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HEIGHT ONE MATRICES

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Abstract. Let $\mathbb{P}^1(\overline{\mathbb{Q}})$ be the projective line over $\overline{\mathbb{Q}}$ and H the Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. A classical result in algebraic number theory, so called Kronecker's theorem, states that $H(1,x)=1$ if and only if $x \in \overline{\mathbb{Q}}$ is 0 or a root of unity. In [4], Talamanca introduced some height functions on $M_n(\overline{\mathbb{Q}})$. The purpose of this paper is to show analogues of Kronecker's theorem for these heights: We determine height one matrices relative to these heights.

1. Introduction

Let K be an algebraic number field. Throughout the paper, we employ the following notation:

- $\mathbb{Q}$: the field of rationals;
- $\overline{\mathbb{Q}}$: the field of algebraic numbers;
- $\mathbb{C}$: the field of complex numbers;
- $\mathcal{O}_K$: the ring of integers of K;
- $\mathcal{M}_K^\infty$: the set of all field homomorphisms from K to $\mathbb{C}$;
- $\mathcal{M}_K^0$: the set of all non-zero prime ideals of $\mathcal{O}_K$;
- $\mathcal{M}_K := \mathcal{M}_K^\infty \sqcup \mathcal{M}_K^0$;
- $|\cdot|_v$: an absolute value on K defined for each $v \in \mathcal{M}_K$ as

$$|x|_v := \begin{cases} |v(x)| & (v \in \mathcal{M}_K^\infty), \\ \#(\mathcal{O}_K/v)^{-\mathrm{ord}_v(x)} & (v \in \mathcal{M}_K^0), \end{cases}$$

  where $\mathrm{ord}_v(x)$ is the order of $x \in K$ for each $v \in \mathcal{M}_K^0$, *i.e.*, if $x \in K^\times$ and $(x) = \prod_{k=1}^n v_k^{e_k}$ is the prime factorization of the fractional ideal (x), we set

$$\mathrm{ord}_v(x) := \begin{cases} e_k & (v = v_k \text{ for some } k), \\ 0 & (v \neq v_k \text{ for any } k), \end{cases}$$

  and $\mathrm{ord}_v(0) := \infty$ for all $v \in \mathcal{M}_K^0$;
- K_v: the completion of K by the absolute value $|\cdot|_v$;

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- $M_n(F)$: the ring of $n \times n$ square matrices over a field F.

For $\vec{x} = {}^t(x_1, \ldots, x_n) \in K^n$, we set

$$H(\vec{x}) := \left(\prod_{v \in \mathcal{M}_K} \max\{|x_1|_v, \ldots, |x_n|_v\} \right)^{1/[K:\mathbb{Q}]}.$$

As usual, we set $H(\vec{0}) = 1$. Now, we note the product formula

$$\prod_{v \in M_K} |x|_v = 1 \text{ for all } x \in K^\times,$$

and hence we have $H(c\vec{x}) = H(\vec{x})$ for all $\vec{x} \in K^n$ and $c \in K^\times$. Since we know that the value of $H(\vec{x})$ is independent of the choice of K, we can consider H to be a function on $\overline{\mathbb{Q}}^n$. The function H is called the *Weil height*, measuring a certain kind of arithmetic complexity of $\vec{x} \in \overline{\mathbb{Q}}^n$. Note that $H(\vec{x}) \geq 1$ for all $\vec{x} \in \overline{\mathbb{Q}}^n$. The following theorem, which determines the vectors $\vec{x} \in \overline{\mathbb{Q}}^n$ which satisfy $H(\vec{x}) = 1$, is an immediate consequence of Kronecker's theorem (see, *e.g.*, [1], Theorem 1.5.9):

Theorem (A) . *Let $\vec{x} \in \overline{\mathbb{Q}}^n$. Then $H(\vec{x}) = 1$ if and only if there exist a constant $r \in \overline{\mathbb{Q}}$ and a vector $\vec{e} \in \overline{\mathbb{Q}}^n$ such that each of entries of $\vec{e}$ is 0 or a root of unity and $\vec{x} = r\vec{e}$.*

Now for each $v \in \mathcal{M}_K$, we set a norm N_v on K^n with respect to $|\cdot|_v$:

$$N_v(\vec{x}) := \begin{cases} \sqrt{|x_1|_v^2 + \cdots + |x_n|_v^2} & (v \in \mathcal{M}_K^\infty), \\ \max\{|x_1|_v, \ldots, |x_n|_v\} & (v \in \mathcal{M}_K^0). \end{cases}$$

In [4], Talamanca introduced some height functions on $M_n(K)$:

$$\mathcal{H}(A) := \left(\prod_{v \in \mathcal{M}_K} \|A\|_v \right)^{1/[K:\mathbb{Q}]},$$

$$\mathcal{H}^{\mathrm{op}}(A) := \sup_{\vec{x} \in \overline{\mathbb{Q}}^n \setminus \{\vec{0}\}} \left\{ \left(\prod_{v \in \mathcal{M}_{K_{\vec{x}}}} \frac{N_v(A\vec{x})}{N_v(\vec{x})} \right)^{1/[K_{\vec{x}}:\mathbb{Q}]} \right\},$$

where $\|A\|_v$ denotes the operator norm of $K_v^n \ni \vec{x} \mapsto A\vec{x} \in K_v^n$ induced by N_v, and $K_{\vec{x}} := K(x_1, \ldots, x_n)$ for each $\vec{x} = {}^t(x_1, \ldots, x_n) \in \overline{\mathbb{Q}}^n \setminus \{\vec{0}\}$. As usual, we set $\mathcal{H}(O) = \mathcal{H}^{\mathrm{op}}(O) = 1$. Since the values of $\mathcal{H}(A)$ and $\mathcal{H}^{\mathrm{op}}(A)$ are also independent of the choice of K, we can also consider $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$ to be functions on $M_n(\overline{\mathbb{Q}})$; see Sections 2 and 3 for more details.

The main purpose of this paper is to show analogues of Theorem (A) for $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$, *i.e.,* to determine the matrices $A \in M_n(\overline{\mathbb{Q}})$ which satisfy $\mathcal{H}(A) = 1$ or $\mathcal{H}^{\mathrm{op}}(A) = 1$. Throughout this paper, we call a matrix *scattered* if no two non-zero entries lie in the same row or the same column.

Theorem 1.1. *Let* $A \in M_n(\overline{\mathbb{Q}})$*. Then:*

(1) $\mathcal{H}(A) = 1$ *if and only if there exist a constant* $r \in \overline{\mathbb{Q}}$ *and a matrix* $B \in M_n(\overline{\mathbb{Q}})$ *such that* B *is scattered, each of entries of* B *is* 0 *or a root of unity, and* $A = rB$*;*

(2) $\mathcal{H}^{\mathrm{op}}(A) = 1$ *if and only if* A *has at most one non-zero row vector, or there exist* $r \in \overline{\mathbb{Q}}$ *and* $B \in M_n(\overline{\mathbb{Q}})$ *such that* B *is scattered, each of entries of* B *is* 0 *or a root of unity, and* $A = rB$*.*

2. The Weil L^2-height

As mentioned in Section 1, we defined $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$ by using the L^2-norm N_v for each $v \in \mathcal{M}_K^\infty$, while H is defined by using the L^∞-norm for each $v \in \mathcal{M}_K$. So the following height functions are more suitable when we study $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$; for $\vec{x} \in K^n$, we set

$$H_2(\vec{x}) := \left(\prod_{v \in \mathcal{M}_K} N_v(\vec{x}) \right)^{1/[K:\mathbb{Q}]},$$

$$H_2^+(\vec{x}) := \left(\prod_{v \in \mathcal{M}_K} \max\{1, N_v(\vec{x})\} \right)^{1/[K:\mathbb{Q}]}.$$

As usual, we set $H_2(\vec{0}) = 1$. By the product formula, we have $H_2(c\vec{x}) = H_2(\vec{x})$ for all $c \in K^\times$. We can also consider H_2 and H_2^+ to be functions on $\overline{\mathbb{Q}}^n$. The function H_2 is called the *Weil* L^2*-height.* Note that $H_2(\vec{x}) \geq 1$ for all $\vec{x} \in \overline{\mathbb{Q}}^n$. Now we have

$$\mathcal{H}^{\mathrm{op}}(A) = \sup_{\vec{x} \in \overline{\mathbb{Q}}^n \setminus \{\vec{0}\}} \left\{ \frac{\left(\prod_{v \in \mathcal{M}_{K_{\vec{x}}}} N_v(A\vec{x}) \right)^{1/[K_{\vec{x}}:\mathbb{Q}]}}{\left(\prod_{v \in \mathcal{M}_{K_{\vec{x}}}} N_v(\vec{x}) \right)^{1/[K_{\vec{x}}:\mathbb{Q}]}} \right\} = \sup_{\vec{x} \in \overline{\mathbb{Q}}^n \setminus \{\vec{0}\}} \left\{ \frac{H_2(A\vec{x})}{H_2(\vec{x})} \right\}.$$

Consequently, we can consider $\mathcal{H}^{\mathrm{op}}$ to be a function on $M_n(\overline{\mathbb{Q}})$; we will show $\mathcal{H}^{\mathrm{op}}(A) < \infty$ in the next section.

The following lemma is an analogue of Theorem (A) for the Weil L^2-height.

Lemma 2.1. *Let* $\vec{x} \in \overline{\mathbb{Q}}^n$*. Then:*

(1) $H_2(\vec{x}) = 1$ *if and only if* $\vec{x}$ *has at most one non-zero entry.*

(2) $H_2^+(\vec{x}) = 1$ *if and only if each of entries of* $\vec{x}$ *is* 0 *or a root of unity and* $\vec{x}$ *has at most one non-zero entry.*

Proof. (1) We know that the former condition implies the latter condition by the inequality $H_2(a, b) > H_2(a)$ for any a, $b \neq 0$, and that the latter condition implies the former condition by the product formula.

(2) We know that the latter condition implies the former condition. So we shall prove the converse. Take any $\vec{x} \in \overline{\mathbb{Q}}^n$ with $H_2^+(\vec{x}) = 1$. By the inequality $H(1, \vec{x}) \leq H_2^+(\vec{x})$ and Theorem (A), each of entries of $\vec{x}$ is 0 or a root of unity. By the inequality $H_2(\vec{x}) \leq H_2^+(\vec{x})$ and (1), $\vec{x}$ has at most one non-zero entry. □

3. Some basic properties of $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$

In this section, we summarize some basic properties of $\mathcal{H}$ and $\mathcal{H}^{\mathrm{op}}$.

First, we shall make some remarks on the operator norms mentioned in Section 1. Let $B = (b_{ij}) \in M_n(K_v)$. It is known that

$$\|B\|_v = \sqrt{\mathrm{sp}(B^*B)} \qquad \text{if } v \in \mathcal{M}_K^\infty, \tag{3.1}$$

$$\|B\|_v = \max_{i,j}\{|b_{ij}|_v\} \qquad \text{if } v \in \mathcal{M}_K^0, \tag{3.2}$$

where B^* is the adjoint of B and $\mathrm{sp}(B^*B)$ is the maximum eigenvalue of B^*B. Thus we find that for any $A \in M_n(K)$, we have $\|A\|_v = 1$ for all but finitely many $v \in \mathcal{M}_K$. Therefore to set $\mathcal{H}$, as is in Section 1, makes sense. We also find that the value of $\mathcal{H}(A)$ is independent of the choice of K, and hence we have $\mathcal{H}^{\mathrm{op}}(A) \leq \mathcal{H}(A) < \infty$. Thus we find that to set $\mathcal{H}^{\mathrm{op}}$, as is in Section 1, also makes sense.

Next, to study $\mathcal{H}$, we introduce an auxiliary height $\mathcal{H}^+$:

$$\mathcal{H}^+(A) := \left(\prod_{v \in \mathcal{M}_K} \max\{1, \|A\|_v\} \right)^{1/[K:\mathbb{Q}]},$$

where $A \in M_n(K)$. By (3.1) and (3.2), we have the following:

Lemma 3.1. *For any $A \in M_n(\overline{\mathbb{Q}})$, we have* $\mathcal{H}^+(A) = \mathcal{H}\begin{pmatrix} 1 & {}^t\vec{0} \\ \vec{0} & A \end{pmatrix}$.

Thus we can consider $\mathcal{H}^+$ to be a function on $M_n(\overline{\mathbb{Q}})$. The following inequalities play important roles in the proof of the main theorem.

Lemma 3.2. *Let $A = (\vec{a}_1, \ldots, \vec{a}_n) \in M_n(\overline{\mathbb{Q}})$. Then for any $1 \leq j \leq n$:*

(1) $H_2(\vec{a}_j) \leq \mathcal{H}^{\mathrm{op}}(A)$;
(2) $H_2^+(\vec{a}_j) \leq \mathcal{H}^+(A)$.

Proof. Let $(a_{ij}) := A$, $K := \mathbb{Q}(a_{11}, \ldots, a_{nn})$ and $\vec{e}_j := {}^t(0, \ldots, \underset{j\text{-th}}{\underset{\wedge}{1}}, \ldots, 0)$.

(1) We have

$$H_2(\vec{a}_j) = \frac{H_2(A\vec{e}_j)}{H_2(\vec{e}_j)} \leq \mathcal{H}^{\mathrm{op}}(A).$$

(2) We have

$$H_2^+(\vec{a}_j) = \left(\prod_{v \in \mathcal{M}_K} \max\left\{1, \frac{N_v(A\vec{e}_j)}{N_v(\vec{e}_j)}\right\} \right)^{1/[K:\mathbb{Q}]} \leq \mathcal{H}^+(A).$$

□

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and $B \in M_n(\mathbb{F})$. We know that

$$\|B^*\| = \|B\|, \tag{3.3}$$

where $\|\cdot\|$ is the operator norm induced by the standard L^2-norm on $\mathbb{F}^n$. Therefore, for any $A = (a_{ij}) \in M_n(K)$ and $\sigma \in \mathcal{M}_K^\infty$, we have

$$\|{}^tA\|_\sigma = \|(\sigma(a_{ji}))\| = \|(\overline{\sigma}(a_{ij}))\| = \|A\|_{\overline{\sigma}}, \tag{3.4}$$

where ${}^tA$ is the transpose of A and $\overline{\sigma}$ is the composition of the complex conjugation and σ. By (3.2) and (3.4), we have the following:

Lemma 3.3. *For any $A \in M_n(\overline{\mathbb{Q}})$, we have $\mathcal{H}(A) = \mathcal{H}({}^tA)$ and $\mathcal{H}^+(A) = \mathcal{H}^+({}^tA)$.*

The following lemma is a key to prove Theorem 1.1.

Lemma 3.4. *Let $A \in M_n(\overline{\mathbb{Q}})$. Then $\mathcal{H}^+(A) = 1$ if and only if each of entries of A is 0 or a root of unity and A is scattered.*

Proof. By (3.1) and (3.2), we know that the latter condition implies the former condition. So we shall prove the converse. Suppose that $\mathcal{H}^+(A) = 1$. Then, by Lemma 3.2 (2) and Lemma 2.1 (2), we find that each of column vectors of A has at most one non-zero entry and it is a root of unity. On the other hand, by Lemma 3.3, we find that each of row vectors of A has at most one non-zero entry. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. (1) We know that the latter condition implies the former condition. So we shall prove the converse. Let $A \in M_n(\overline{\mathbb{Q}}) \setminus \{O\}$ with $\mathcal{H}(A) = 1$. By the inequality $\mathcal{H}^{\mathrm{op}}(A) \leq \mathcal{H}(A)$, Lemma 3.2 (1) and Lemma 2.1 (1), we find that each of column vectors of A has at most one non-zero entry. On the other hand, by Lemma 3.3, we find that each of row vectors of A has at most one non-zero entry. Therefore A is scattered. Now note that multiplying A by any permutation matrix does not change the value of $\mathcal{H}(A)$. So we may assume that $a_{11} \neq 0$. Furthermore, by the product formula, we may also assume that $a_{11} = 1$. Thus we may assume that A is a form of

$$\begin{pmatrix} 1 & {}^t\vec{0} \\ \vec{0} & A' \end{pmatrix},$$

where $A' \in M_{n-1}(\overline{\mathbb{Q}})$. By Lemma 3.1, we have $\mathcal{H}^+(A') = \mathcal{H}(A)$. Therefore, by Lemma 3.4, each of entries of A' is 0 or a root of unity and A' is scattered. This completes the proof of (1).

(2) First, we shall prove that the latter condition implies the former condition. Suppose that A has at most one non-zero row vector:

$$A = \begin{pmatrix} & O & \\ a_1 & \cdots & a_n \\ & O & \end{pmatrix}.$$

Then for any $\vec{x} = {}^t(x_1, \dots, x_n) \in \overline{\mathbb{Q}}^n \setminus \{\vec{0}\}$, we have

$$\begin{aligned} &H_2(A\vec{x}) = H_2(a_1x_1 + \cdots + a_nx_n) = 1, \\ &H_2(\vec{x}) \geq 1. \end{aligned}$$

Therefore we have $\mathcal{H}^{\mathrm{op}}(A) = 1$. Suppose that there exist $r \in \overline{\mathbb{Q}}$ and $B \in M_n(\overline{\mathbb{Q}})$ such that B is scattered, each of entries of B is zero or a root of unity, and $A = rB$. Then we have $1 \leq \mathcal{H}^{\mathrm{op}}(A) \leq \mathcal{H}(A) = 1$ by (1).

Next, we shall prove the converse. Let $A = (a_{ij}) \in M_n(\overline{\mathbb{Q}})$ with $\mathcal{H}^{\mathrm{op}}(A) = 1$. By Lemma 3.2 (1) and Lemma 2.1 (1), we find that each of column vectors of A has at most one non-zero entry. Suppose that A has more than one non-zero entries a_{ij}, a_{kl} with $j < l$ and $i \neq k$. We set

$$\vec{c} := {}^t(0, \dots, 0, \underset{j\text{-th}}{\underset{\wedge}{1}}, 0, \dots, 0, \underset{l\text{-th}}{\underset{\wedge}{1}}, 0, \dots, 0).$$

Then we have

$$\begin{aligned} &H_2(A\vec{c}) = H_2\left(\frac{1}{a_{ij}}A\vec{c}\right) = H_2(1, a), \\ &H_2(\vec{c}) = \sqrt{2}, \end{aligned}$$

where $a := a_{kl}/a_{ij}$. By the inequality $H_2(A\vec{c})/H_2(\vec{c}) \leq \mathcal{H}^{\mathrm{op}}(A)$, we find that $H_2(1, a) \leq \sqrt{2}$. Let $K := \mathbb{Q}(a_{11}, \dots, a_{nn})$ and $d := [K : \mathbb{Q}]$. Now, note that

$$\max\{1, t\} \geq \sqrt{t}, \tag{4.1}$$

$$\sqrt{1 + t^2} \geq \sqrt{2t} \tag{4.2}$$

for all $t \geq 0$, and both the inequalities become equalities if and only if $t = 1$. Hence we have

$$\begin{aligned}
\sqrt{2} &\geq H_2(1, a) \\
&\geq \left(\prod_{v \in \mathcal{M}_K^0} |a|_v \right)^{1/2d} \left(2^d \prod_{v \in \mathcal{M}_K^\infty} |a|_v \right)^{1/2d} && \text{(by (4.1), (4.2))} \qquad (4.3) \\
&= \sqrt{2} && \text{(by the product formula).}
\end{aligned}$$

Therefore, the inequality (4.3) must be an equality. So we have $|a|_v = 1$ for all $v \in \mathcal{M}_K$. Thus, by Kronecker's theorem, we find that a is a root of unity. Therefore if A is scattered, then there exist $r \in \overline{\mathbb{Q}}$ and $B \in M_n(\overline{\mathbb{Q}})$ such that B is scattered, each of entries of B is zero or a root of unity, and $A = rB$. Now, suppose that A is not scattered and that A has more than one non-zero row vectors. Then there exist non-zero entries a_{ij}, a_{kl} and a_{km} of A with $i \neq k$ and $j \neq l \neq m \neq j$. For simplicity, we assume that $j < l < m$. By the argument described above, $u_1 := a_{kl}/a_{ij}$ and $u_2 := a_{km}/a_{ij}$ must be roots of unity. Taking positive integers p and q such that $u_1^p = u_2^q = 1$, we set

$$\vec{d} := {}^t(0, \dots, 0, \underset{j\text{-th}}{\underset{\wedge}{1}}, 0, \dots, 0, \underset{l\text{-th}}{\underset{\wedge}{u_1^{p-1}}}, 0, \dots, 0, \underset{m\text{-th}}{\underset{\wedge}{u_2^{q-1}}}, 0, \dots, 0).$$

Then we have

$$\begin{aligned}
H_2(A\vec{d}) &= H_2\left(\frac{1}{a_{ij}} A\vec{d}\right) = H_2(1, 2) = \sqrt{5}, \\
H_2(\vec{d}) &= \sqrt{3}.
\end{aligned}$$

Hence $\mathcal{H}^{\mathrm{op}}(A) \geq H_2(A\vec{d})/H_2(\vec{d}) = \sqrt{5/3} > 1$, which contradicts the assumption $\mathcal{H}^{\mathrm{op}}(A) = 1$. $\square$

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References

[1] E. Bombieri and W. Gubler, Heights in Diophantine Geometry, New Mathematical Monographs, 4. Cambridge University Press, Cambridge, 2006.

[2] J. Chahal, Topics in Number Theory, The University Series in Mathematics, Plenum Press, New York, 1988.

[3] V. Talamanca, *Height preserving linear transformations on linear spaces*, Ph.D. thesis, Brandeis University, 1995.

[4] V. Talamanca, *A Gelfand-Beurling type formula for heights on endomorphism rings*, J. Number Theory **83** (2000), no. 1, 91–105.

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