$$\sum_{(i_1,i_2,i_3)\sim(1,2,3)} A_{i_1i_2i_3} h_{\iota(i_1)} h_{\iota(i_2)} h_{\iota(i_3)} = C_i h_1 h_2 h_i + C_j h_1 h_2 h_j = 0,$$

where $C_i := A_{123} (\neq 0)$ and $C_j := A_{124} (\neq 0)$ are the constants given in the proof of Proposition 4.3. Since Proposition 2.10 remains valid if we replace p_j's by $-p_j$, we can apply the same arguments to combinations $J := (i_1, i_2, i_3) \in \mathcal{I}_{6,3}$ with $p_{\iota(4)} + p_{\iota(5)} + p_{\iota(6)} = p_{\iota(i_1)} + p_{\iota(i_2)} + p_{\iota(i_3)}$. So, we have

$$C'_i h_i h_6 h_7 + C'_j h_j h_6 h_7 = 0$$

for $C'_i := A_{356}$ and $C'_j := A_{456}$. These conclude that

$$C_i h_i + C_j h_j = C'_i h_i + C'_j h_j = 0,$$

whence we have $h_i = h_j$ except the case where $C_i C'_j = C'_i C_j$. Since we can choose i, j with $3 \leq i < j \leq 5$ arbitrarily, we can conclude that $h_3 = h_4 = h_5$ and hence $f = g$ for 'generically' given hyperplanes. Theorem 6.1 is valid in this case. Therefore, we may assume $p_2 = p_3$ or $p_5 = p_6$. Replacing p_j by $-p_j$ if necessary, we assume that

$$p_1 \leq p_2 \leq p_3 = p_4 = p_5 = p_6 \leq p_7.$$

Next, we study the case

$$p_1 \leq p_2 < p_3 = p_4 = p_5 = p_6 < p_7.$$

Choose i, j, k with $3 \leq i < j < k \leq 6$ and choose

$$\iota(1) := 1, \ \ \iota(2) := 2, \ \ \iota(3) := i, \ \ \iota(4) := j, \ \ \iota(5) := k, \ \ \iota(6) := 7.$$

By the same arguments as the above, for combinations $J := (j_1, j_2, j_3)$ with $p_{\iota(1)} + p_{\iota(2)} + p_{\iota(3)} = p_{\iota(j_1)} + p_{\iota(j_2)} + p_{\iota(j_3)}$, we have $J = (1,2,3)$, $J = (1,2,4)$ or $J = (1,2,5)$. It then follows that there are some nonzero polynomials D_i, D_j, D_k in the coefficients of the defining equations for H_j's such that $D_1 h_i + D_2 h_j + D_3 h_k = 0$. Moreover, observing combinations $J := (j_1, j_2, j_3)$ with $p_{\iota(4)} + p_{\iota(5)} + p_{\iota(6)} = p_{\iota(j_1)} + p_{\iota(j_2)} + p_{\iota(j_3)}$, we have $J = (3,4,6)$, $J = (4,5,6)$ or $J = (3,5,6)$ and so we have $D'_1 h_i h_j + D'_2 h_j h_k + D'_3 h_k h_i = 0$ for nonzero D'_j. This concludes that h_i/h_k and h_j/h_k are constants for 'generically' given hyperplanes H_j's. Since we can choose i, j, k arbitrarily, all h_i/h_6 $(3 \leq i \leq 6)$ are constants, whence $f = g$. Theorem 6.1 is true in this case too.

It remains to study the following two cases;

(a) $p_1 < p_2 = p_3 = p_4 = p_5 = p_6 < p_7$,

(b) $p_1 \leq p_2 < p_3 = p_4 = p_5 = p_6 = p_7$.

We first study the case (a). For our purpose, take four indices i, j, k, ℓ with $2 \leq i < j < k < \ell \leq 6$ arbitrarily, and we choose

$$\iota(1) = 1, \ \ \iota(2) = i, \ \ \iota(3) = j, \ \ \iota(4) = k, \ \ \iota(5) = \ell, \ \ \iota(6) = 7.$$

Consider (i_1, i_2, i_3) and (j_1, j_2, j_3) in $\mathcal{I}_{6,3}$ with

$$\begin{aligned} p_{\iota(1)} + p_{\iota(2)} + p_{\iota(3)} &= p_{\iota(i_1)} + p_{\iota(i_2)} + p_{\iota(i_k)}, \\ p_{\iota(4)} + p_{\iota(5)} + p_{\iota(6)} &= p_{\iota(j_1)} + p_{\iota(j_2)} + p_{\iota(j_k)}. \end{aligned}$$

By the same argument as the above, we have two homogeneous linear relations among ten functions $h_{mm'} := h_m h_{m'}$ $(2 \leq m < m' \leq 6)$. Since there are five possible choices of indices i, j, k, ℓ among $2, \dots, 6$, we obtain ten linear homogeneous relations among ten functions $h_{mm'}$'s. In this situation, it is not difficult to show that these linear relations are linearly independent for 'generically' chosen hyperplanes H_j's. This shows that the case (a) is impossible for 'generically' chosen hyperplanes H_j's, because each h_j's is nonzero. Theorem 6.1 is true in this case too.

Lastly, we study the case (b). In this case, choosing four indices among $3, 4, \dots, 7$, say $3, 4, 5, 6$, we choose $\iota(j) = j$ $(j = 1, 2, \dots, 6)$ and consider (i_1, i_2, i_3) with

$$p_1 + p_2 + p_3 = p_{i_1} + p_{i_2} + p_{i_3}.$$

Then, we can see easily $i_1 = 1, i_2 = 2$ and i_3 is equal to $3, 4, 5$ or 6, whence we have a homogeneous linear relation among h_3, h_4, h_5, h_6. In this way, we have five homogeneous linear relations among five functions $h_3, \dots, h_7$. In this case too, we can easily show that these linear relations are linearly independent for 'generically' chosen hyperplanes H_j's. The case (b) is also impossible for such hyperplanes. The proof of Theorem 6.1 is completed.

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