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COMPUTATIONAL METHODS FOR SET-RELATION-BASED SCALARIZING FUNCTIONS

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Abstract. In this research, we propose computational methods to evaluate scalarizing functions, which are defined via set-relations. In recent years, many theoretical results of the scalarizing functions for sets have been published. The aim of this paper is to show that each value of the scalarizing functions can be computed and to introduce computational algorithms of them for convex polytopes in a finite dimensional space.

1. Introduction

Generally speaking, multiobjective programming and vector optimization are studied based on "multicriteria" evaluation like some partial orderings or preorders. Gerstewitz's (Tammer's) sublinear scalarizing function for vectors [4, 5] is one of important mathematical tools in those areas to get optimal solutions for multicriteria decision problems without convexity assumptions. This kind of scalarizing functions is the smallest strictly monotonic (increasing) function with respect to the ordering structure; see [6, 12]. Based on this property, Georgiev and Tanaka [2, 3] applied it to obtain Fan's type inequalities for multifunctions with vector-valued images. Also, Nishizawa and Tanaka [13] studied certain characterizations of set-valued mappings by using the inherited properties of the scalarizing function on cone-convexity and cone-semicontinuity.

On the other hand, Sonda, Tanaka, and Yamada [15] proposed a certain computational scheme to calculate practically four types of scalarizing functions for a given set. Their approach is based on the idea of Gerstewitz's sublinear scalarizing function.

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Recently, theoretical results on set-relation-based scalarizing functions have been researched; see [14] and references cited therein. Especially, sublinear scalarizing functions for sets are proposed in [11] based on the binary relations introduced by Kuroiwa, Tanaka, and Ha in [9]. In this study, we give a computational scheme to calculate these scalarizing functions from the viewpoint of set-oriented and vector-oriented versions.

The organization of this paper is as follows. In Section 2, we recall the definitions of set-relations and scalarizing functions for sets. In Section 3, we give reformation of these functions. In Section 4, we propose computational algorithms in a finite-dimensional Euclidean space for certain polytope cases with a convex polyhedral cone.

2. Preliminaries

Throughout this paper, let X be a real topological vector space and X^* the continuous linear functional space of X. The topological interior and the convex hull of a set $A \subset X$ are denoted by $\operatorname{int} A$ and $\operatorname{co} A$, respectively. Let C be a non-empty convex solid ($\operatorname{int} C \neq \emptyset$) cone in X. The ordering $\leq_C$ induced by C is as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x,\, y \in X.$$

Obviously, the ordering $\leq_C$ has reflexive and transitive properties since C is a convex cone.

Firstly, let us introduce binary relations between two non-empty sets as follows.

Definition 2.1 (set-relations, [9])**.** For any non-empty sets $A,\, B \subset X$, we write

(i) $A \leq_C^{(1)} B$ by $\forall a \in A,\ \forall b \in B,\ a \leq_C b$, equivalently $A \subset \bigcap_{b \in B}(b - C)$;

(ii) $A \leq_C^{(2)} B$ by $\exists a \in A$ s.t. $\forall b \in B,\ a \leq_C b$, equivalently $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \leq_C^{(3)} B$ by $\forall b \in B,\ \exists a \in A$ s.t. $a \leq_C b$, equivalently $B \subset A + C$;

(iv) $A \leq_C^{(4)} B$ by $\exists b \in B$ s.t. $\forall a \in A,\ a \leq_C b$, equivalently $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;

(v) $A \leq_C^{(5)} B$ by $\forall a \in A,\ \exists b \in B$ s.t. $a \leq_C b$, equivalently $A \subset B - C$;

(vi) $A \leq_C^{(6)} B$ by $\exists a \in A,\ \exists b \in B$ s.t. $a \leq_C b$, equivalently $A \cap (B - C) \neq \emptyset$.

Proposition 2.1. *For any non-empty sets $A, B \subset X$, the following statements hold:*

$$A \leq_C^{(1)} B \textit{ iff } B \leq_{-C}^{(1)} A; \quad A \leq_C^{(2)} B \textit{ iff } B \leq_{-C}^{(4)} A;$$
$$A \leq_C^{(3)} B \textit{ iff } B \leq_{-C}^{(5)} A; \quad A \leq_C^{(4)} B \textit{ iff } B \leq_{-C}^{(2)} A;$$
$$A \leq_C^{(5)} B \textit{ iff } B \leq_{-C}^{(3)} A; \quad A \leq_C^{(6)} B \textit{ iff } B \leq_{-C}^{(6)} A.$$

Proof. By Definition 2.1, the statements are clear. □

The six binary relations $\leq_C^{(1)}, \ldots, \leq_C^{(6)}$ are certain generalizations of an ordering for vectors induced by a convex cone in a vector space. Especially, $\leq_C^{(3)}$ and $\leq_C^{(5)}$ are preorders for sets. If B is a singleton set in Definition 2.1, set-relations $\leq_C^{(2)}$, $\leq_C^{(3)}$, $\leq_C^{(6)}$ are coincident with each other, and the others $\leq_C^{(1)}$, $\leq_C^{(4)}$, $\leq_C^{(5)}$ coincide. Based on these binary relations, we introduce the following scalarizing functions for sets.

Definition 2.2 (scalarizing functions, [11])**.** Let A and V be non-empty subsets of X and a direction $k \in \operatorname{int} C$. For each $j = 1, \ldots, 6$, we define the scalarizing function $I_{k,V}^{(j)} : 2^X \backslash \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$I_{k,V}^{(j)}(A) := \inf \left\{ t \in \mathbb{R} \;\middle|\; A \leq_C^{(j)} (V + tk) \right\}.$$

These scalarizing functions measure how far a set needs to be moved toward a specific direction to fulfill each set-relation. They are a kind of generalization of the scalarizing functions for sets introduced in [8] and inspired by the sublinear scalarizing function originally introduced in [4, 5].

3. Reformation of set-relation-based scalarizing functions

In this section, we reform the scalarizing functions in Definition 2.2 to vector-oriented expressions by using the ordering $\leq_C$ for vectors.

Proposition 3.1. *The scalarizing functions for sets can be reformed as follows:*

$$\begin{aligned}
\text{(i)}\ \ I_{k,V}^{(1)}(A) &= \sup_{a \in A} \sup_{v \in V} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}; \\
\text{(ii)}\ \ I_{k,V}^{(2)}(A) &= \inf_{a \in A} \sup_{v \in V} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}; \\
\text{(iii)}\ \ I_{k,V}^{(3)}(A) &= \sup_{v \in V} \inf_{a \in A} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}; \\
\text{(iv)}\ \ I_{k,V}^{(4)}(A) &= \inf_{v \in V} \sup_{a \in A} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}; \\
\text{(v)}\ \ I_{k,V}^{(5)}(A) &= \sup_{a \in A} \inf_{v \in V} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}; \\
\text{(vi)}\ \ I_{k,V}^{(6)}(A) &= \inf_{a \in A} \inf_{v \in V} \inf\{t \in \mathbb{R} \mid a \leq_C v + tk\}.
\end{aligned}$$

Proof. It follows from Definition 2.1 that the scalarizing function on the left-hand side of each statement above is represented by an elementwise formula with $\leq_C$.

(i) At first we shall prove that

$$\inf\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} = \sup_{a \in A} \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\}. \tag{1}$$

By selecting elements $a' \in A$ and $v' \in V$, it is easy to see

$$\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} \subset \{t \mid a' \leq_C v' + tk\}.$$

Taking the infimum for both sides of this formula, we have

$$\inf\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} \geq \inf\{t \mid a' \leq_C v' + tk\}.$$

Since a' and v' are arbitrary,

$$\inf\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} \geq \sup_{a \in A} \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\}.$$

Now, we assume that there exists $\bar{s} \in \mathbb{R}$ such that

$$\inf\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} > \bar{s} > \sup_{a \in A} \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\}. \tag{2}$$

Since $\sup_{a \in A} \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\} < \bar{s}$, for all $a \in A$ and $v \in V$ there exists $s_{a,v} < \bar{s}$ such that $a \leq_C v + s_{a,v}k$. It follows that $v - a \in C - s_{a,v}k \subset C - \bar{s}k$. This implies that $a \leq_C v + \bar{s}k$. Therefore, $\inf\{t \mid \forall a \in A,\ \forall v \in V,\ a \leq_C v + tk\} \leq \bar{s}$, which contradicts (2). Thus, (1) holds.

(ii) In a similar way to the proof of (i), we get

$$\inf\{t \mid \exists a \in A \text{ s.t. } \forall v \in V,\ a \leq_C v + tk\} = \inf_{a \in A} \inf\{t \mid \forall v \in V,\ a \leq_C v + tk\}. \tag{3}$$

From (1), we have

$$\inf\{t \mid \forall v \in V,\ a \leq_C v + tk\} = \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\} \tag{4}$$

for each $a \in A$.

Hence, by (3) and (4), we obtain

$$\inf\{t \mid \exists a \in A \text{ s.t. } \forall v \in V,\ a \leq_C v + tk\} = \inf_{a \in A} \sup_{v \in V} \inf\{t \mid a \leq_C v + tk\}.$$

(iii) In a similar way to the proof of (i), we get

$$\inf\{t \mid \forall v \in V,\ \exists a \in A \text{ s.t. } a \leq_C v + tk\} = \sup_{v \in V} \inf\{t \mid \exists a \in A \text{ s.t. } a \leq_C v + tk\}. \tag{5}$$

From (3), we have

$$\inf\{t \mid \exists a \in A \text{ s.t. } a \leq_C v + tk\} = \inf_{a \in A} \inf\{t \mid a \leq_C v + tk\} \tag{6}$$

for each $v \in V$.

Thus, by (5) and (6), we complete the proof.

(iv) By Proposition 2.1,

$$\inf\left\{t \,\middle|\, A \leq_C^{(4)} (V + tk)\right\} = \inf\left\{t \,\middle|\, V \leq_{-C}^{(2)} (A + t(-k))\right\}. \tag{7}$$

From the result (ii), we have

$$\inf\left\{t \,\middle|\, V \leq_{-C}^{(2)} (A + t(-k))\right\} = \inf_{v \in V} \sup_{a \in A} \inf\{t \mid v \leq_{-C} a + t(-k)\}. \tag{8}$$

By (7) and (8),

$$\begin{aligned}\inf\left\{t \,\middle|\, A \leq_C^{(4)} (V + tk)\right\} &= \inf_{v \in V} \sup_{a \in A} \inf\{t \mid v \leq_{-C} a + t(-k)\} \\ &= \inf_{v \in V} \sup_{a \in A} \inf\{t \mid a \leq_C v + tk\},\end{aligned}$$

which completes the proof.

(v) By Proposition 2.1,

$$\inf\left\{t \,\middle|\, A \leq_C^{(5)} (V+tk)\right\} = \inf\left\{t \,\middle|\, V \leq_{-C}^{(3)} (A+t(-k))\right\}. \tag{9}$$

From the result (iii), we have

$$\inf\left\{t \,\middle|\, V \leq_{-C}^{(3)} (A+t(-k))\right\} = \sup_{a\in A}\inf_{v\in V}\inf\{t \mid v \leq_{-C} a+t(-k)\}. \tag{10}$$

By (9) and (10),

$$\begin{aligned}\inf\left\{t \,\middle|\, A \leq_C^{(5)} (V+tk)\right\} &= \sup_{a\in A}\inf_{v\in V}\inf\{t \mid v \leq_{-C} a+t(-k)\}\\ &= \sup_{a\in A}\inf_{v\in V}\inf\{t \mid a \leq_C v+tk\},\end{aligned}$$

which completes the proof.

(vi) In a similar way to the proof of (i), the equality

$$\inf\{t \mid \exists a \in A,\ \exists v \in V \text{ s.t. } a \leq_C v+tk\} = \inf_{a\in A}\inf_{v\in V}\inf\{t \mid a \leq_C v+tk\}$$

is clear.

□

4. Calculation algorithms of the scalarizing functions

In this section, we consider algorithms for computing the scalarizing functions under certain assumptions. To this end, we consider a Euclidean space $\mathbb{R}^n$ with the usual inner product $\langle a,b\rangle := \sum_{i=1}^n a_i b_i$ for $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)\in\mathbb{R}^n$. Also, in this section, we assume that $C\subset\mathbb{R}^n$ is a solid convex polyhedral cone defined as $C := \bigcap_{i=1}^m \{x\in\mathbb{R}^n \mid \langle p_i,x\rangle \geq 0\}$, where $p_1,\dots,p_m\in\mathbb{R}^n\backslash\{\theta^n\}$, and that A, $V\subset\mathbb{R}^n$ are convex polytopes defined as $A=\mathrm{co}\,\{a_1,\dots,a_\alpha\}$ and $V=\mathrm{co}\,\{v_1,\dots,v_\beta\}$, where $a_1,\dots,a_\alpha,v_1,\dots,v_\beta\in\mathbb{R}^n$.

Firstly, we consider a sort of reformation of a scalarizing function for vectors using the convex polyhedral cone.

Proposition 4.1 (see Proposition 1.44 and Corollary 1.45 of [1]). *Assume that $k\in\operatorname{int} C$. We have*

$$\inf\{t\in\mathbb{R} \mid x\leq_C tk\} = \max_{i=1,\dots,m}\left\langle \frac{p_i}{\langle p_i,k\rangle}, x\right\rangle \quad \textit{for } x\in\mathbb{R}^n.$$

Proof. Since $k\in\operatorname{int} C$, $\langle p_i,k\rangle > 0$ for each $i=1,\dots,m$. It follows from the definition of C that for all $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$,

$$x\leq_C tk \iff \langle p_i, tk-x\rangle \geq 0,\ \forall i=1,\dots,m$$

$$\iff t \geq \left\langle \frac{p_i}{\langle p_i, k\rangle}, x \right\rangle, \ \forall i = 1, \ldots, m$$

$$\iff t \geq \max_{i=1,\ldots,m} \left\langle \frac{p_i}{\langle p_i, k\rangle}, x \right\rangle.$$

Thus,

$$\inf \{t \in \mathbb{R} \mid x \leq_C tk\} = \max_{i=1,\ldots,m} \left\langle \frac{p_i}{\langle p_i, k\rangle}, x \right\rangle.$$

□

By replacing x in Proposition 4.1 with $a - v$ for $a \in A$ and $v \in V$, we can give a computational scheme to calculate the six scalarizing functions in Proposition 3.1.

Theorem 4.1. *Assume that $k \in \operatorname{int} C$. For each $q \in \mathbb{N}$, let $I(q) := \{1, \ldots, q\}$ and*

$$M^q := \left\{ (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^n \;\middle|\; \sum_{r=1}^{q} \lambda_r = 1, \ \lambda_r \geq 0 \textit{ for } r \in I(q) \right\}.$$

Then, we get

(i) $I_{k,V}^{(1)}(A) = \max_{s \in I(\alpha)} \max_{j \in I(\beta)} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a_s - v_j \right\rangle;$

(ii) $I_{k,V}^{(2)}(A) = \min_{\lambda \in M^\alpha} \max_{j \in I(\beta)} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_j \right\rangle;$

(iii) $I_{k,V}^{(3)}(A) = \max_{j \in I(\beta)} \min_{\lambda \in M^\alpha} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_j \right\rangle;$

(iv) $I_{k,V}^{(4)}(A) = \min_{\mu \in M^\beta} \max_{s \in I(\alpha)} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a_s - \sum_{j=1}^{\beta} \mu_j v_j \right\rangle;$

(v) $I_{k,V}^{(5)}(A) = \max_{s \in I(\alpha)} \min_{\mu \in M^\beta} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a_s - \sum_{j=1}^{\beta} \mu_j v_j \right\rangle;$

(vi) $I_{k,V}^{(6)}(A) = \min_{\lambda \in M^\alpha} \min_{\mu \in M^\beta} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - \sum_{j=1}^{\beta} \mu_j v_j \right\rangle.$

Proof. By Propositions 3.1 and 4.1, we get

$$I_{k,V}^{(1)}(A) = \sup_{a \in A} \sup_{v \in V} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle; \quad I_{k,V}^{(2)}(A) = \inf_{a \in A} \sup_{v \in V} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle;$$

$$I_{k,V}^{(3)}(A) = \sup_{v \in V} \inf_{a \in A} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle; \quad I_{k,V}^{(4)}(A) = \inf_{v \in V} \sup_{a \in A} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle;$$

$$I_{k,V}^{(5)}(A) = \sup_{a \in A} \inf_{v \in V} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle; \quad I_{k,V}^{(6)}(A) = \inf_{a \in A} \inf_{v \in V} \max_{i \in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle.$$

For all $a \in A$ and $v \in V$, there exist $\lambda \in M^{\alpha}$ and $\mu \in M^{\beta}$ such that $a = \sum_{s=1}^{\alpha} \lambda_s a_s$, $v = \sum_{j=1}^{\beta} \mu_j v_j$.

Firstly, we prove statement (i).

$$\begin{aligned}
I_{k,V}^{(1)}(A) &= \sup_{a\in A} \sup_{v\in V} \max_{i\in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle \\
&= \sup_{a\in A} \max_{i\in I(m)} \sup_{\mu\in M^{\beta}} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{j=1}^{\beta} \mu_j (a - v_j) \right\rangle \\
&= \sup_{a\in A} \max_{i\in I(m)} \max_{j\in I(\beta)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v_j \right\rangle \\
&= \max_{i\in I(m)} \max_{j\in I(\beta)} \sup_{\lambda\in M^{\alpha}} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s (a_s - v_j) \right\rangle \\
&= \max_{i\in I(m)} \max_{j\in I(\beta)} \max_{s\in I(\alpha)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a_s - v_j \right\rangle \\
&= \max_{s\in I(\alpha)} \max_{j\in I(\beta)} \max_{i\in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a_s - v_j \right\rangle.
\end{aligned}$$

Next, we prove statement (ii).

$$\begin{aligned}
I_{k,V}^{(2)}(A) &= \inf_{a\in A} \sup_{v\in V} \max_{i\in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v \right\rangle \\
&= \inf_{a\in A} \max_{i\in I(m)} \sup_{\mu\in M^{\beta}} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{j=1}^{\beta} \mu_j (a - v_j) \right\rangle \\
&= \inf_{a\in A} \max_{i\in I(m)} \max_{j\in I(\beta)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, a - v_j \right\rangle \\
&= \inf_{\lambda\in M^{\alpha}} \max_{i\in I(m)} \max_{j\in I(\beta)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_j \right\rangle \\
&= \min_{\lambda\in M^{\alpha}} \max_{j\in I(\beta)} \max_{i\in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_j \right\rangle.
\end{aligned}$$

Actually, since the real-valued function

$$f(\lambda) := \max_{j\in I(\beta)} \max_{i\in I(m)} \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_j \right\rangle$$

is continuous on the compact set M^{α}, it must attain its minimum on M^{α}. Therefore, the last equality above holds.

Then, we can prove statements (iv) and (vi) similarly.

Next, we prove (iii). Using the convexity of $A+C$,

$$\inf\{t\in\mathbb{R}\mid (V+tk)\subset A+C\}=\max_{j\in I(\beta)}\inf\{t\in\mathbb{R}\mid v_j+tk\in A+C\}. \tag{11}$$

Then, for all $j\in I(\beta)$, the following equality holds:

$$\inf\{t\in\mathbb{R}\mid v_j+tk\in A+C\}=\inf\{t\in\mathbb{R}\mid A\cap(v_j-C+tk)\neq\emptyset\}. \tag{12}$$

Taking $V:=\{v_j\}$ in (ii),

$$\inf\{t\in\mathbb{R}\mid A\cap(v_j-C+tk)\neq\emptyset\}=\min_{\lambda\in M^\alpha}\max_{i\in I(m)}\left\langle \frac{p_i}{\langle p_i,k\rangle},\sum_{s=1}^{\alpha}\lambda_s a_s-v_j\right\rangle. \tag{13}$$

By (11), (12) and (13), we have

$$I^{(3)}_{k,V}(A)=\max_{j\in I(\beta)}\min_{\lambda\in M^\alpha}\max_{i\in I(m)}\left\langle \frac{p_i}{\langle p_i,k\rangle},\sum_{s=1}^{\alpha}\lambda_s a_s-v_j\right\rangle.$$

At last, we can prove statement (v) similarly. □

From Theorem 4.1, we can easily calculate $I^{(1)}_{k,V}(A)$ by finding the maximum of at most $\alpha\times\beta\times m$ real numbers. In contrast, since $I^{(2)}_{k,V}(A),\ldots,I^{(6)}_{k,V}(A)$ include the form of convex combination, it is necessary to solve certain linear programming problems. In fact, the value of $I^{(2)}_{k,V}(A)$ is obtained by solving the following linear programming problem for $(t,\lambda_1,\ldots,\lambda_\alpha)$:

$$\text{(LP(2)):}\quad \boxed{\begin{array}{ll} \text{Minimize} & t\in\mathbb{R}\\ \text{subject to} & t\geq\left\langle \frac{p_i}{\langle p_i,k\rangle},\sum_{s=1}^{\alpha}\lambda_s a_s-v_j\right\rangle,\ \text{for all } i\in I(m),\ j\in I(\beta),\\ & \displaystyle\sum_{s=1}^{\alpha}\lambda_s=1,\\ & \lambda_s\geq 0\quad (s=1,\ldots,\alpha).\end{array}}$$

Besides, we reform the above problem as follows:

$$\text{(LP(2)):}\quad \boxed{\begin{array}{ll} \text{Minimize} & t\in\mathbb{R}\\ \text{subject to} & \displaystyle\min_{j\in I(\beta)}\langle p_1,v_j\rangle\geq\left\langle p_1,\sum_{s=1}^{\alpha}\lambda_s a_s-tk\right\rangle,\\ & \vdots\\ & \displaystyle\min_{j\in I(\beta)}\langle p_m,v_j\rangle\geq\left\langle p_m,\sum_{s=1}^{\alpha}\lambda_s a_s-tk\right\rangle,\\ & \displaystyle\sum_{s=1}^{\alpha}\lambda_s=1,\\ & \lambda_s\geq 0\quad (s=1,\ldots,\alpha).\end{array}}$$

For the value of $I^{(3)}_{k,V}(A)$, we need to solve the following linear programming problems LP(3-1), LP(3-2),...,LP(3-β) for $(t, \lambda_1, \ldots, \lambda_\alpha)$ and take the maximum of β optimal values of these subproblems.

$$(\text{LP}(3\text{-}1)): \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & t \geq \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_1 \right\rangle, \text{ for all } i \in I(m), \\ & \displaystyle\sum_{s=1}^{\alpha} \lambda_s = 1, \\ & \lambda_s \geq 0 \quad (s = 1, \ldots, \alpha). \end{array}}$$

$$\vdots$$

$$(\text{LP}(3\text{-}\beta)): \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & t \geq \left\langle \frac{p_i}{\langle p_i, k\rangle}, \sum_{s=1}^{\alpha} \lambda_s a_s - v_\beta \right\rangle, \text{ for all } i \in I(m), \\ & \displaystyle\sum_{s=1}^{\alpha} \lambda_s = 1, \\ & \lambda_s \geq 0 \quad (s = 1, \ldots, \alpha). \end{array}}$$

Then, we reform the above problems as follows:

$$(\text{LP}(3\text{-}1)): \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & \langle p_1, v_1 \rangle \geq \langle p_1, \sum_{s=1}^{\alpha} \lambda_s a_s - tk \rangle, \\ & \vdots \\ & \langle p_m, v_1 \rangle \geq \langle p_m, \sum_{s=1}^{\alpha} \lambda_s a_s - tk \rangle, \\ & \displaystyle\sum_{s=1}^{\alpha} \lambda_s = 1, \\ & \lambda_s \geq 0 \quad (s = 1, \ldots, \alpha). \end{array}}$$

$$\vdots$$

$$(\text{LP}(3\text{-}\beta)): \quad \boxed{\begin{array}{ll} \text{Minimize} & t \in \mathbb{R} \\ \text{subject to} & \langle p_1, v_\beta \rangle \geq \langle p_1, \sum_{s=1}^{\alpha} \lambda_s a_s - tk \rangle, \\ & \vdots \\ & \langle p_m, v_\beta \rangle \geq \langle p_m, \sum_{s=1}^{\alpha} \lambda_s a_s - tk \rangle, \\ & \displaystyle\sum_{s=1}^{\alpha} \lambda_s = 1, \\ & \lambda_s \geq 0 \quad (s = 1, \ldots, \alpha). \end{array}}$$

The values of $I^{(4)}_{k,V}(A)$, $I^{(5)}_{k,V}(A)$, $I^{(6)}_{k,V}(A)$ are similarly obtained.

5. Conclusions

We reform six types of scalarizing functions for sets in an ordered vector space to vector-oriented expressions by using the vector ordering. Also, we propose computational algorithms of the functions in a finite-dimensional Euclidean space for certain polytope cases with a convex polyhedral cone inducing the ordering. As a result, we show that the problem to calculate each scalarizing function can be decomposed into finite numbers of linear programming subproblems.

In addition to the six types of scalarizing functions (called "inf types") , we remark that other types (called "sup types") given in [11] can be applied to our reformation (by setting $C := -C,\ k := -k$). This implies that our result is a generalization of the computational methods proposed in [15].

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